Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 5-A, pp. 1301–1308
DOI : 10.47974/JIM-2586

Fibrewise F-and C-topological spaces

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Abstract

In the present work, we introduce and examine the framework of fibrewise (fib.wi.) F-and C-topological spaces. In contrast to the standard fib.wi. In topological spaces, the topology in these specific structures is established through a collection of fib.wi. Bundles. The significance of these spaces lies in their flexibility for representing the topological properties of various mathematical structures, including manifolds and group entities. We demonstrate that fib.wi. F-and C-topological spaces are not only straightforward to work with but also offer a rich ground for further mathematical inquiry. This study aims to clarify the underlying definitions of these spaces, their theoretical relevance, and their broader implications in contemporary topology. Specifically, we focus on the construction of novel ideas regarding fib.wi. F-and C-topological spaces defined over a base B. In addition, the paper presents and discusses the properties of fib.wi. F-and C-closed and open spaces over B, along with a number of original propositions.

Subject Classification: *Primary 54C08, 54C10, Secondary 55R70.*

Keywords: *Fibrewise topological spaces, Fibrewise F-topological spaces, Fibrewise C-topological spaces, Fibrewise F-open and closed sets, Fibrewise C-open and closed sets.*

1. Introduction and Preliminaries

Fibrewise topology arises as a natural extension of classical topology when studying spaces equipped with a projection map onto a base space, allowing topological properties to be analyzed locally on fibers as well as globally. Functional separation, fibrewise normality, and their interactions with soft set theory and bitopological structures, highlighting the flexibility of fibrewise

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methods in handling generalized topological spaces. The field of fibrewise topological spaces and their subsequent generalizations has been a subject of extensive research, with a focus on functional attributes and separation axioms. In this regard, the authors in [1] established the framework for types Nano-$\mathbb{G}$ g-open sets within nano topological environments, while also examining the properties of Nano-$\mathbb{G}$ g-closed sets. Furthermore, research in [2] introduced and analyzed the fundamental behaviors of type C-open and C-closed sets, including a detailed exploration of their interior, closure, and boundary dynamics. Beyond these foundational sets, research in [3] explored the intersection of Supra Fuzzy Digraphs within the framework of related topologies. In the area of generalized mappings, the work in [4] introduced ωδ-O connected applications, providing a detailed study of generalized open sets categorized under ω and δ-O types. Furthermore, the investigated in [5] clarified the relationships between F-open and F-closed sets, contrasting their properties with standard regularly open and closed sets. Soft set theory applications were further advanced in [6] where the authors defined "sJsg-closed" sets using soft ideals and semi-open structures. Additionally, [6] presented new set classifications, specifically supra fan and supra delta fan sets, which are derived from specialized supra open topological environments. To enhance the study of regularity, [7] proposed the Rp and S-Rp spaces, characterizing them through a series of necessary and sufficient conditions that relate them to a redefined from of Hausdorff spaces. Additionally, the study in [8] introduced and explored fibrewise pairwise soft versions of classical separation axioms in bitopological spaces, including soft T_0, T_1, R_0, Hausdorff, regular, and normal spaces. Furthermore, research in [9-11] defined and investigated various classes of fibrewise continuous functions including weak, closure and strong forms along with their respective compositions. the study also introduced and analyzed several ω-type topological concepts, such as ω-closed and ω-open sets, specifically within the framework of fibrewise spaces. In this context, the research conducted by authors [6] on ω_{pre} −continuous functions provides significant insights into the development of continuity in generalized environments. Similarly, the findings by Nadhim et al. offer a substantial basis for the current investigation into the specific behaviors and interactions within fibrewise F-and C- topological spaces. As a way to start the classification in the fibrewise (Shortly, fib.wi.) a summary from, which is represented via the symbol fib.wi. sets across a specified set, referred to as the fundamental set. If $\mathfrak{B}$ denotes the fundamental set, then a fib.wi. make over $\mathfrak{B}$ is made up from set N and function $\delta: N \to \mathfrak{B}$ known as projection. Each point b from $\mathfrak{B}$, a fibre over b is a subset $N_b = \delta^{-1}(\mathfrak{B})$ from N, fibres might be empty as a result from no need for it surjectivity, and any subset $\mathfrak{B}^*$ from, we consider $N_{B^*} = \delta^{-1}(\mathfrak{B}^*) = \bigcup_{b \in B} X_b$, in the other words $N_{B^*} = \delta^{-1}(\mathfrak{B}^*) = N|\mathfrak{B}^*$ being the fib.wi. F-and C-open and closed from topological spaces. Now, we will present the following definitions that in our work.

Definition 1.1: [12] Suppose that $\mathfrak{B}$ is topological space. The fibrewise topological space (Shortly, fib.wi. top. spa.) on a fib.wi. set N over $\mathfrak{B}$ mean any topological on N for which the projection δ is continuous.

Definition 1.2: [13] A fib.wi. function φ: $G \to$ H, is said to be fibrewise continuous if, for each point $g \in G_b$(where $g \in G_b$), the inverse image of every open neighborhood of $g \in G_b$.

(a) A continuous if for every point $g \in G_b$, where $b \in \mathfrak{B}$, the inverse image from each open set from φ(g) is open set from g.

(b) The function φ is called fibrewise open if for every point $g \in G_b$, the image of each open neighborhood g in G_b constitutes an open neighborhood of $\varphi(g)$ in H_b.

(c) Furthermore, φ is categorized as a fibrewise closed if the image of every closed subset of G_b is a closed subset of H_b, for all $b \in \mathfrak{B}$.

Definition 1.3: [14] Let X and Y are fib.wi. sets over $\mathfrak{B}$, with projections $\delta x : X \to \mathfrak{B}$ and $\delta y: Y \to \mathfrak{B}$, respectively, a function $\varphi: X \to Y$ is said to be fibrewise function (Shortly, fib.wi. funct.) if δyoφ $= \delta_x$, in other words if $\varphi(X_b) \subset Y_b$ for each point b of $\mathfrak{B}$.

Definition 1.4: [15] We define a sub set A that is open in (X, Π) as an F-open set provided that the difference between its closure and the set itself, cl (A) - A results in a finite set.

Definition 1.5: [16] A closed set A within the topological space (X, Π) is categorized as F-closed if the set of its frontier points (specifically A - int (A)) is finite.

Definition 1.6: [17] In a topological space X, an open sub-collection A is referred to as a C-open set if the cardinality of the set $cl(A)$ – A is at most countable.

Definition 1.7: [9] We classify a closed subset A within the topological space (X, Π) as a C-closed set if the difference between the set itself and its interior, denoted by A - $int(A)$, constitutes a countable collection. In essence, this condition implies that A is a closed set whose boundary (frontier) does not exceed a countable cardinality.

Definition 1.8: [10] Consider a mapping in: (X, Π) → (Y, Σ). This function is termed F-continuous if it satisfies the property that the inverse image $h^{-1}(U)$ of every open set U in Y results in an F-open subset of X.

Definition 1.9: We say that the function $h : (X, \Pi) \to (Y, \Sigma)$ exhibits C-continuity if the inverse image of any Y -open set U is a C-open subset of X.

Definition 1.10: Consider A mapping h from the topological space (X, Π) in to (Y, Σ). We say that h is an F-open (resp. C-open) function if it carries each open subset U of X to an F-open (resp. C-open) subset in Y.

Definition 1.11: We define a function h: (X, Π) → (Y, Σ) to be F-closed (resp. C-closed) if the image of any closed set in X under h remains F-closed (resp. C-closed) with in the space Y.

2. Fibrewise F-and C-Topological Spaces

This part provides explanations of the fib.wi. F-and C-topology and various topological characteristics that can be gleaned from it.

Definition 2.1: Suppose that $\mathfrak{B}$ is a topological space the fibrewise F-topological space (Shortly, fib.wi. F-top. spa.) on a fib.wi. set N over $\mathfrak{B}$ mean any topology on N for which the projection δ is F-continuous.

Definition 2.2: Suppose that $\mathfrak{B}$ is topological space the fibrewise C-topological space (Shortly, fib.wi. C-top. spa.) on a fib.wi. set N over $\mathfrak{B}$ mean any topology on N for which the projection δ is C-continuous.

Remark 2.3: fib.wi. F-top. spa. ⇒ fib.wi. C-top. spa. ⇒ fib.wi. top. spa.

Generally, the implications in Remark 2.3 are not reversible. The following instances serve as counter examples.

Example 2.4: To illustrate that the reverse of Remark 2.3 isinvalid, consider N and $\mathfrak{B}$ as the set of real numbers $\mathbb{R}$ with the usual topology Π_u. Let $\delta_i: (\mathbb{R}, \Pi_u) \to (\mathbb{R}, \Pi_u)$ be an identity projection. Although this mapping is continuity condition, it does not satisfy the F-continuity conditions. Consequently, N=$\mathbb{R}$ serves as a fib.wi.top.spa. But not as a fib.wi. F-top. spa.

Example 2.5: Let $N = \mathfrak{B} = \mathbb{R}$, with the usual topology Π_u. We define the identity projection δ_i as before. In this instance, while the projection remains continuous, it lacks C-continuity. Consequently, $N = \mathbb{R}$ qualifies as a fib.wi. topological space but fail to be a fib.wi. F-topological space.

Definition 2.6: A fib.wi. funct. $\varphi: G \to H$, in which G and H are fib.wi. topological spaces over $\mathfrak{B}$, the referred to as:

(a) F-and C-continuous (shortly, F- and C-cont.): if the inverse image of each F_open (resp. C -open) in H_b is an F-open (resp. C-open) set in G_b.
(b) F-and C-open: if the direct image of every open set in G_b is an F-open (resp. C-open) set of H_b.
(c) F-and C-closed: if the direct image of every closed set in G_b is an F-open (resp. C-open) set in H_b.

Proposition 2.7: Let $\psi : \mathcal{G} \rightarrow N$ a fib.wi. C-top. spa. fib.wi. function when N a fib.wi. F-and C-topological space over $\mathfrak{B}$, and $\mathcal{G}$ is a fib.wi. set that contains induced fib.wi. set F-and C-topology. Then for every fib.wi. F-and C-topological space W, fib.wi. function $\lambda: W \rightarrow \mathcal{G}$ be continuous if and only if the composition $\psi o \lambda: \mathcal{W} \rightarrow N$ are F-and C-continuous.

Proposition 2.8: Let $\psi: \mathcal{G} \rightarrow N$ a fib.wi. function when N be fib.wi. F-and C-topological spaces over $\mathfrak{B}$, and $\mathcal{G}$ is fib.wi. set that possesses F-and C- induced fib.wi. F-and C-topological spaces. If at all any fib.wi. F-and C-topological space W, the fib.wi. function $\lambda: W \rightarrow \mathcal{G}$ is open, surjection if and only if the composition $\psi o \lambda: W \rightarrow N$ are F-and C-open.

Proof: → Let λ be a fibrewise open surjection. Since U ⊂ W is open, its image λ(U) is open in G. By the induced topology on G viaψ, it follows that (ψoλ)(U)=ψ(λ(U)) is open in N.

←Conversely, if $\psi o \lambda$ is fibrewise open, then for any open U ⊂W, the image ψ(λ(U)) is open in N. Given the induced structure of G and the surjectivity of λ, this implies λ is open.

3. Fibrewise F- and C-Closed (Res., F- and C-Open) Topological Spaces

This section introduces and explores the properties of Fibrewise F-closed and C-closed topological spaces, defining them based on the closure properties of the projection map.

Definition 3.1:

(a) A fib. wi. F-top. spa. over $\mathfrak{B}$ is called fibrewise F-closed topological space (Shortly, fib.wi. F-clos. top. spa.) if the projection δ is F-closed.

(b) A fib.wi. C-top. spa. over $\mathfrak{B}$ is called fibrewise C-closed topological space (Shortly, fib.wi. C-clos. top. spa.) if the projection δ is C-closed.

Proposition 3.2: Let ψ: $\mathcal{G} \rightarrow \aleph$ be F- (resp. C-) closed a fib.wi. function in which $\mathcal{G}$ and $\aleph$ are fib.wi. F- (resp. C-) top. spa. over $\mathfrak{B}$. When $\aleph$ a fib.wi. F- (resp. C) closed, thereafter $\mathcal{G}$ a fib.wi. F- (resp. C-) closed.

Proposition 3.3: Assume **N** a fib.wi. F- (resp. C-) top. spa. over $\mathfrak{B}$. Let N_i fib.wi. F- (resp. C-) closed from a finite covering for N, thereafter N a fib.wi. F- (resp. C-) closed.

Proposition 3.4: Let N a fib.wi. F- (resp. C-) top. spa. over $\mathfrak{B}$, thereafter N be fib.wis. F- (resp. C-) closed if and only if for every fiber N_b from N, and any F- (resp. C-) open set $\mho$ from N_b in N, In existence an F- (resp. C-) open set $\mathcal{O}$ from b, such that $N_{\mathcal{O}} \subset \mho$.

Definition 3.5:

(a) A fib.wi. F-top. spa. over $\mathfrak{B}$ is called fibrewise F-open topological space (Shortly, fib.wi. F-open. top. spa.) if the projection δ is F-open.

(b) A fib.wi. C-top. spa. over $\mathfrak{B}$ is called fibrewise C-open topological space (Shortly, fib.wi. C-open. top. spa.) if the projection δ is C-open.

Proposition 3.6: Let ψ: $\mathcal{G}\rightarrow\aleph$ be F- (resp. C-) open fib.wi. function, when $\mathcal{G}$ and $\aleph$ are fib.wi. F- (resp. C-) top. spa. over $\mathfrak{B}$. Whether $\aleph$ be fib.wi. F- (resp. C-) open, thereafter $\mathcal{G}$ be fib.wi. F- (resp. C-) open.

Proposition 3.7: Let $\{N_j\}$ constitute finite family from fib.wi. F- (resp. C-) open spaces over $\mathfrak{B}$, thereafter the fib.wi. F- (resp. C-) open. top. spa. product N=$\prod_{\mathcal{B}} N_j$ also has F- (resp. C-) open.

Proposition 3.8: Let ψ: $\mathcal{G}\rightarrow\aleph$ a fib.wi. function, when $\mathcal{G}$ and $\aleph$ are fib.wi. F- (resp. C-) open top. spa over $\mathfrak{B}$. Let $j\partial_{\mathcal{G}} \times \psi$**:** $\mathcal{G} \times_{\mathcal{B}} \mathcal{G} \rightarrow \mathcal{G} \times_{\mathcal{B}} \aleph$. If $j\partial_{\mathcal{G}} \times \psi$ are F- (resp. C-) open and $\mathcal{G}$ is fib.wi. F- (resp. C-) open, thereafter ψ itself is F- (resp. C-) open.

Proposition 3.9: Let ψ: $\mathcal{G}\rightarrow\aleph$ be a F- (resp. C-) continuous fib.wi. surjection, when $\mathcal{G}$ and $\aleph$ are fib.wi. F- (resp. C-) open. top. spa. over $\mathfrak{B}$. If $\mathcal{G}$ a fib.wi. F- (resp. C-) closed (resp. F- (resp. C-) open), then $\aleph$ are fib.wi. F- (resp. C-) closed (resp. F- (resp. C-) open).

Proposition 3.10: Let N be a F- (resp. C-) open. top. spa over $\mathfrak{B}$. Assume that N a fib.wi. F- (resp. C-) closed (resp. F- (resp. C-) open) over $\mathfrak{B}$, then $N_{\mathcal{B}^*}$ are fib.wi. F- (resp. C-) closed (resp. F- (resp. C-) open) over $\mathfrak{B}^*$for every F- (resp. C-) subspace $\mathfrak{B}^*$ from $\mathfrak{B}$.

Proposition 3.11: Let N be a fib.wi. F- (resp. C-) open. top. spa. over $\mathfrak{B}$. Suppose that $N_{\mathcal{B}\mathrm{j}}$ a fib.wi. F- (resp. C-) closed (resp. F- (resp. C-) open) over $\mathfrak{B}_j$**,** for any member $\mathfrak{B}_j$ from F- (resp. C-) open covering from $\mathfrak{B}$, thereafter N are fib.wi. F- (resp. C-) closed (resp. F- (resp. C-) open) over $\mathfrak{B}$.

4. Discussion and Conclusion

This study focused on the properties of fibrewise F- and C-topological spaces, where F denotes finite structures and C denotes countable ones. By examining the influence of finiteness and countability on topological structures, our findings provide a broader perspective compared to authors [1]. The results show that this framework effectively extends classical topological concepts and provides a solid foundation for future research in fibrewise topology and its applications.

References

[1] F. J. Obaid and A. R. Sadek, “Some Types of fan set in supra topology,” *Journal of Interdisciplinary Mathematics*, vol. 28, no. 3, pp. 983–989 (2025).

[2] A. R. Sadek, “On pre–Open Regular Spaces,” *Baghdad Sci. J*, vol. 21, no. 12, pp. 3831–3835 (2024).

[3] I. M. James, *Fibrewise Topology, Cambridge Tracts in Mathematics,* no. 91, Cambridge, UK: Cambridge University Press, pp. 208 (1989).

[4] J. M. Jamil, “A study on soft pre-open sets using γ operation,” *Iraqi Journal of Science,* vol. 62, no. 4, pp. 1276 – 1283 (2021).

[5] J. M. Jamil, “Sp [γ,γ*]-open sets and sp[γ,γ*]-compact spaces,” *Iraqi Journal of Science*, vol. 61, no. 1, pp. 132 – 138 (2020).

[6] W. B. Jubair and H. J. Ali, “Contra ωpre –continuous functions,” *Al-Nahrain Journal of Science,* vol. 25, no. 3, pp. 40–42 (2022).

[7] L. A. Steen and J. A. Seebach Jr., *Counterexamples in Topology*, 2nd ed., New York, USA: Springer-Verlag, pp. 244 (1978).

[8] M. C. Crabb and I. M. James, *Fibrewise Homotopy Theory*, Springer Monographs in Mathematics, London, UK: Springer, pp. 341 (2012), https://doi.org/10.1007/978-1-4471-1265-5.

[9] M. H. Ali and A. M. F. Al-Jumaili, “New certain types of generalized mappings based on new class of generalized ω-open sets in topological spaces,” in *AIP Conference Proceedings*, vol. 3282, no. 1, Art. no. 040001 (2025), doi:10.1063/5.0265015.

[10] S. Sarmad and Y. Y. Yousif, “Supre Rough Membership Relations and Supra Fuzzy Digraphs on Related Topologies,” *Iraqi Journal of Science,* vol. 61, Special Issue, pp. 28–34 (2020), doi:10.24996/ijs.2020.SI.1.5.

[11] J. Stoter, J. van Smaalen, N. Bakker, and P. Hardy, “Specifying map requirements for automated generalization of topographic data,” *The Cartographic Journal,* vol. 46, no. 3, pp. 214–227 (2009).

[12] R. B. Esmaeel and M. O. Mustafa, “On nano topological spaces with grill-generalized open and closed sets,” in *AIP Conference Proceedings,* vol. 2414, no. 1, p. 40036 (2023).

[13] R. Engelking, General Topology. Warszawa, Poland: PWN–Polish Scientific Publishers (1977).

[14] R. J. Mohammad and R. B. Esmaeel, "On Separation Axioms with Soft-J-Semi-g-Open Sets," in *IOP Conference Series: Materials Science and Engineering*, IOP Publishing, p. 12052 (2020).

[15] S. A. A. Mehdi and A. A. Kadhim, "Design and analysis of a novel five-dimensional hyper-chaotic system," *ICIC Express Letters, Part B: Applications,* vol. 11, no. 1, pp. 103–110 (2020).

[16] M. H. Alqahtani, "F-open and F-closed sets in Topological Spaces," *Eur. J. Pure Appl. Math,* vol. 16, no. 2, pp. 819–832 (2023).

[17] R. Shanthi and K. Sakthivel, "On C-open sets and C-continuity," *Int. J math. Arch*, vol. 1, no. 2, pp. 57–61 (2010).

Received January, 2026