

Fully pseudo-stable gamma acts

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Abstract

In this work, we introduce the concept of fully pseudo-stable gamma acts as a generalization of fully stable gamma acts. We investigate various properties and characterizations of this class. In particular, we show that an M_Γ – act A is fully pseudo-stable if and only if every cyclic M_Γ – subact of A satisfies Baer’s M_Γ – criterion. Finally, we propose a further generalization, called essentially fully pseudo-stable, which is a very widely occurring property in any M_Γ – act.

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1. Introduction

Consider two nonempty sets M and Γ . The set M is referred to as a Γ -semigroup, if there is a map $M \times \Gamma \times M$ to M , $u_1 \alpha u_2$ is the element (u_1, α, u_2) such that $(u_1 \alpha u_2) \beta u_3 = u_1 \alpha (u_2 \beta u_3)$ for all $u_1, u_2, u_3 \in M$ and $\alpha, \beta \in \Gamma$ [1]. Let $\gamma \in \Gamma$ fixed element, an element 1_γ in M is said to be γ -identity if $u \gamma 1_\gamma = u = 1_\gamma \gamma u$ for all $u \in M$ [2], M is an γ -monoid if M having γ -identity. A subset $J \neq \emptyset$ of M is right (left) Γ -ideal, if $J\Gamma M \subseteq J$ ($M\Gamma J \subseteq J$), and M is commutative if $u_1 \alpha u_2 = u_2 \alpha u_1$ for all $u_1, u_2 \in M$ and $\alpha \in \Gamma$. The concept of Γ -semigroup has been generalized in [1] as follows. A nonempty set A is called a right Γ -act over Γ -semigroup M , denoted by M_Γ -act, if there is a mapping $A \times \Gamma \times M \rightarrow A$ such that $(q \alpha u_1) \beta u_2 = q \alpha (u_1 \beta u_2)$ for all $q \in A$, $\alpha, \beta \in \Gamma$ and $u_1, u_2 \in M$. A right M_Γ -act A is γ -unitary, for arbitrary fixed $\gamma \in \Gamma$, if M has γ -identity 1_γ with $q = q \gamma 1_\gamma$. A subset $W \neq \emptyset$ of an M_Γ -act A is M_Γ -subact, denoted by $W \leq A$ if $W\Gamma M \subseteq W$. Let $Z \neq \emptyset$ be a subset of A . Then $[Z]_A = \cup_{q \in Z} q\Gamma M$ where $q\Gamma M = \{q \alpha u \mid \alpha \in \Gamma \text{ and } u \in M\}$, The set Z is called a generating set of A if $A = [Z]_A$. And A is a cyclic if $A = [\{q\}]_M$ for some q in A [1]. By [1] we have the union of two subacts is also subact. A mapping $f: A \rightarrow B$ (where A and B M_Γ -acts) is M_Γ -homomorphism if $f(q \alpha u) = f(q) \alpha u$ for all $q \in A$, $\alpha \in \Gamma$ and $u \in M$. For an additional, if f is injection (projection, bijective), then f is said to be M_Γ -monomorphism (M_Γ -epimorphism, M_Γ -isomorphism), A and B is said to be isomorphic, if there exist an M_Γ -isomorphism between them and denoted by $A \cong B$, and $Mon_{M_\Gamma}(A, B)$ ($Iso_{M_\Gamma}(A, B)$) is the set of all M_Γ -monomorphism (M_Γ -isomorphism) from A to B . An M_Γ -act A is said to be decomposable, if there are two M_Γ -subacts Z, X of A such that $A = Z \cup X$, where $\cup$ is the disjoint union, otherwise A is called indecomposable. Let Z be an M_Γ -subact of A , a subset $[Z : \Gamma : A] = \{u \in M \mid q \alpha u \in Z \text{ for all } \alpha \in \Gamma \text{ and } q \in A\}$ of M is the annihilator of Z , in particular for δ in Γ , $[Z : \delta : A] = \{u \in M \mid q \delta u \in Z \text{ for all } q \in A\}$. An equivalent relation ρ on A is said to be a congruence on A , if for all $(q_1, q_2) \in \rho$, it follows that $(u \alpha q_1, u \alpha q_2) \in \rho$ for all $u \in M$ and $\alpha \in \Gamma$ [1]. For a nonempty subsets $J \subseteq M$, $Z \subseteq A$, and $H \subseteq M \times M$, we define $R_M(Z) = \{(u_1, u_2) \in M \times M \mid q \alpha u_1 = q \beta u_2 \text{ for all } q \text{ in } Z \text{ and } \alpha, \beta \text{ in } \Gamma\}$, $R_M^\alpha(Z) = \{(u_1, u_2) \in M \times M \mid q \alpha u_1 = q \alpha u_2 \text{ for all } q \text{ in } Z \text{ and } \alpha \text{ in } \Gamma\}$, $L_A(H) = \{q \in A \mid q \alpha u_1 = q \beta u_2 \text{ for all } (u_1, u_2) \in H \text{ and } \alpha, \beta \text{ in } \Gamma\}$ [3]. Let $\{A_b \mid b \in B\}$ be a family of M_Γ -acts, the Cartesian product $\prod_{b \in B} A_b$ of A_b has been shown in $\prod_{b \in B} A_b$ is M_Γ -act component wisely and it is called the direct product of $\{A_b \mid b \in B\}$. An M_Γ -subact Z of A is called Γ -essential in A , if for any $f \in Hom_{M_\Gamma}(A, B)$, (B is any M_Γ -act) such that $f|_Z \in Mon_{M_\Gamma}(Z, B)$. Implies that $f \in Mon_{M_\Gamma}(A, B)$ [2].

2. Fully Pseudo-Stable on Gamma Acts.

In this section we present some fundamental general facts and introduce a generalized notion of complete stability for γ -acts.

Definition 2.1: An M_Γ -subact Z of A is called **pseudo-stable** (abbreviated ps - M_Γ -subact) if for any $f \in Mon_{M_\Gamma}(Z, A)$. Implies $f(Z) \subseteq Z$. The M_Γ -act A is called

fully pseudo-stable M_I -act (abbreviated **fps - M_I -act** if every M_I -subact of A is pseudo-stable.

We provide a characterization of fully pseudo-stable M_I -subact, restricted to only cyclic indecomposable M_I -subacts.

Proposition 2.2: The following conditions are equivalent for any M_I -act A

(1) A is fps - M_I -act. (2) Every cyclic M_I -subact of A is ps- M_I -subact. (3) If $q \in A$, then any $f \in \text{Mon}_{M_I}(q\Gamma M, A)$ is a multiplication by some element q in M with respect to some element α in Γ .

Proof: (1) $\rightarrow$ (2) Straightforward. (2) $\rightarrow$ (3) Let $q \in A$ and $f \in \text{Mon}_{M_I}(q\Gamma M, A)$, then by hypotheses implies $f(q\Gamma M) \subseteq q\Gamma M$. Hence $f(q) = f(q\gamma 1_\gamma) \in f(q\Gamma M) \subseteq q\Gamma M$, then there exists some $u \in M$ and $\alpha \in \Gamma$ that is, $f(q) = q\alpha u$. (3) $\rightarrow$ (1) Consider an M_I -subact Z of A and $f \in \text{Mon}_{M_I}(Z, A)$. Now for any $q \in Z \subseteq A$, then there exists some $u \in M$ and $\alpha \in \Gamma$ in this way, $f(q) = f|_{q\Gamma M}(q) = q\alpha u$, and hence $f(q) = q\alpha u \in q\Gamma M \subseteq Z$ implies $f(Z) \subseteq Z$.

The next result provide another characterization of fully pseudo-stable M_I -act.

Proposition 2.3: The following conditions are equivalent for any M_I -act A .

(1) A is fps - M_I -act. (2) Any M_I -subact of A is fps - M_I -act. (3) Any 2-generated M_I -subact of A is fps - M_I -act.

Proof: (1) $\rightarrow$ (2) $\rightarrow$ (3) directly. (3) $\rightarrow$ (1) Let Z is an M_I -subact of A , and $f \in \text{Mon}_{M_I}(Z, A)$. Additionally, let $q \in Z$. Consider the M_I -subact $X = q\Gamma M \cup f(q)\Gamma M$, and $g = f|_{q\Gamma M} : q\Gamma M \rightarrow A$. By hypotheses X is fps - M_I -act, hence $f(q) \in q\Gamma M$. It follows that A is fps - M_I -act.

In the category of semigroups, proposition 2.2 in [4] states that for an S -act M_S the following are equivalent (1) M_S is fully pseudo-stable (2) if Z, X are subact of M_S and $Z \cong X$, then $Z = X$. However, in the category of Gamma Acts this equivalence does not necessarily hold in the general case. To ensure this property holds we must impose an additional condition.

Following [2] an M_I -act A is said to **strongly α - faithful** (for a fixed element $\alpha \in \Gamma$) if there exists some $a \in A$ with $q\alpha u_1 = q\alpha u_2$ implies $u_1 = u_2$, the next result provides a version of this property for acts.

Theorem 2.4: The following are equivalent for a strongly δ - faithful M_I -act A for some $\delta \in \Gamma$

(1) A is fps - M_I -act. (2) Every two isomorphic M_I -subacts of A is equal. (3) $R_M(q\delta M) \subseteq R_M(q\delta M)$. For all $q \in A$ (4) $L_A(R_M(q\delta M)) = q\delta M$, for any $q \in A$

Proof: (1) $\Rightarrow$ (2) Consider M_I -subacts Z and X of A with $f \in \text{Isom}(Z, X)_{M_I}$. By (1), $X = f(Z) \subseteq Z$ and $Z = f^{-1}(X) \subseteq X$. (2) $\Rightarrow$ (3) it is assumed that there is $q, w \in A$ with $w \notin q\Gamma M$ and $R_M(q\delta M) \subseteq R_M(w\delta M)$. Define the mapping $f: q\delta M \rightarrow$

$w\delta M$ by $f(q\delta m) = w\delta m$, for all $q\delta u \in q\delta M$ [to show f well defined, let $q\delta u_1 = q\delta u_2$, then $(u_1, u_2) \in R_M(q\delta M)$, hence $(u_1, u_2) \in R_M(w\delta M)$ implies $w\delta u_1 = w\delta u_2$]. By strongly δ - faithful property implies $f \in \text{Isom}(q\delta M, w\delta M)_{M_\Gamma}$. By (2), $w \in w\delta M \subseteq q\delta M$ is a contradiction. **(3) $\Rightarrow$ (4)** Let $w \in L_M(R_M(q\delta M))$ and $w \notin q\delta M$. By (3) $R_M(q\delta M) \not\subseteq R_M(w\delta M)$ hence there is $(u_1, u_2) \in R_M(q\delta M)$ and $(u_1, u_2) \notin R_M(w\delta M)$ so $w\delta u_1 \neq w\delta u_2$, which is a contradiction. **(4) $\Rightarrow$ (1)** Let $f \in \text{Mon}_{M_\Gamma}(q\Gamma M, A)$ and let $(u_1, u_2) \in R_M(q\Gamma M)$, then $qau_1 = qau_2$, $f(q) = f(q)\gamma 1_\gamma = f(q\gamma 1_\gamma) \in f(q\Gamma M)$, $f(q)au_1 = f(qau_1) = f(qau_2) = f(q)au_2$, then $f(q) \in L_A(R_M(q\Gamma M)) = q\Gamma M$.

Examples 2.5:

- (1) Clearly that a stable M_Γ -subact is pseudo-stable, consequently, a fully stable M_Γ -act is fps - M_Γ -act. However, the convers may not be true in general. See 2. Let $M = \{q, w, e\}$ and $\Gamma = \{\alpha, \beta\}$. The action defined by the following Table 1.

Table 1
Action tables defining the $M\Gamma$ -act with $M = \{q, w, e\}$ and $\Gamma = \{\alpha, \beta\}$.

α	q	w	e	β	q	w	e
q	w	w	e	q	w	q	e
w	w	q	e	w	q	w	e
e	e	w	e	e	w	q	e

The M_Γ -act M , is fps - M_Γ -act. However, it is not a fully stable M_Γ -act.

- (3) Any simple M_Γ -act is fps - M_Γ -act.
 (4) Let $\Gamma = \{0, 1\}$. It is known that $\mathbb{N}$ is Γ - semigroup with a multiplication of integer. However, $\mathbb{N}$ is not fps - N_Γ - act. Indeed, for an N_Γ - monomorphism $f: 2\Gamma\mathbb{N} \rightarrow \mathbb{N}$ by $f(2an) = 3an$ for all $a \in \Gamma$ and $n \in \mathbb{N}$. But $f(2\Gamma\mathbb{N}) \not\subseteq 2\Gamma\mathbb{N}$.

In the following propositions, we explain the behavior of pseudo-stability under direct products and unions.

Proposition 2.6:

- (1) If a direct product $\prod_{b \in B} A_b$ of M_Γ -act A_b be a fps. Then A_b is fps - M_Γ -act for all $b \in B$.
 (2) Let $\{Z_b \mid b \in B\}$ be family M_Γ -subacts of A . If Z_b is a ps - M_Γ -subact for all $b \in B$, then $\bigcup_{b \in B} Z_b$ is ps - M_Γ -subact.

Proof: (1). For an M_Γ -subact Z of A_μ with $f \in \text{Mon}_{M_\Gamma}(Z, A_\mu)$. Since A is a direct product, there is the inclusion $i_\mu: A_\mu \rightarrow A$, then by (proposition 2.3) implies $Z = i_\mu(Z)$ which is an M_Γ -subact of A . By fps - M_Γ -act of A implies that $f(Z) = i_\mu \circ f(Z) \subseteq Z$. (2). Let $f \in \text{Mon}_{M_\Gamma}(\bigcup_{b \in B} Z_b, A)$, Thus $f \circ i_{Z_b}: Z_b \rightarrow A$ is a Γ -monomorphism, hence $f(Z_b) \subseteq Z_b$ for all $b \in B$. Now, let $q \in f(\bigcup_{b \in B} Z_b)$ then

there exists $b_0 \in B$ with $w \in Z_{b_0} \subseteq \bigcup_{b \in B} Z_b$ such that $q = f(w) \in f(Z_{b_0}) \subseteq \bigcup_{b \in B} Z_b$.

The concept of Baer's M -criterion has been introduced into the act category [5].

Definition 2.7: [6] Let Z be an M_I -subact of A . Then Z is said to satisfy Baer's $M\Gamma$ -criterion if for $f \in \text{Mon}_{M_\Gamma}(Z, A)$, there exists an element $u \in M$ and $\alpha \in \Gamma$ such that $f(q) = q\alpha u$ for any $q \in Z$. If any M_I -subact of A satisfying Baer's M_I -criterion, then A is said to satisfy Baer's M_I -criterion.

In the following theorem, we prove the equivalence of $\text{fps} - M_I$ -act by using Baer's M_I -criterion.

Theorem 2.8: An M_I -act A is $\text{fps} - M_I$ -act iff any cyclic M_I -subact of A satisfies Baer's $M\Gamma$ -criterion.

Proof: [7] Let $Z = q\Gamma M$ be a cyclic M_I -subact of A and $f \in \text{Mon}_{M_\Gamma}(Z, A)$. By the pseudo-stability of Z , impels $f(q) \in f(q\Gamma M) \subseteq q\Gamma M$, then there exists an element u in M and an α in Γ such that $f(q) = q\alpha u$. The other direction of proof is straightforward.

Corollary 2.9: If A satisfies Baer's M -criterion, then A is $\text{fps} - M_I$ -act.

3. Γ -Essentially Pseudo-stable

The following definition provides a generalization of the concept of $\text{fps} - M_I$ -act with respect to Γ -essentially.

Definition 3.1: [8] An M_I -act A is said to be essentially fully pseudo-stable if any Γ -essential M_I -subact of A is pseudo-stable.

It is evident that any fully pseudo-stable M_I -subact is essentially fully pseudo-stable; however, the converse is not true in general.

Proposition 3.2: An M_I -act A is Γ -essential $\text{fps} - M_I$ -act if and only if every cyclic Γ -essential M_I -subact is $\text{ps} - M_I$ -subact.

Proof: The first direction is clear. Now let Z be a Γ -essential M_I -subact of A , stable Γ -act. $f \in \text{Mon}_{M_\Gamma}(Z, A)$ let $f(q) \in f(Z)$. Define $g: f(q)\gamma M \rightarrow A$ by $f(q)\gamma u \mapsto q\gamma u$. By hypotheses $f(q) = g(f(q)\gamma 1_\gamma) = q\gamma 1_\gamma \in q\gamma M \subseteq Z$.

Proposition 3.3: For an M_I -act A , if every cyclic M_I -subact of A of the form $a\delta M$ is Γ -essential; the following conditions are equivalent. (1) A is essentially $\text{fps} - M_I$ -act. (2) Distinct Γ -essential cyclic M_I -subact of A are not isomorphic. (3) $R_M^\delta(q) = R_M^\delta(w)$ and $q\delta M, w\delta M$ are Γ -essential, implies that $q\delta M = w\delta M$ where $q, w \in A$, (where 1_δ is δ -identity).

Proof: (1) to (2). Assume there is an M_I -isomorphism Let $f: Z \rightarrow X$ where Z and X are distinct essential cyclic M_I -subact of A . Now $i_X \circ f \in \text{Mon}_{M_I}(Z, A)$, $i_Z \circ f^{-1} \in \text{Mon}_{M_I}(X, A)$, where i_Z, i_X are the inclusion maps to A . By Γ -Essentially fps - M_I -act of A implies $i_x \circ f(Z) = i_X(f(Z)) = i_X(X) = X \subseteq Z$, and similarly, we have another inclusion direction. And hence $Z = X$, which is a contradiction. (2) to (3) Let $R_M^\delta(q) = R_M^\delta(w)$ for some $q, w \in A$ where are Γ -essential. Define $f: q\delta M \rightarrow w\delta M$ by $f(q\delta u) = w\delta u$ for $u \in M$. it clear that f is M_I -isomorphism and so $q\delta M = w\delta M$. (3) to (1) Let Z be Γ -essential M_I -subact of A and $f \in \text{Mon}_{M_I}(Z, A)$. Then for any $q \in Z$ implies $R_M^\delta(q) = R_M^\delta(f(q))$ (since $f \in \text{Mon}_{M_I}(Z, A)$) and from (3) we have $f(q) \in f(q)\delta M = q\delta M \subseteq Z$. Let Z be an Γ -essential M_I -subact of A . Define $\hat{Z} := \{f(q): f \in \text{Mon}_{M_I}(Z, A) \text{ and } q \in Z\}$. Then $\hat{Z}$ is an M_I -subact of A generated by the image of all elements in $\text{Mon}_{M_I}(Z, A)$. Clearly $Z \subseteq \hat{Z}$, by properties of Γ -essentially in [4] implies $\hat{Z}$ is a Γ -essential in A and Z is Γ -essential in $\hat{Z}$. In fact, an M_I -act A is essentially fps - M_I -act iff $Z = \hat{Z}$ for any Γ -essential M_I -subact Z of A iff $Z = \cup \{f(Z) \mid f \in \text{Mon}_{M_I}(Z, A)\}$.

For any M_I -act A . Consider the following subset $\hat{A}$ of $E(A)$ $\hat{A} = \{f(a): f \in \text{Mon}_{M_I}(A, E(A)) \text{ and } a \in A\}$.

Proposition 3.4: For any M_I -act A , $\hat{A}$ is essentially ps - M_I -subact of $E(A)$

Proof: Let $\beta \in \text{Mon}_{M_I}(\hat{Z}, A)$ and $f(q) \in \hat{Z}$ where $f \in \text{Mon}_{M_I}(Z, A)$. Then $\beta(f(q)) \in \hat{Z}$.

Conclusion

In conclusion, this work introduced fully pseudo-stable gamma acts as a generalization of fully stable gamma acts and established several related properties and characterizations. Additionally, the concept of essentially fully pseudo-stable gamma acts was proposed as a broader and widely applicable generalization in M_I -acts.

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