

Finite time stability for multi- ψ -caputa nonlinear fractional integro-differential systems

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Abstract

In this work, a solution for multi- ψ -caputa nonlinear fractional integro-differential systems have been studied by using ψ -Laplace transformation with Mittag-Leffler function and some necessary conditions of a nonlinear function involving ψ -R-K fractional integral to compute the finite time stability of the presented system, and all interesting conditions are illustrated by many examples.

Subject Classification: Primary 93A30, Secondary 49K15.

Keywords: Finite time stability, Multi- ψ -caputo fractional derivatives, Riemann-katugampola.

1. Introduction

The first calculus of fractions began in the seventeenth century with the work of Leibniz and Newton, who together established the classical calculus, the basis of modern calculus. The operators of fractional calculus, which is most classical, are called the integral and derivative of R–L fractional [1-2]. Also, Caputo is introducing another concept of this issue, which is very suitable for physical conditions, see [3-4]. Moreover, there are many others of fractional operators' definitions which explained in details and until now still studied, such as authors, we can see [4-5]. So since that are many definitions implied that to suggested to establish some generalized fractional operators for the particular cases of definitions. This The motivation is important for equations included fractional differential state in models of physic, economic, engineering and other branch for the science, see [5]. There are many methods are presented by many

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authors for FDEs and their analytically solving or numerically solving, and they interesting on developing which is methods use it to obtains solutions analytically for some class for the fractional differential equations and was considered as a one of tasks where it is challenger in the interesting field which is fractional calculus. Some authors in the last years interested on the extensions the concepts of fractional to classical integral transforms, such as Laplace and other transformations, [5]. In particular, the extensions of Laplace transform appeared in [4-5] has been used as an effective tool for finding analytic of FDEs. The traditional stability and more best behaviors such as asymptotical stability and other best behaviors as exponential stability by Lyapunov approach, system which has this property was considered in an infinite-time interval The stability analysis is important issues of control systems field, also the time- delay systems investigation needed this problem many years ago [4]. In [6-7], the researchers introduced the finite time related to satisfy the stability for the fractional system contained delay time equation by using Mittag-Leffler delay type matrix or other suitable conditions. In [8], nonlinear fractional order stability of finite time in system studied through Mittag Leffler function containing both of orders $0 < \alpha_1 - \alpha_2 < 1$ and $1 \leq \alpha_1 - \alpha_2 < 2$. In [9], have been introduced Black-Scholes fractional equation contain Katugampola Fractional Derivative is a one of Caputo type. The solutions for time - Black – Scholes fractional equations in analytic processer solved with technique contained of homotopy perturbation with important transformation which is generalized Laplace method all details appeared in [5]. Many a branch of science such as, neural networks, control systems and missile systems and other fields, the more practical method is the finite-time stability, see [2], which be called Laplace transforms. These issue of operators suitable with a fractional integral and derivative, and which used to solve differential equations. In [3] the authors studied the nonlinear stochastic Ψ –Hilfer fractional system and their finite time stability.

2. Preliminaries:

Definition 2.1: [10] The finite times stabled relative to $t \in [0, T]$ and $\delta, \varepsilon > 0$ iff $\sigma < \delta$ s.t $\|x(t)\| < \varepsilon$ for $\sigma = \max\{\|x(0)\|, \|x'(0)\|\}$ the initials time for systems differential.

Definition 2.2: [11] Let $\beta > 0, \rho > 0$ & $[a, b]$ in R . A left $R - K$ FDs be define respectively by

$${}^{RK}D_{t-}^{\beta, \rho} h(t) = \frac{\rho^\beta}{\Gamma(1-\beta)} (t^{1-\rho} \frac{d}{dt}) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\beta} h(\tau) d\tau.$$

Definition 2.3: [12], [13] A one parameters Mittag Leffler functions be the

$E_{\beta_1}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\beta_1 + 1)}$, With $\beta_1 > 0$, $Re(\beta_1) > 0$ & z in C . For parameter & β_2 , $E_{\beta_1, \beta_2}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\beta_1 + \beta_2)}$, With $\beta_1, \beta_2 \in C$, $Re(\beta_1) > 0$, $Re(\beta_2) > 0$. Taken $\beta_2 = 1$, $E_{\beta_1}(z) = E_{\beta_1, 1}(z)$.

Definition 2.4: [14] For the $f, g : [a, \infty) \rightarrow \mathbb{R}$ real value function when $g(t)$ be the continuous & $g'(t) > 0$ on $[a, \infty)$, $a > 0$. Let generalize Laplace transformation is $\mathcal{L}_g\{f(t)\}(s) = \int_a^\infty e^{-s(g(t)-g(a))} f(t) g'(t) dt$, (1).

Remark 2.5: [2] If $g(t) = \psi(t)$ then (3) become $\psi(t)$ – Laplace transform hence, $\mathcal{L}_{\psi(t)}\{f(t)\}(s) = \int_0^\infty e^{-s(\psi(t)-\psi(a))} f(t) \psi'(t) dt$. (2)

If $a > 0$ then $\psi(t)$ undefinition, If $a = 0$ then $\psi(t)$ definition.

Lemma 2.6: [1-3]

- 1- $\mathcal{L}_{\psi(t)}\{{}_0^C D_t^{\beta, \psi} x(t)\}(s) = s^\beta \mathcal{L}_{\psi(t)}\{x(t)\}(s) - s^{\beta-1} x(0)$ $0 < \beta \leq 1$
- 2- $\mathcal{L}_{\psi(t)}\{{}_0^C D_t^{\beta, \psi} x(t)\}(s) = s^\beta \mathcal{L}_{\psi(t)}\{x(t)\}(s) - s^{\beta-1} x(0) - s^{\beta-2} x'(0)$ $1 < \beta \leq 2$
- 3- $\mathcal{L}_{\psi(t)}\{E_{\beta_1, 1}(\pm \lambda(\psi(t))^{\beta_1})\}(s) = \frac{s^{\beta_1-1}}{s^{\beta_1} \pm \lambda}$, $\text{Re}(\beta_1) > 0$
- 4- $\mathcal{L}_{\psi(t)}\{E_{\beta_2, \beta_2}(\pm \lambda(\psi(t))^{\beta_2})\}(s) = \frac{s^{\beta_2-1}}{s^{\beta_2} \pm \lambda}$, $\text{Re}(\beta_1) > 0, \text{Re}(\beta_2) > 0$

3. Multi- ψ – Caputa fractional integro-differential nonlinear system

Consider the following Multi- ψ – Caputa fractional integro-differential nonlinear system involving the nonlinear function contained ψ – $R - K$ fractional integral which explained as follows:

$$\begin{cases} {}_0^C D_{\psi(t)}^{\beta_1} x(t) - A {}_0^C D_{\psi(t)}^{\beta_2} x(t) = g(t, x(t), {}_0^{RK} I_{\psi(t)}^{\eta, \rho} x(t)), t \in I \quad (3) \\ x(0) = x_0, x'(0) = x_1 \quad (4) \end{cases}$$

where ${}_0^C D_{\psi(t)}^{\beta_1}$ indicates the ψ – Caputo Fractional Derivatives with β_1, β_2 and $I = [0, T]$, $0 < \beta_2 \leq 1$, $1 < \beta_1 \leq 2$, $0 < \beta_3 \leq 1$, $g: I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ the continuous functions. ${}_0^{RK} I_{\psi(t)}^{\eta, \rho}$ is ψ – $R - K$ fractional integral.

Assumption 3.1:

1. By lemma (3.4), the function $g(t, x(t), {}_0^{RK} I_{\psi(t)}^{\eta, \rho} x(t))$ satisfies $\left\| g(t, x_1(t), {}_0^{RK} I_{\psi(t)}^{\eta, \rho} x_1(t)) - g(t, x_2(t), {}_0^{RK} I_{\psi(t)}^{\eta, \rho} x_2(t)) \right\| \leq W \|x_1(t) - x_2(t)\| + \tilde{w}(t) \|x_1(t) - x_2(t)\|$ for W and $\tilde{w}(t) > 0$.
2. The function $g(t, x(t), {}_0^{RK} I_{\psi(t)}^{\eta, \rho} x(t))$ satisfies the following $\left\| g(t, x(t), {}_0^{RK} I_{\psi(t)}^{\eta, \rho} x(t)) \right\| \leq (M + \tilde{w}(t)) \|x(t)\|$, for $t \in I, x \in \mathbb{R}^n, M > 0$.

Lemma 3.2: The solution of Multi- C fractional integro-differential nonlinear system (3), (4), using $\psi(t)$ – Laplace and inverse Laplace Transform explained as following:

$$x(t) = X_0 E_{\beta_1 - \beta_2} \left(A(\psi(t))^{(\beta_1 - \beta_2)} \right) E_{\beta_1 - \beta_2, \beta_1 - \beta_2 + 1} (A((\psi(t)))^{(\beta_1 - \beta_2)})$$

$$+ X_1 E_{\beta_1 - \beta_2, 2} (A ((\psi(t))^{\beta_1 - \beta_2})) + \int_0^t (\psi(t) - \psi(\theta))^{\beta_1 - \beta_2 - 1} E_{\beta_1 - \beta_2, \beta_1} (A ((\psi(t) - \psi(\theta))^{\beta_1 - \beta_2 - 1})) g(\theta, x(\theta), {}^{\text{RK}} I_{\psi(t)}^{\eta, \rho} \theta(t)) d\theta \quad (5)$$

Lemma 3.3: [3] If $b(t) > 0$ & $c(t) > 0$ the locally integrals on the $[0, T], \beta_1 > 0, d(t) \leq W$ & $b(t) \leq c(t) + d(t) \int_0^t (\psi(t) - \psi(\alpha))^{\beta_1 - 1} b(\alpha) \psi'(\alpha) d(\alpha), 0 \leq t < T$. Then $b(t) \leq c(t) + d(t) \int_0^t [\sum_{n=1}^{+\infty} \frac{[d(t)r(\beta_1)]^n}{r(n\beta_1)} (\psi(t) - \psi(\alpha))^{(n\beta_1 - 1)} c(\alpha)] \psi'(\alpha) d(\alpha)$

Remark 3.4: From above lemma (3.3) implies, $b(t) \leq c(t) E_{\beta_1} (d(t) r(\beta_1)) (\psi(t))^{\beta_1}$.

Lemma 3.5: Let $\alpha > 0, \mu > 0$, and $(t) \in L^1([a, b], R), 0 < a < b < \infty$
 $\left\| {}^{\text{RK}} D_{\psi(t)}^{-\alpha, \mu} x(t) - {}^{\text{RK}} D_{\psi(t)}^{-\alpha, \mu} y(t) \right\| \leq W(t) \|x(t) - y(t)\|$, such that $W(t) = \frac{1}{\mu \alpha \Gamma(\alpha)} \left(\frac{\psi(t)^\mu - a^\mu}{\mu} \right)$

Lemma 3.6:

1. For some $W_1, W_2 \geq 0$ such that, for $\beta_1, \beta_2 \in R^+$ and $\beta_1 - \beta_2 < 1$, we have that $\|E_{\beta_1 - \beta_2, 1} (A \psi(t)^{\beta_1 - \beta_2})\| \leq W_1 \|e^{A \psi(t)}\|$, and $\|E_{\beta_1 - \beta_2, \beta_1 - \beta_2} (A \psi(t)^{\beta_1 - \beta_2})\| \leq W_2 \|e^{A \psi(t)}\|$, where A is a constant matrix.
2. Suppose that $\beta_1, \beta_2 \in R^+$ and $\beta_1 - \beta_2 \geq 1$ then for $= 1, 2, \beta_1$, we have that, $\|E_{\beta_1 - \beta_2, \gamma} (A (\frac{t^{\rho(\beta_1 - \beta_2)}}{\rho^{\beta_1 - \beta_2}}))\| \leq \|e^{A (\frac{t^{\rho(\beta_1 - \beta_2)}}{\rho^{\beta_1 - \beta_2}})}\|$
3. For A the stabile matrix, so $\exists K \geq 1$ s.t for $\psi(t) > 0$, then $\|E_{\beta_1 - \beta_2, \gamma} (A \psi(t)^{\beta_1 - \beta_2})\| \leq K e^{-\eta \psi(t)}$ for $0 < \beta_1 - \beta_2 < 1$, and $\|E_{\beta_1 - \beta_2, \gamma} (A \psi(t)^{\beta_1 - \beta_2})\| \leq e^{-\eta \psi(t)}$ for $1 \leq \beta_1 - \beta_2 < 2$, where the greatest eigenvalue of A is η .

Theorem 3.7: Let $0 > \beta_1 - \beta_2 < 1$ and $t \in I, x(.) \in R^n$. Then the $R - K$ nonlinear fractional integro-differential system (3), (4) is a finite times stabled provide condition:

$$K e^{-\eta(\psi(t))} [1 + \|A\| \psi(t)^{\beta_1 - \beta_2} + (\psi(t))] E_{\beta_1 - \beta_2} (K(M + \tilde{w}(t)) r(\beta_1 - \beta_2)) (\psi(t))^{\beta_1 - \beta_2} < \frac{\varepsilon}{\delta} \quad (6)$$

Proof: Taking norm function of solution equation (3), we obtain following,
 $\|x(t)\| = \|x_0\| \|E_{\beta_1 - \beta_2} (A \psi(t)^{\beta_1 - \beta_2})\| + \|A\| \|x_0\| \|(\psi(t))^{\beta_1 - \beta_2}\|$
 $\|E_{\beta_1 - \beta_2, \beta_1 - \beta_2 + 1} (A \psi(t)^{\beta_1 - \beta_2})\| + \|x_1\| \|E_{\beta_1 - \beta_2, 2} (A \psi(t)^{\beta_1 - \beta_2})\| + \int_0^t \|(\psi(t) - \psi(\theta))^{\beta_1 - \beta_2 - 1}\| \|E_{\beta_1 - \beta_2, \beta_1} (A \psi(t)^{\beta_1 - \beta_2})\| \|g(\theta, x(\theta), {}^{\text{RK}} I_{\psi(t)}^{\eta, \rho} \theta(t))\| d\theta \quad (7)$

by using lemma (3.6), equation (7) become that, $\|x(t)\| = \|x_0\| K e^{-\eta(\Psi(t)) + \|A\|} \|x_0\| \|(\Psi(t))^{(\beta_1 - \beta_2)}\| K e^{-\eta(\Psi(t)) + \|x_1\|} K e^{-\eta(\Psi(t)) + \int_0^t |(\Psi(t) - \Psi(\theta))^{(\beta_1 - \beta_2 - 1)}| K e^{-\eta(\Psi(t) - \Psi(\theta))} \|g(\theta, x(\theta), {}^{RK}I_{\Psi(t)}^{\eta, \rho} \theta(t))\| d\theta$ (8) By using assumption (3.1), (2), the equation (8), become that,

$$\|x(t)\| = \|x_0\| K e^{-\eta(\Psi(t))} + \|A\| \|x_0\| \|\Psi(t)^{(\beta_1 - \beta_2)}\| K e^{-\eta(\Psi(t))} + \|x_1\| K e^{-\eta(\Psi(t))} + \int_0^t |(\Psi(t) - \Psi(\theta))^{(\beta_1 - \beta_2 - 1)}| K e^{-\eta(\Psi(t) - \Psi(\theta))} (M + \tilde{w}(\theta)) \|x(\theta)\| d\theta. \quad (9)$$

Multiply equation (9) by $e^{\eta(\Psi(t))}$, we have that, $e^{\eta(\Psi(t))} \|x(t)\| = K[\|x_0\| + \|A\| \|x_0\| \|\Psi(t)^{(\beta_1 - \beta_2)}\| + \|x_1\|] + \int_0^t |(\Psi(t) - \Psi(\theta))^{(\beta_1 - \beta_2 - 1)}| K e^{\eta\theta} (M + \tilde{w}(\theta)) \|x(\theta)\| d\theta$. (10) According on lemma (3.3), we put $b(t) = e^{\eta(\Psi(t))} \|x(t)\|$, & we consider $c(t) = K[\|x_0\| + \|A\| \|x_0\| \|\Psi(t)^{(\beta_1 - \beta_2)}\| + \|x_1\|]$, and $d(t) = K(M + \tilde{w}(\theta))$, by remark (3.4), the equation (10), become to $b(t) \leq c(t) E_{\beta_1 - \beta_2} (d(t) r(\beta_1 - \beta_2) (\Psi(t)^{(\beta_1 - \beta_2)}))$

$$\leq K[\|x_0\| + \|A\| \|x_0\| \|\Psi(t)^{(\beta_1 - \beta_2)}\| + \|x_1\|] E_{\beta_1 - \beta_2} (K(M + \tilde{w}(\theta)) r(\beta_1 - \beta_2) (\Psi(t)^{(\beta_1 - \beta_2)})) \quad (11)$$

then $\|x(t)\| \leq K e^{-\eta(\Psi(t))} [\|x_0\| + \|A\| \|x_0\| \|\Psi(t)^{(\beta_1 - \beta_2)}\| + \|x_1\|] E_{\beta_1 - \beta_2} (K(M + \tilde{w}(\theta)) r(\beta_1 - \beta_2) (\Psi(t)^{(\beta_1 - \beta_2)}))$ (12) From definition (2.1), we obtain $\|x_0\| \leq \delta, \|x_1\| \leq \delta$, then

$$\|x(t)\| \leq K \delta e^{-\eta(\Psi(t))} [1 + \|A\| \|\Psi(t)^{(\beta_1 - \beta_2)}\| + |(\Psi(t))|] E_{\beta_1 - \beta_2} (K(M + \tilde{w}(\theta)) r(\beta_1 - \beta_2) (\Psi(t)^{(\beta_1 - \beta_2)})),$$

Therefore, $\|x(t)\| \leq \varepsilon$, for all $t \in [0, T]$. Which implies that formula (3) is a finite time stable.

Theorem 3.8: Let $1 \leq \beta_1 - \beta_2 < 2$ and $t \in I, x(.) \in R^n$. Then the $R - K$ nonlinear fractional integro-differential system (3), (4) is a finite times stabled provide $e^{-\eta(\Psi(t))} [1 + \|A\| (\Psi(t)^{(\beta_1 - \beta_2)}) + (\Psi(t))] E_{\beta_1 - \beta_2} (K(M + \tilde{w}(t)) r(\beta_1 - \beta_2) (\Psi(t)^{(\beta_1 - \beta_2)})) < \frac{\varepsilon}{\delta}, t \in [0, T]$. (13)

Corollary 3.9: A multi terms fractionals linear equations,

$$\begin{cases} {}^C_0 D_{\Psi(t)}^{\beta_1} x(t) - A {}^C_0 D_{\Psi(t)}^{\beta_2} x(t) = Bx(t) + {}^{RK}I_t^{\eta, \rho} x(t), \text{ for } t \in I \quad (14) \\ x(0) = x_0, x'(0) = x_1 \quad (15) \end{cases}$$

Be the finite times stabled where $0 < \beta_1 - \beta_2 < 1$,

Corollary 3.10: The $R - K$ linear fractional integro- differential system (14), Is finite times stabled where $1 \leq \beta_1 - \beta_2 < 2$, if $e^{-\eta(\Psi(t))} [1 + \|A\| ((\Psi(t)^{(\beta_1 - \beta_2)}) + (\Psi(t)))] E_{\beta_1 - \beta_2} (\|B\| + \tilde{w}(t)) r(\beta_1 + \beta_2) ((\Psi(t)^{(\beta_1 - \beta_2)})) < \frac{\varepsilon}{\delta}$.

4. Illustrated Examples

Example 4.1: Put a Multi- ψ – Caputa fractional integro-differential nonlinear system involving the ψ – $R - K$ fractional integral

$$\begin{cases} {}^C_0D_{\psi(t)}^{\beta_1} x(t) - A {}^C_0D_{\psi(t)}^{\beta_2} x(t) = \frac{x_1}{1+|x_1|} + {}^{RK}_0I_{\psi(t)}^{\eta,\rho} x_1(t) & (16) \\ {}^C_0D_{\psi(t)}^{\beta_2} x_2(t) - {}^C_0D_{\psi(t)}^{\beta_2} x_1(t) = 0 & (17) \end{cases}$$

Where $\beta_1 = 4.75$ and $\beta_2 = 4.25$. by using considered system (3) for (16), (17), yield $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $g(t, x(t), {}^{RK}_0I_t^{\beta_3,\rho} x(t)) = \begin{bmatrix} \frac{x}{1+|x|} + {}^{RK}_0I_t^{\beta_3,\rho} x_1(t) \\ 0 \end{bmatrix}$, assumption (3.1), we obtain $\|g(t, x(t), {}^{RK}_0I_t^{\beta_3,\rho} x(t))\| \leq (1 + \dot{w}(t)) \|x(t)\|$, for $t \in I, x \in R^2$, Now to compute the condition (4) w.r.t $\eta = 1.5, 1.8, 1.25, 0.25, 0.5$ and $\|A\| = 1$ and $\psi(t) = t$, and choose $\delta = 0.05$ and $K = 1.5, 2, 1.75, 1, 0.5, \varepsilon = 1.5, \rho = 1$,

$$Ke^{-\eta(\psi(t))} [1 + \|A\|(\psi(t)^{(\beta_1-\beta_2)} + (\psi(t))) E_{\beta_1-\beta_2}(K(M + \tilde{w}(t)) \Gamma(\beta_1 - \beta_2)) (\psi(t)^{(\beta_1-\beta_2)})] < \frac{\varepsilon}{\delta}, \text{ with } \dot{w}(t) = \frac{1}{\beta_2 \beta_1 \Gamma(\beta_1)} \left(\frac{\psi(t)^{\beta_2 - \alpha \beta_2}}{\beta_2} \right)$$

Hence the estimated finite time stable of $R - K$ nonlinear fractional integro-differential system $T \approx 0.9231, 1.3207, 0.0951, 0.2980, 2.0787$.

Example 4.2: Put a systems (14) with a parameter $\beta_1 = 4.75, \beta_2 = 4.25$, $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$, by choosing many value of $K = 1, 3, 4, 2, 1.5$ and $\varepsilon = 1.5$, and $\delta = 0.05$ and $\psi(t) = t$. Now to compute condition (32) with $\rho = 1, \eta = 0.5, 1, 0.95, 1.24, 1.56$, and $\|A\| = 3.7417, \|B\| = 2$ with

$$Ke^{-\eta(\psi(t))} [1 + \|A\|((\psi(t)^{(\beta_1-\beta_2)}) + (\psi(t))) E_{\beta_1-\beta_2}((\|B\| + \tilde{w}(t)) K \Gamma(\beta_1 - \beta_2)) ((\psi(t)^{(\beta_1-\beta_2)})) < \frac{\varepsilon}{\delta}, \dot{w}(t) = \frac{1}{\beta_2 \beta_1 \Gamma(\beta_1)} \left(\frac{t^{\beta_2 - \alpha \beta_2}}{\beta_2} \right),$$

the estimated finite time stable of (14), (15) are $T \approx 0.4712, 0.1094, 0.2263, 0.8660, 0.4511$.

5. Discussion and Conclusion

We conclude that the following Multi- ψ – Caputa fractional integro-differential nonlinear system involving the nonlinear function contained ψ – $R - K$ fractional integral is stable under a finite time which can be determined by using some ψ suitable to the presented fractional system and ψ – Laplace transformation.

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Received May, 2025