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Generalized homoderivations and their orthogonal properties

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Abstract

This paper investigates the orthogonality of generalized homoderivation mappings in semiprime rings, with an emphasis on specific identities involving their product and composition. The concept of orthogonal generalized homoderivations is introduced, broadening the scope of previously established results. Two Generalized homoderivation mappings $(\mathcal{K}, ꝺ)$ and (∂, μ) of a ring $\mathbb{M}$ are considered orthogonal if the following conditions hold for all $\wp, \tau \in \mathbb{M}$:

i. If $\mathcal{K}$ is a zero-power valued mapping and the products $\mathcal{K}(\wp)\partial(\tau)$ and $ꝺ(\wp)\partial(\tau)$ are identical to zero.

ii. If the product $\mathcal{K}(\wp)\partial(\tau)$ and the compositions $ꝺ\partial$ and $ꝺ\mu$ are all zero.

These results offer new insights and generalizations on homoderivations in semiprime rings.

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Keywords: *Homoderivation, Orthogonal generalized homoderivation, Semiprime ring, Derivation.*

1. Introduction

Throughout, $\mathbb{M}$ will represent an associative ring with center $\mathcal{G}$. A mapping $\xi: \mathbb{M} \rightarrow \mathbb{M}$ is called zero-power valued ($\mathcal{Z}$, for short) on a subset $\mathfrak{S}$ of $\mathbb{M}$ if $\xi(\mathfrak{S})$ is a subset of $\mathfrak{S}$ and $\xi^{\mathfrak{y}(\sigma)}(\sigma)$ is equal to zero, for some a positive integer $\mathfrak{y}(\sigma)$ >1 [1]. For all elements $\mathcal{k}$ of $\mathbb{M}$, if $2\mathcal{k} = 0$ implies $\mathcal{k} = 0$, then $\mathbb{M}$ is considered 2-torsion free. Additionally, $\mathbb{M}$ is prime if $\varkappa\mathbb{M}\wp = 0$ implies $\varkappa = 0$ or $\wp = 0$, and $\mathbb{M}$ is semiprime if $\wp\mathbb{M}\wp = 0$ implies $\wp = 0$ [2]. In 2025 definition of a derivation sparked significant interest among many authors, leading to extensive exploration of derivation theory across various algebraic structures, all grounded

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in his original concept of a derivation on a ring [3]. A derivation (Ɗ, for short) is an additive mapping $\mathrm{d}: \mathbb{M} \to \mathbb{M}$ satisfying $\mathrm{d}(\varkappa\wp) = \mathrm{d}(\varkappa)\,\wp + \varkappa\,\mathrm{d}(\wp)$ for all elements $\varkappa, \wp \in \mathbb{M}$. The Ɗ-mapping, developed over the past half-century, is crucial for calculating matrix eigenvalues, impacting mathematics, science, business, engineering, and quantum physics. Furthermore, many authors have expanded on it in various ways. In [2], author was seemingly the first to introduce the concept of generalized derivation (ƓƊ, for short) as one of these expansion methods. An additive mapping $\mathcal{K}$ from $\mathbb{M}$ into itself is called a ƓƊ-mapping associated with a Ɗ-mapping d of $\mathbb{M}$ if $\mathcal{K}(\varkappa_1\varkappa_2) = \mathcal{K}(\varkappa_1)\varkappa_2 + \varkappa_1\mathrm{d}(\varkappa_2)$ for all $\varkappa_1, \varkappa_2 \in \mathbb{M}$. Several types of Ɗ and ƓƊ -mappings are discussed in ([4-9]). According to [1], author introduced the concept of homoderivations ($\mathcal{H}$Ɗ, for short), which generalizes the familiar notions of a ring endomorphism and a Ɗ-mapping. An additive mapping d from $\mathbb{M}$ into itself is said to be $\mathcal{H}$Ɗ -mapping if for all elements $\varkappa_1, \varkappa_2 \in \mathbb{M}$, the following condition holds: $\mathrm{d}(\varkappa_1\varkappa_2) = \mathrm{d}(\varkappa_1)\mathrm{d}(\varkappa_2) + \mathrm{d}(\varkappa_1)\varkappa_2 + \varkappa_1\,\mathrm{d}(\varkappa_2)$. It is important to note that $\mathcal{H}$Ɗ-mappings play a crucial role in algebra, especially in analyzing the structural properties of rings and algebras, including semiprime and prime rings. They are also valuable for studying functional identities, extending classical derivations, and examining the relationship between derivations and endomorphisms. Many authors studied the commutativity of prime and semiprime rings via homoderivations [1], [10-11]. Further, in 2016, the authors [12] investigated the commutativity of the quotient ring $\mathbb{M}/\mathcal{B}$ with respect to $\mathcal{H}$Ɗ-mappings, where $\mathcal{B}$ is a prime ideal, and extended results of author [1] and other authors [13]. The authors presented the term of generalized homoderivation (Ɠ$\mathcal{H}$Ɗ, for short) as an extension of $\mathcal{H}$Ɗ-mapping to an additive mapping $\mathcal{K}$ from $\mathbb{M}$ into itself that satisfies $\mathcal{K}(\mathfrak{ab}) = \mathcal{K}(\mathfrak{a})\mathcal{K}(\mathfrak{b}) + \mathcal{K}(\mathfrak{a})\mathfrak{b} + \mathfrak{a}\mathrm{d}(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in \mathbb{M}$, where d is an $\mathcal{H}$Ɗ-mapping of $\mathbb{M}$ [14]. In the same reference, the authors explored several algebraic identities involving Ɠ$\mathcal{H}$Ɗ-mappings to study the commutativity of prime rings. For simplicity, the notation $(\mathcal{K}, \mathrm{d})$ will refer to a Ɠ$\mathcal{H}$Ɗ-mapping $\mathcal{K}$ associated with an $\mathcal{H}$Ɗ-mapping d. Orthogonal derivations enhance the concept of derivations by introducing orthogonality conditions, providing a tool for studying the structure of rings and algebras. They are particularly useful in analyzing the commutativity of prime rings and exploring algebraic identities. In [15], author defined orthogonality for a pair of Ɗ-mappings on a semiprime ring, presenting various necessary and sufficient conditions for their orthogonality. Two Ɗ-mappings d_1 and d_2 of $\mathbb{M}$ are termed orthogonal (Ơ, for short) if $\mathrm{d}_1(\wp_1)\mathbb{M}\mathrm{d}_2(\wp_2) = 0 = \mathrm{d}_2(\wp_2)\,\mathbb{M}\mathrm{d}_1(\wp_1)$ for all $\wp_1, \wp_2 \in \mathbb{M}$. Additionally, the authors in [16] proposed the concept of Ơ-mappings on semirings and proved several straightforward yet insightful results. In 2012, other authors introduced the concept of orthogonal generalized derivations (ƠƓƊ, for short) as follows: Two ƓƊ -mappings $(\mathcal{K}, \mathrm{d})$ and (∂, μ) of $\mathbb{M}$ are considered ƠƓƊ-mappings if $\mathcal{K}(\varkappa_1)\,\mathbb{M}\,\partial(\varkappa_2) = 0 = \partial(\varkappa_2)\mathbb{M}\,\mathcal{K}(\varkappa_1)$ for all $\varkappa_1, \varkappa_2 \in \mathbb{M}$ [17]. Their results generalize the findings of authors in [16], which are connected to E. Posner's theorem on the product of derivations in a prime ring. Various findings involving ƠƊ and ƠƓƊ -mappings are discussed in [18] and [19]. This paper examines the orthogonality of $\acute{G}\mathcal{H}\mathcal{D}$-mappings in semiprime rings, introducing $\acute{O}\acute{G}\mathcal{H}\mathcal{D}$-mappings and expanding on previous work. It identifies conditions for the orthogonality of $\acute{G}\mathcal{H}\mathcal{D}$-mappings $(\mathcal{K}, d)$ and (∂, μ), offering new perspectives in the study of $\mathcal{H}\mathcal{D}$-mappings. Throughout this paper, $\mathbb{M}$ is assumed to be a 2-torsion-free semiprime ring, unless stated otherwise.

2. The Main Results

In this section, ÓǴHĎ-mappings are presented to support the development of the main results.

Definition 2.1: Two $\acute{G}\mathcal{H}\mathcal{D}$-mappings $(\mathcal{K}, d)$ and (∂, μ) of $\mathbb{M}$ are considered orthogonal ($\acute{O}\acute{G}\mathcal{H}\mathcal{D}$, for short) if the following conditions hold for all $\wp, \tau \in \mathbb{M}$:

$$\mathcal{K}(\wp)\ \mathbb{M}\ \partial(\tau) = \partial(\tau)\ \mathbb{M}\ \mathcal{K}(\wp) = 0.$$

The following example illustrates the concept of $\acute{O}\acute{G}\mathcal{H}\mathcal{D}$-mappings:

Example 2.2: Let d and μ be two $\mathcal{H}\mathcal{D}$-mappings of $\mathbb{M}$. Define $\Delta = \mathbb{M} \oplus \mathbb{M}$, and let the maps d_1 and μ_1 on Δ be defined by $d_1\big((\wp, \tau)\big) = (d(\wp), 0)$ and $\mu_1\big((\wp, \tau)\big) = (0, \mu(\tau))$ for all $\wp, \tau \in \mathbb{M}$. Then d_1 and μ_1 are $\mathcal{H}\mathcal{D}$-mappings of Δ. Furthermore, if $(\mathcal{K}, d)$ and (∂, μ) are $\acute{G}\mathcal{H}\mathcal{D}$-mappings of $\mathbb{M}$, then by putting $\mathcal{K}_1\big((\wp, \tau)\big) = (\mathcal{K}(\wp), 0)$ and $\partial_1\big((\wp, \tau)\big) = (0, \partial(\tau))$ for all $\wp, \tau \in \mathbb{M}$, then it follows that $(\mathcal{K}_1, d)$ and (∂_1, μ) are $\acute{G}\mathcal{H}\mathcal{D}$-mappings of Δ, with such that $\mathcal{K}_1$ and $\mathcal{S}_1$ being $\acute{O}\acute{G}\mathcal{H}\mathcal{D}$- mappings.

The following note will be used freely as needed.

Note 2.3: For a Z- mapping ξ defined on a subset $\mathfrak{S}$ of $\mathbb{M}$. If $\sigma + f(\sigma) \in \mathfrak{S}$, then by Theorem 2.1 in [12], $\sum_{i=1}^{n} \xi^{i-1}(\sigma)$ can replace σ. Since ξ is Z- mapping, there exists a positive integer $\mathfrak{y}(\sigma) > 1$ such that $\xi^{\mathfrak{y}(\sigma)}(\sigma)$=0 and $\xi^{o}(\mathfrak{S})$ denotes the identity mapping on $\mathfrak{S}$. Consequently, $\sigma + f(\sigma) = \sigma$, for all $\sigma \in \mathfrak{S}$.

Building on the properties of ÓĎ- and ÓǴĎ-mappings studied in ([18], [19]), ÓHĎ- and ÓǴHĎ-mappings are introduced within semiprime rings. The following lemmas extend earlier results and support the main proofs.

Lemma 2.4: *Let d and μ be Z-$\mathcal{H}\mathcal{D}$ mappings. The mappings d and μ are $\acute{O}$-mappings if and only if the condition $d(\wp)\mu(\tau) + \mu(\wp)d(\tau) = 0$ holds for all $\wp, \tau \in \mathbb{M}$.*

Proof: Suppose $d(\wp)\mu(\tau) + \mu(\wp)d(\tau) = 0$ is satisfied for all elements $\wp$ and τ of $\mathbb{M}$. Replacing τ with $\tau\wp$ in this equation gives $d(\wp)\mu(\tau\wp) + \mu(\wp)$ $d(\tau\wp) = 0$ holds for all $\wp, \tau \in \mathbb{M}$. Since d and μ are Z-$\mathcal{H}\mathcal{D}$-mappings, this simplifies to $d(\wp)\ \tau\ \mu(\wp) + \mu(\wp)\ \tau\ d(\wp) = 0$ holds for all $\wp, \tau \in \mathbb{M}$. By (Lemma 1, [15]) it follows that $d(\wp)\ \tau\ \mu(\wp) = \mu(\wp)\ \tau\ d(\wp)$ for all $\wp, \tau \in \mathbb{M}$. Using [Proposition 1.1.7, [17]], this extends to $d(\wp)\ \tau\ \mu(\varkappa) = \mu(\wp)\ \tau\ d(\varkappa)$ for all $\wp, \tau, \varkappa \in \mathbb{M}$. Thus d and μ are $\acute{O}$-mappings.

Conversely, assume that d and μ are $\mho$-mappings, then for all $\wp, \tau \in \mathbb{M}$, $\mathrm{d}(\wp)\, \varkappa\, \mu(\tau) = \mu(\wp)\, \varkappa\, \mathrm{d}(\tau) = 0$. Applying (Lemma 1, [17]) implies that $\mathrm{d}(\wp)\, \mu(\tau) = \mu(\tau)\, \mathrm{d}(\wp) = 0$, which leads to $\mathrm{d}(\wp)\, \mu(\tau) + \mu(\tau)\, \mathrm{d}(\wp) = 0$ for all $\wp, \tau \in \mathbb{M}$.

Lemma 2.5: *Let* d *and* μ *are* $\mathcal{Z}$-$\mathcal{HD}$*-mappings. Then* d *and* μ *are* $\mho$*-mappings if and only if their composition* $\mathrm{d}\mu = 0$.

Proof: Suppose that $\mathrm{d}\mu = 0$. Then for all elements $\wp$ and τ of $\mathbb{M}$, the following equation holds $\mathrm{d}\mu(\wp\tau) = \mathrm{d}\big(\mu(\wp)\mu(\tau) + \mu(\wp)\tau + \wp\mu(\tau)\big) = 0$. Expanding the right-hand side of this equation to have:

$$0 = \mathrm{d}\big(\mu(\wp)\big)\, \mathrm{d}\big(\mu(\tau)\big) + \mathrm{d}\big(\mu(\wp)\big)\, \mu(\tau) + \mu(\wp)\mathrm{d}\big(\mu(\tau)\big) + \mathrm{d}\big(\mu(\wp)\big)\mathrm{d}(\tau)$$
$$+\mathrm{d}\big(\mu(\wp)\big)\tau + \mu(\wp)\mathrm{d}(\tau) + \mathrm{d}(\wp)\mathrm{d}\big(\mu(\tau)\big) + \mathrm{d}(\wp)\mu(\tau) + \wp\mathrm{d}\big(\mu(\tau)\big)$$

$$0 = \mu(\wp)\, \mathrm{d}(\tau) + \mathrm{d}(\wp)\, \mu(\tau).$$

By Lemma 2.4, it follows that d and μ are $\mho\mathcal{HD}$-mappings.

Conversely, if d and μ are $\mho\mathcal{HD}$-mappings, then $0 = \mathrm{d}\big(\mathrm{d}(\wp)\mu(\tau)\big) = \mathrm{d}\big(\mathrm{d}(\wp)\big)\mathrm{d}\big(\mu(\tau)\big) + \mathrm{d}(\wp)\mathrm{d}\big(\mu(\tau)\big) + \mathrm{d}\big(\mathrm{d}(\wp)\big)\mu(\tau) = \mathrm{d}(\mathrm{d}(\wp) + \wp)\mathrm{d}\mu(\tau)$ for all $\wp, \tau \in \mathbb{M}$. Since d is $\mathcal{Z}$-$\mathcal{HD}$-mapping, then the last equation gives $\mathrm{d}(\wp)\mathrm{d}\mu(\tau) = 0$. Substituting $\wp$ by $\wp\varkappa$, where $\varkappa \in \mathbb{M}$ in this equation to have $\mathrm{d}(\wp)\varkappa\mathrm{d}\mu(\tau) = 0$. Next, replace $\mu(\tau)$ instead of $\wp$ in this equation and, using the semiprimeness property of $\mathbb{M}$, conclude that $\mathrm{d}\mu(\tau) = 0$ for all $\tau \in \mathbb{M}$.

Lemma 2.6: *Let* $(\mathcal{K}, \mathrm{d})$ *be a* $\mathcal{G}\mathcal{HD}$*-mapping of* $\mathbb{M}$*. If* $\mathcal{K}(\wp)\, \mathcal{K}(\tau) = 0$ *for all* $\wp, \tau \in \mathbb{M}$*, then* $\mathcal{K} = \mathrm{d} = 0$.

Proof: Assume $\mathcal{K}(\wp)\, \mathcal{K}(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$. (1)

Putting $\tau = \tau\varkappa$, where $\varkappa \in \mathbb{M}$, into equation (1) gives $\mathcal{K}(\wp)\tau\, \mathrm{d}(\varkappa) = 0$ for all $\wp, \tau, \varkappa \in \mathbb{M}$. Using (Lemma 1, [18]) it follows that

$\mathcal{K}(\wp)\, \mathrm{d}(\varkappa) = \mathrm{d}(\varkappa)\mathcal{K}(\wp) = 0$ for all $\wp, \varkappa \in \mathbb{M}$. Substituting $\wp = \wp\varkappa$ into $\mathrm{d}(\varkappa)\mathcal{K}(\wp) = 0$, yields $\mathrm{d}(\varkappa)\wp\, \mathrm{d}(\varkappa) = 0$ holds for all $\wp, \varkappa \in \mathbb{M}$. Using the semiprimeness of $\mathbb{M}$ to conclude that $\mathrm{d} = 0$. Similarly, replacing $\wp$ by $\tau\wp$ in equation (1) and using the property of $\mathbb{M}$, it follows that $\mathcal{K} = 0$.

Lemma 2.7: *Let* $(\mathcal{K}, \mathrm{d})$ *and* (∂, μ) *be* $\mho\mathcal{G}\mathcal{HD}$*-mappings of* $\mathbb{M}$ *with* ∂ *being a* $\mathcal{Z}$*-mapping. Then, for all* $\wp, \tau \in \mathbb{M}$*, the following relations hold:*

(i) $\mathcal{K}(\wp)\, \partial(\tau) = \partial(\wp)\mathcal{K}(\tau) = 0$.

(ii) d and ∂ are $\mho$-mappings, and $\mathrm{d}(\wp)\partial(\tau) = \partial(\tau)\mathrm{d}(\wp) = 0$.

(iii) μ and $\mathcal{K}$ are $\mho$-mappings, and $\mu(\wp)\mathcal{K}(\tau) = \mathcal{K}(\tau)\mu(\wp) = 0$.

(iv) d and μ are $\mho\mathcal{HD}$-mappings.

(v) $\mathrm{d}\partial = \partial\mathrm{d} = 0$ and $\mu\mathcal{K} = \mathcal{K}\mu = 0$.

(vi) $\mathcal{K}\partial = \partial\mathcal{K} = 0$.

Proof:

(i) Since $\mathcal{K}$ and ∂ are $\mho\mathcal{GHD}$-mappings of $\mathbb{M}$, it follows that $\mathcal{K}(\wp)\, c\, \partial(\tau) = 0$ for all $c, \wp, \tau \in \mathbb{M}$. By applying (Lemma 1, [16]), it can be concluded that $\mathcal{K}(\wp)\, \partial(\tau) = \partial(\wp)\mathcal{K}(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$.

(ii) Since $\mathcal{K}$ and ∂ are $\mho\mathcal{GHD}$-mappings of $\mathbb{M}$, it follows that $\mathcal{K}(\wp)\, c\, \partial(\tau) = 0$ for all $c, \wp, \tau \in \mathbb{M}$. By (Lemma 1, [16]), the following equation is true $\mathcal{K}(\wp)\, \partial(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$. (2)

Substituting $\wp$ with $t\wp, t \in \mathbb{M}$ in equation (2) and applying equation (2),

$$t\, \mathrm{d}(\wp)\, \partial(\tau) = 0 \text{ for all } \wp, \tau, t \in \mathbb{M}. \tag{3}$$

By semiprimeness of $\mathbb{M}$, then $\mathrm{d}(\wp)\, \partial(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$. (4)

Thus, d and ∂ are $\mho$-mappings. Substituting $\wp = \wp t$ in equation (4) gives $\mathrm{d}(\wp)\mathrm{d}(t)\partial(\tau) + \mathrm{d}(\wp)t\partial(\tau) + \wp\mathrm{d}(t)\partial(\tau) = 0$. Using equation (4), implies that $\mathrm{d}(\wp)t\partial(\tau) = 0$ for all $\wp, \tau, t \in \mathbb{M}$. By (Lemma 1, [16]), it follows that $\partial(\tau)\mathrm{d}(\wp) = 0$ for all $\wp, \tau \in \mathbb{M}$.

(iii) The proof follows the same approach as in (ii).

(iv) Since $\mathcal{K}$ and ∂ are $\mho\mathcal{GHD}$-mappings of $\mathbb{M}$, it follows that $\mathcal{K}(\wp)\, c\, \partial(\tau) = 0$ for all $\wp, \tau, c \in \mathbb{M}$. By applying (Lemma 1, [16]), then

$$\mathcal{K}(\wp)\, \partial(\tau) = 0 \text{ for all } \wp, \tau \in \mathbb{M}. \tag{5}$$

Substituting $\wp$ with $\wp\varkappa$ and τ with τw, where $\varkappa, w \in \mathbb{M}$ in equation (5) results in $\big(\mathcal{K}(\wp)\, \mathcal{K}(\varkappa) + \mathcal{K}(\wp)\varkappa + \wp\, \mathrm{d}(\varkappa)\big)\big(\partial(\tau)\partial(w) + \partial(\tau)w + \tau\mu(w)\big) = 0$.

Using (i), (ii), and (iii), it simplifies to $\wp\, \mathrm{d}(\varkappa)\tau\mu(w) = 0$ for all $\wp, \tau, \varkappa, w \in \mathbb{M}$. By the semiprimeness property of $\mathbb{M}$, it follows that $\mathrm{d}(\varkappa)\tau\mu(w) = 0$. Thus d and μ are $\mho$-mappings.

(v) Using (ii) implies that for all $\wp, \tau, \varkappa \in \mathbb{M}$,

$$\partial(\mathrm{d}(\wp)\varkappa)\, \partial\big(\partial(\tau)\big) + \partial(\mathrm{d}(\wp)\varkappa)\partial(\tau) + \mathrm{d}(\wp)\varkappa\, \mu\big(\partial(\tau)\big) = 0. \tag{6}$$

By applying (iv) and since ∂ is a $\mathcal{GHD}$-mapping, then equation (6) becomes

$$\Big(\partial\big(\mathrm{d}(\wp)\big)\partial(\varkappa) + \partial\big(\mathrm{d}(\wp)\big)\varkappa + \mathrm{d}(\wp)\partial(\varkappa)\Big)\, \partial\big(\partial(\tau)\big)$$
$$+ \Big(\partial\big(\mathrm{d}(\wp)\big)\partial(\varkappa) + \partial\big(\mathrm{d}(\wp)\big)\varkappa + \mathrm{d}(\wp)\partial(\varkappa)\Big)\, \partial(\tau) = 0.$$

Again using (ii) and since ∂ is a Z - mapping, the equation simplifies to

$$0 = \partial\big(\mathrm{d}(\wp)\big)\, \big(\varkappa + \partial(\varkappa)\big)\, \Big(\partial\big(\partial(\tau)\big) + \partial(\tau)\Big) = \partial\big(\mathrm{d}(\wp)\big)\, \varkappa\, \partial(\tau). \tag{7}$$

Putting $\tau = \mathrm{d}(\wp)$ in equation (7) and using the property of $\mathbb{M}$ yields $\partial\big(\mathrm{d}(\wp)\big) = 0$ for all $\wp \in \mathbb{M}$. A similar approach can be used to show that $\mathrm{d}\partial = 0$, $\mu\mathcal{K} = 0$, and $\mathcal{K}\mu = 0$.

(vi) Since $\mathcal{K}$ and ∂ are $\mho$-mappings, then $\mathcal{K}(\wp)\, \varkappa\, \partial(\tau) = 0$ for all $\wp, \tau, \varkappa \in \mathbb{M}$. Hence,

$$0 = \partial\big(\mathcal{K}(\wp)\, \varkappa\, \partial(\tau)\big) = \partial(\mathcal{K}(\wp)\, \varkappa\,)\, \partial\big(\partial(\tau)\big) + \partial(\mathcal{K}(\wp)\, \varkappa\,)\, \partial(\tau) + \mathcal{K}(\wp)\, \varkappa\, \mu\big(\partial(\tau)\big) \text{ for all } \wp, \tau, \varkappa \in \mathbb{M}. \quad (8)$$

By using (v) and since ∂ is a ƓℋƊ-mapping, equation (8) becomes $0=\Big(\partial\big(\mathcal{K}(\wp)\big)\partial(\varkappa) + \partial\big(\mathcal{K}(\wp)\big)\varkappa + \mathcal{K}(\wp)\mu(\varkappa)\Big)\,\partial\big(\partial(\tau)\big) + \Big(\partial\big(\mathcal{K}(\wp)\big)\partial(\varkappa) + \partial\big(\mathcal{K}(\wp)\big)\varkappa + \mathcal{K}(\wp)\mu(\varkappa)\Big)\,\partial(\tau) = \partial\big(\mathcal{K}(\wp)\big)(\partial(\varkappa) + \varkappa)\ \Big(\partial\big(\partial(\tau)\big) + \partial(\tau)\Big)$ (9)

Applying the fact that ∂ is a Ʒ -mapping to equation (9) yields $\partial\big(\mathcal{K}(\wp)\big)\, \varkappa\, \partial(\tau) = 0$ for all $\wp, \tau, \varkappa \in \mathbb{M}$. In this equation, replacing τ by $\mathcal{K}(\wp)$ and using the semiprimeness of $\mathbb{M}$, it follows that $\partial\big(\mathcal{K}(\wp)\big) = 0$ for all $\wp \in \mathbb{M}$, which implies that $\partial\mathcal{K} = 0$. Similarly, it can be shown that $\mathcal{K}\partial = 0$.

The following theorems generalize the results in [18], utilizing ƓℋƊ -mappings of $\mathbb{M}$.

Theorem 2.8: *Let* $(\mathcal{K}, \text{ȡ})$ *and* (∂, μ) *be* ƓℋƊ *-mappings of* $\mathbb{M}$. *Then* $(\mathcal{K}, \text{ȡ})$ *and* (∂, μ) *are* Ơ*-mappings. If* ∂ *and* $\mathcal{K}$ *are* Ʒ*-mappings and the following conditions are satisfied for all* $\wp, \tau \in \mathbb{M}$*:*

(i) $\mathcal{K}(\wp)\partial(\tau) + \partial(\wp)\mathcal{K}(\tau) = 0.$

(ii) $\text{ȡ}(\wp)\partial(\tau) + \mu(\wp)\mathcal{K}(\tau) = 0.$

Proof: Replacing $\wp$ by $\wp\varkappa$, where $\varkappa \in \mathbb{M}$, in (i) gives:

$$= \mathcal{K}(\wp)\, \mathcal{K}(\varkappa)\partial(\tau) + \mathcal{K}(\wp)\varkappa\partial(\tau) + \partial(\wp)\partial(\varkappa)\mathcal{K}(\tau) + \partial(\wp)\varkappa\mathcal{K}(\tau) + \wp\, \big(\text{ȡ}(\varkappa)\partial(\tau) + \mu(\varkappa)\mathcal{K}(\tau)\big) \quad \text{for all } \wp, \tau, \varkappa \in \mathbb{M}. \quad (10)$$

Applying the condition (ii) to the last term of equation (10) leads to

$$\mathcal{K}(\wp)(\, \mathcal{K}(\varkappa) + \varkappa)\partial(\tau) + \partial(\wp)(\partial(\varkappa) + \varkappa)\mathcal{K}(\tau) = 0 \text{ for all } \wp, \tau, \varkappa \in \mathbb{M}. \quad (11)$$

Since ∂ and $\mathcal{K}$ are Ʒ–mappings, equation (11) simplifies to $\mathcal{K}(\wp)\varkappa\partial(\tau) + \partial(\wp)\varkappa\mathcal{K}(\tau) = 0$ for all $\wp, \tau, \varkappa \in \mathbb{M}$. Using (Lemma 1, [16]) and (Proposition 1.1.7,[16]), then the last equation yields $\mathcal{K}(\wp)\mathbb{M}\partial(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$. Hence, $\mathcal{K}$ and ∂ are Ơ-mappings.

Theorem 2.9: *Let* $(\mathcal{K}, \text{ȡ})$ *and* (∂, μ) *be* ƓℋƊ *-mappings of* $\mathbb{M}$. *If* $\mathcal{K}(\wp)\partial(\tau) = d(\wp)\partial(\tau) = 0$ *for all* $\wp, \tau \in \mathbb{M}$, *then* $(\mathcal{K}, \text{ȡ})$ *and* (∂, μ) *are* Ơ*-mappings.*

Proof: Assume that, $\mathcal{K}(\wp)\partial(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$. Substituting $\wp = \wp c$, where $c \in \mathbb{M}$, into this relation, gives:

$$\mathcal{K}(\wp)\, \mathcal{K}(c)\partial(\tau) + \mathcal{K}(\wp)c\, \partial(\tau) + \wp\, \text{ȡ}(c)\partial(\tau) = 0 \quad \text{for all } c, \wp, \tau \in \mathbb{M}.$$

By the given hypothesis, $\mathcal{K}(\wp)c\, \partial(\tau) = 0$ for all $c, \wp, \tau \in \mathbb{M}$. By (Lemma 1, [16]), it follows that $\partial(\tau)\, \mathcal{K}(\wp) = 0$. Thus, $\mathcal{K}$ and ∂ are Ơ-mappings.

The following theorem involves a composition of HƊ and ƓHƊ-mappings.

Theorem 2.10: *Let* $(\mathcal{K}, \text{ȡ})$ *and* (∂, μ) *be* ƓℋƊ *-mappings of* $\mathbb{M}$. *If* $\mathcal{K}(\wp)\partial(\tau) = \text{ȡ}\partial = \text{ȡ}\mu = 0$, *then* $(\mathcal{K}, \text{ȡ})$ *and* (∂, μ) *are* Ơ*-mappings.*

Proof: Since $\mathrm{d}\mu = 0$, Lemma 2.5 ensures that d and μ are orthogonal mappings. By (Lemma 1, [16]), the following holds:

$$\mathrm{d}(\wp)\,\mu(\tau) = \mu(\tau)\mathrm{d}(\wp) = 0 \text{ for all } \wp, \tau \in \mathbb{M}. \tag{12}$$

Assume $\mathrm{d}\partial = 0$, which implies $\mathrm{d}\partial(\wp) = 0$ for all $\wp \in \mathbb{M}$. In this equation, replacing $\wp$ with $\wp\tau$ and using the hypothesis leads to

$$\partial(\wp)\mathrm{d}(\tau) = 0 \text{ for all } \wp, \tau \in \mathbb{M}. \tag{13}$$

Substituting $\wp c$ instead of $\wp$ in equation (13) yields $\partial(\wp)\partial(c)\mathrm{d}(\tau) + \partial(\wp)c\mathrm{d}(\tau) + \wp\mu(c)\mathrm{d}(\tau) = 0$ for all $c, \wp, \tau \in \mathbb{M}$. Using equations (12) and (13) in the last equation results in $\partial(\wp)c\mathrm{d}(\tau) = 0$ for all $c, \wp, \tau \in \mathbb{M}$. By (Lemma 1, [16]), this implies $\mathrm{d}(\tau)\,\partial(\wp) = 0$ for all $\wp, \tau \in \mathbb{M}$. Thus, applying the hypothesis and Theorem 2.9 confirms the desired result.

The following result considers ƓĦƊ-mappings with zero product.

Theorem 2.11: *Let* $(\mathcal{K}, \mathrm{d})$ *and* (∂, μ) *be* $Ɠ\mathcal{H}Ɗ$*-mappings of* $\mathbb{M}$*, where* d, ∂ *and* $\mathcal{K}$ *are* Ȥ*-mappings. Suppose that the mapping* $(\mathcal{K}\partial, \mathrm{d}\mu)$ *is a* $Ɠ\mathcal{H}Ɗ$*-mapping and that* $\mathcal{K}(\wp)\partial(\tau) = 0$ *for all* $\wp, \tau \in \mathbb{M}$*. Then:*

(i) $\mathcal{K}$ and μ are Ơ-mappings,

(ii) $\mathrm{d}\mathcal{K}$ and μ are Ơ-mappings.

Proof: Since $\mathcal{K}(\wp)\partial(\tau) = 0$ for all $\wp, \tau \in \mathbb{M}$. (14)

Replacing τ with τz, where $z \in \mathbb{M}$, in equation (14), gives $\mathcal{K}(\wp)\partial(\tau)\partial(z) + \mathcal{K}(\wp)\partial(\tau)z + \mathcal{K}(\wp)\tau\mu(z) = 0$. Using equation (14), it follows that $\mathcal{K}(\wp)\tau\mu(z) = 0$ for all $\wp, \tau, z \in \mathbb{M}$. Applying (Lemma 1, [16]), it implies that $\mathcal{K}$ and μ are Ơ-mappings. Next, replacing $\wp$ by $\wp z$ in equation (14) and using it yields: $\mathcal{K}(\wp)z\partial(\tau) + \wp\mathrm{d}(z)\partial(\tau) = 0$ for all $\wp, \tau, z \in \mathbb{M}$. Putting $z = z\,\mathcal{K}(\ell)$, where $\ell \in \mathbb{M}$, in this equation gives: $\mathcal{K}(\wp)\,z\,\mathcal{K}(\ell)\,\partial(\tau) + \wp\left(\mathrm{d}(z)\mathrm{d}(\mathcal{K}(\ell)) + \mathrm{d}(z)\mathcal{K}(\ell) + z\mathrm{d}(\mathcal{K}(\ell))\right)\partial(\tau) = 0$ for all $\ell, \wp, \tau, z \in \mathbb{M}$. Applying equation (14) to get $\wp(z + \mathrm{d}(z))\,\mathrm{d}\mathcal{K}(\ell)\partial(\tau) = 0$ for all $\ell, \wp, \tau, z \in \mathbb{M}$. Since d is a Ȥ-mapping and $\mathbb{M}$ is a semiprime ring, then this equation simplifies to:

$$\mathrm{d}\mathcal{K}(\ell)\partial(\tau) = 0 \text{ for all } \ell, \tau \in \mathbb{M}. \tag{15}$$

Replacing τ with τn, where $n \in \mathbb{M}$, in equation (15) gives:

$$\mathrm{d}\mathcal{K}(\ell)\big(\partial(\tau)\partial(n) + \partial(\tau)n + \tau\,\mu(n)\big) = 0 \text{ for all } \ell, n, \tau \in \mathbb{M}\,. \tag{16}$$

Using equation (15) in equation (16) yields: $\mathrm{d}\mathcal{K}(\ell)\,\tau\,\mu(n) = 0$ for all $\ell, n, \tau \in \mathbb{M}$. Therefore, $\mathrm{d}\mathcal{K}$ and μ are Ơ-mappings.

4. Conclusion

This paper examines orthogonal generalized homoderivations in semiprime rings, emphasizing identities involving their products and compositions. Orthogonality emerges as key to understanding derivation-like mappings. Future work may extend these results to other ring classes or algebraic structures.

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