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Effective improved Runge-Kutta Nystrom for second order ODEs with diagonally implicit

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Abstract

This study constructs a novel three-stages, diagonally implicits improved Runge-Kutta nystrom technique. The suggested approach was created to address second - order ordinary differentials equation (ODEs). A derived methods involves fewer steps than the current Runge-Kutta-Nystrom (RKN) methods, resulting in fewer function evaluations per step. A few numericals experiment is carried out to demonstrate effectiveness and robustness for a novel approaches.

Subject Classification: *Primary 65L05, Secondary 65L06.*

Keywords: *Improved Runge–Kutta Nystrom, Diagonally implicit methods, Differential equations, Initial value problems, Oscillatory problems.*

1. Introduction

Numerous approaches have proposed of numericals resolution for specials second ODEs for a type

$$q''(t) = g(t, q), q(t_0) = q_0, q'(t_0) = q'_0, \tag{1}$$

Problem (1) often arises in a variety of fields, including astronomy, quantum and physical sciences, fluid dynamics, and a lot more [1-5]. To find approximates solutions to problems (1), the RKN) and the improved Runge–Kutta Nystrom (IRKN) approaches have typically been taken into consideration. Taking the values of $k_i, i \geq 2$ from earlier stages and applying new terms k_{-i}, to solve second-order ODEs, the IRKN approach was developed [6]. The authors

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in [7-11] have all examined various techniques of the RKN methods which are diagonally implicit with regard to find the numerical solution of (1). [12] Constructs a novel embedded four-stage RKN method that is diagonally implicit. In order to address the problem in (1), we create a novel, effective diagonally implicit IRKN method (DIIRKN). The new method's performance is demonstrated through numerical experiments in comparison to some implicit RKN techniques found in the scientific literature.

This is the format for rest for work: Section 2 provides IRKN method which is diagonally implicit. Section 3 gives the derivation of a suggested DIIRKN method. Section 4 presents a few numerical tests. Finally, we provide a discussion and conclusion in Section 5.

2. Basic Concepts

The general formulas for an IRKN approach with s-stages to solve the problem in (1) are as follows: [6]

$$q_{n+1} = q_n + \frac{3}{2}hq'_n + \frac{1}{2}hq'_{n-1} + h^2\sum_{i=2}^{s} b_i(p_i - p_{-i}), \tag{2}$$

$$q'_{n+1} = q'_n + h(b_1p_1 - b_{-1}p_{-1} + \sum_{i=2}^{s} d_i\,(p_i - p_{-i}), \tag{3}$$

$$p_1 = g(t_n, q_n), \tag{4}$$

$$p_{-1} = g(t_{n-1}, q_{n-1}), \tag{5}$$

$$p_i = g(t_n + c_ih, q_n + hc_iq'_n + h^2\sum_{i=2}^{s} a_{ij}p_i), \tag{6}$$

$$p_{-i} = g(t_{n-1} + c_ih, q_{n-1} + hc_iq'_{n-1} + h^2\sum_{i=2}^{s} a_{ij}p_{-i}), \tag{7}$$

where , $j = 1,2,\dots,s$, b_{-1}, d_i, b_i, c_i and a_{ij} are real numbers. Present the vectors in s dimensions $b = [b_1, d_2, \dots, b_s]^T, d = [d_2, d_3, \dots, b_s]^T, c = [c_1, c_2, \dots, c_s]^T$ and $A = \left[a_{ij}\right]_{s\times s}$ matrix. The Butcher array can be used to formulate the IRKN approach as follows:

c	A
b_{-1}	b
	d

There are two types of IRKN methods: explicit and implicit. When A be the lower triangulars matrix, which considered to be explicit; if not, it is referred to as implicit. The includes the class of IRKN procedures that are diagonally implicit and for which every entry in the diagonal of A is equal. According to [6], the following order conditions for an IRKN approach in eqs (2)–(7) up to order five

Order (for short O) condition of q:

O 1: $b_1 - b_{-1} = 1$,

O 2: $b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}$,

O 3: $\sum_{i=2}^{s} b_ic_i = \frac{5}{12}$,

O 4: $\sum_{i=2}^{s} b_i c_i^2 = \frac{5}{12}$,
O 5: $\sum_{i=2}^{s} b_i c_i^3 = \frac{31}{120}$, $\sum_{i=2}^{s} b_i a_{ij} c_j = \frac{31}{720}$.

Order condition of q':
O 3: $\sum_{i=2}^{s} d_i = \frac{5}{12}$,
O 4: $\sum_{i=2}^{s} d_i c_i = \frac{1}{6}$,
O5: $\sum_{i=2}^{s} d_i c_i^2 = \frac{31}{360}$.

c_i in IRKN methods require to satisfy the following row conditions [1]

$$\sum_{i,j=1}^{s} a_{ij} = \frac{1}{2} c_i^2 \tag{8}$$

3. Construction of the New Method

Following the previously mentioned order restrictions, we build DIIRKN, a novel diagonally implicit improved Runge–Kutta Nystrom technique. The following is the definition of the global truncated error for the $5th$ order DIIRKN technique:

$$\left\| \tau_g^{(5)} \right\|_2 = \left(\sum_{i=1}^{n_p+1} \left(\tau_i^{(5)} \right)^2 + \sum_{i=1}^{n_p'+1} \left(\tau_i'^{(5)} \right)^2 \right)^{\frac{1}{2}} \tag{9}$$

where $\tau_i^{(5)}$ and $\tau_i'^{(5)}$ are the truncation error norms for q and q' .

By combining the row conditions listed in (8) with the equations created by order condition upward to fourth orders, A fourth order DIIRKN technique can be obtained. This affords an eleven-equation system with thirteen unknowns. By fixing $c_1 = \frac{1}{10}$, and pick a_{32} as free parameter, then after solving we get,

$$c_2 = \frac{67}{64} - \frac{1}{64}\sqrt{3017}, c_3 = \frac{67}{64} + \frac{1}{64}\sqrt{3017}, a_{11} = a_{22} = a_{33} = \frac{1}{200},$$

$$a_{31} = \frac{93313}{102400} + \frac{67}{4096}\sqrt{3017} - a_{32},$$

$$d_2 = \frac{5}{6} + \frac{101}{6034}\sqrt{3017}, d_3 = \frac{5}{6} - \frac{101}{6034}\sqrt{3017},$$

$$b_2 = \frac{3}{4} + \frac{443}{36204}\sqrt{3017}, b_3 = \frac{3}{4} - \frac{443}{36204}\sqrt{3017}, a_{21} = \frac{93313}{102400} - \frac{67}{4096}\sqrt{3017},$$

Using Maple software to minimize the global error equations in (9) with regard to the free parameter a_{32}, as per Dormand, El-Mikkawy, and Prince [13], we get $a_{32} = -\frac{68682}{9431}$, which produces $\left\| \tau_g^5 \right\|_2 = 9.06 \times 10^{-1}$. Thus, all of the coefficients of the DIIRKN technique can be represented as Table 1.

Table 1
New DIIRKN method

$\frac{1}{10}$	$\frac{1}{200}$
$\frac{67}{64}-\frac{1}{64}\sqrt{3017}$	$\frac{63313}{102400}-\frac{67}{4096}\sqrt{3017}\ \frac{1}{200}$
$\frac{67}{64}+\frac{1}{64}\sqrt{3017}$	$\frac{63313}{102400}+\frac{67}{4096}\sqrt{3017}\ -\frac{68682}{9431}\ \frac{1}{200}$
-1	$0\ \frac{3}{4}+\frac{443}{36204}\sqrt{3017}\ \frac{3}{4}-\frac{443}{36204}\sqrt{3017}$
	$\frac{5}{6}+\frac{101}{6034}\sqrt{3017}\ \frac{5}{6}-\frac{101}{6034}\sqrt{3017}$

4. Test Problems

Several RKN common techniques for the numerical comparisons have been taken into account to evaluate the efficacy of the suggested approach:

- **DIIRKN:** The constructed diagonally implicit improved Runge–Kutta Nystrom method in this paper [11].
- **DIRKNS**: Fourth orders RKN methods given in [7].
- **DIRKNI**: RKN approach of order four proposed in [9].
- **DIRKNSF**: RKN technique of order four with dispersive order six given by [10].
- **MAXERR:** Maximum Absolute Errors;

Problem 1: [14]

$$q''(t) = 25q, q(0) = 0, q'(0) = 5. t \in [0,5].$$

analytic solutions is $q(t) = \sin(5t)$.

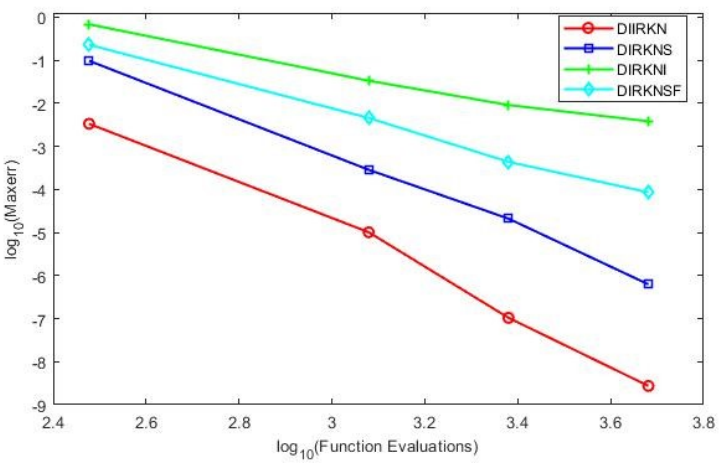


Figure 1
Problem 1 competency graph

Table 2

Problem 1 outcomes

h	Methods	MAXERR
0.1	DIIRKN	$3.3554e^{-3}$
	DIRKNS	$9.7432e^{-2}$
	DIRKNI	$6.8633e^{-1}$
	DIRKNSF	$2.2944e^{-1}$
0.025	DIIRKN	$1.0123e^{-5}$
	DIRKNS	$2.8589e^{-4}$
	DIRKNI	$3.3501e^{-2}$
	DIRKNSF	$4.5717e^{-3}$
0.0125	DIIRKN	$1.0422e^{-7}$
	DIRKNS	$2.0955e^{-5}$
	DIRKNI	$9.1442e^{-3}$
	DIRKNSF	$4.3704e^{-4}$
0.00625	DIIRKN	$2.7303e^{-9}$
	DIRKNS	$6.2429e^{-7}$
	DIRKNI	$3.8301e^{-3}$
	DIRKNSF	$8.5760e^{-5}$

Problem 2: [15]

$$q_1''(t) = -q_1 + \frac{1}{1000}\cos(t), q_1(0) = 1, q_2'(0) = 0,$$

$$q_2''(t) = -q_2 + \frac{1}{1000}\sin(t), q_1(0) = 0, q_2'(0) = \frac{9995}{1000},\ t \in [0,5].$$

analytic solutions be $\cos(t) + \frac{1}{2000}\, t\sin(t) = q_1(t)$ and we have

$$q_2(t) = \sin(t) - \frac{1}{2000}\, t\cos(t)\,.$$

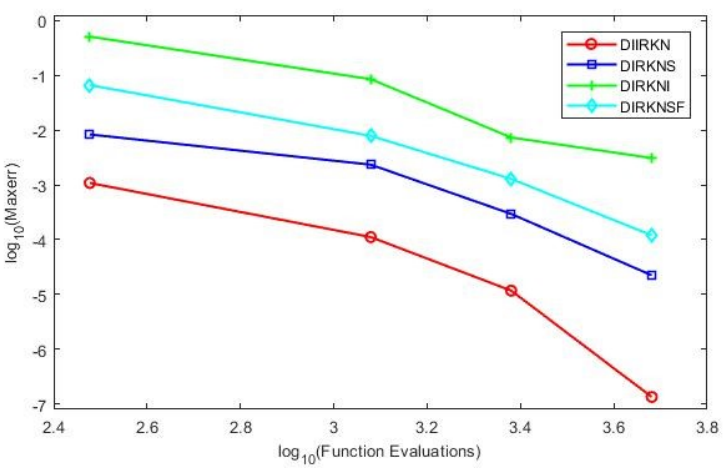


Figure 2

Problem 2 competency graph

Table 3

Problem 2 outcomes

h	Methods	MAXERR
0.1	DIIRKN	$1.0842e^{-3}$
	DIRKNS	$8.3393e^{-3}$
	DIRKNI	$5.1102e^{-1}$
	DIRKNSF	$6.6471e^{-2}$
0.025	DIIRKN	$2.1136e^{-4}$
	DIRKNS	$2.3401e^{-3}$
	DIRKNI	$8.5228e^{-2}$
	DIRKNSF	$7.8616e^{-3}$
0.0125	DIIRKN	$1.1693e^{-5}$
	DIRKNS	$2.9457e^{-4}$
	DIRKNI	$7.3437e^{-3}$
	DIRKNSF	$1.2854e^{-3}$
0.00625	DIIRKN	$1.3370e^{-7}$
	DIRKNS	$2.2287e^{-5}$
	DIRKNI	$3.0943e^{-3}$
	DIRKNSF	$1.1973e^{-4}$

Problem 3: [13]

$$q_1(0) = 1, \text{ and } q_1''(t) = -\frac{q_1}{(q_1^2+q_2^2)^{\frac{3}{2}}}, q_1'(0) = 0,$$

$$q_2(0) = 0, \text{ and } q_2''(t) = -\frac{q_2}{(q_1^2+q_2^2)^{\frac{3}{2}}}, q_2'(0) = 1, t \in [0,5].$$

analytic solutions be the $\cos(t) = q_1(t)$ and we get the

$$q_2(t) = \sin(t).$$

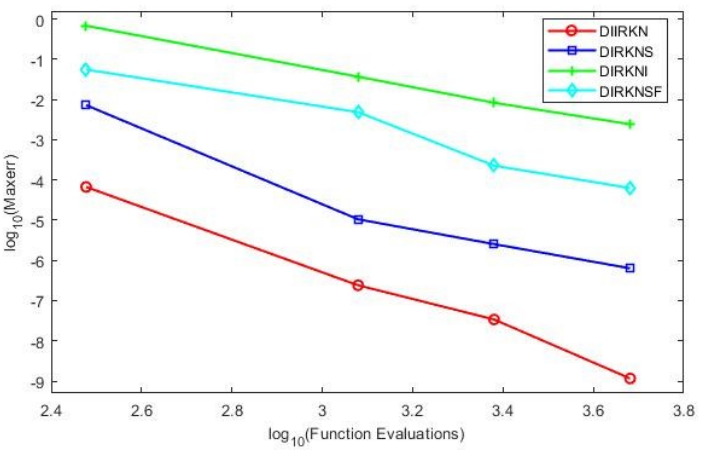


Figure 3

Problem 3 competency graph

Table 4

Problem 3 outcomes

h	Methods	MAXERR
0.1	DIIRKN	$6.7208e^{-5}$
	DIRKNS	$7.2543e^{-3}$
	DIRKNI	$6.8513e^{-1}$
	DIRKNSF	$5.5670e^{-2}$
0.025	DIIRKN	$2.4106e^{-7}$
	DIRKNS	$1.0522e^{-5}$
	DIRKNI	$3.6975e^{-2}$
	DIRKNSF	$4.8502e^{-3}$
0.0125	DIIRKN	$3.4003e^{-8}$
	DIRKNS	$2.5453e^{-6}$
	DIRKNI	$8.3110e^{-3}$
	DIRKNSF	$2.2744e^{-4}$
0.00625	DIIRKN	$1.1620e^{-9}$
	DIRKNS	$6.3614e^{-7}$
	DIRKNI	$2.4294e^{-3}$
	DIRKNSF	$6.3181e^{-5}$

5. Discussion and Conclusions

We have developed a diagonally implicit IRKN method in this research, which we have named the DIIRKN method. The maximum absolute errors for each method and step size are shows in Table 2-4. Figure 1–3 show competence graph for problems, with respect to the function evaluations each technique requires and the maximum absolute errors. The newly derived approach, DIIRKN, is more accurate and efficient at addressing all the given problems, as evidenced by the lowest error norm, as demonstrated in Tables 2-4 and Figures 1-3. The DIIRKN surpassed the well-known DIRKN method and be hence computationally more effective of solve second - orders ODEs.

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