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Cohomology theory for fuzzy bornological groups

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Abstract

This work satisfies cohomology theory within the framework of fuzzy bornological groups. The primary objective is to establish a robust formalization of cohomology for these fuzzy groups and to extend the fundamental properties of classical cohomology to the fuzzy group setting. To achieve this, we construct a cohomology theory for a fuzzy bornological group $(\tilde{G},\tilde{\delta})$ and a fuzzy bornological $\tilde{G}$-modules $(\tilde{M},\tilde{\delta})$, by utilizing the concept of fuzzy bounded cochains.

Subject Classification: *55N35, 54A40.*

Keywords: *Bornological set, Bornological group, Fuzzy bornological group, Action of fuzzy bornological group, Cohomology of group.*

1. Introduction

In this paper, we examine the cohomology of fuzzy bornological groups through the lens of fuzzy bounded cochains. Building upon the foundational work on fuzzy bornological sets and fuzzy bornological groups, reported in [1], and for cohomology of fuzzy groups, see [2]. We define a fuzzy bornology $\tilde{\delta}$ on a fuzzy set $\tilde{X}$ as a collection of fuzzy subsets $\tilde{\delta}$ that covers $\tilde{X}$ and maintains stability under hereditary subsets and finite unions. Furthermore, we characterize a fuzzy bounded map and isomorphism, where a bijective fuzzy bounded function is deemed a fuzzy bornological isomorphism if its inverse remains fuzzy bounded. The fuzzy group bornology is groups & fuzzy

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bornology s.t products & inverse maps is fuzzy bound. See [3], [4], [5], [6], [7]. Our primary objective be cohomology of fuzzy groups bornology & to explore algebraic properties within the context of action theory. Specifically, we consider the scenario where a fuzzy bornological group $(\tilde{G}, \tilde{\delta})$ acts on an abelian fuzzy bornological group $(\tilde{N}, \tilde{\delta})$. We say $(\tilde{N}, \tilde{\delta})$ is a fuzzy bornological $\tilde{G}$-module if $(\tilde{G}, \tilde{\delta}) * (\tilde{N}, \tilde{\delta}) \longrightarrow (\tilde{N}, \tilde{\delta})$, be fuzzy bounded, i,e. if $(\tilde{G}, \tilde{\delta}) * (\tilde{N}, \tilde{\delta}) \longrightarrow (\tilde{N}, \tilde{\delta})$ is fuzzy bounded for the product bornology. For more study in fuzzy see [8].

2. Cohomology Theory for Fuzzy Bornological Group

Cohomology theory for fuzzy group bornology are considered. First of all, let remember action theory for fuzzy bornological group, where fuzzy groups bornology $(\tilde{G}, \tilde{\delta})$ acts over fuzzy sets bornology $(\tilde{X}, \tilde{\delta}_X)$ such that action map is defined as $\lambda: (\tilde{G}, \tilde{\delta}) \times (\tilde{X}, \tilde{\delta}_X) \longrightarrow (\tilde{X}, \tilde{\delta}_X)$. Is fuzzy bounded map and $(\tilde{X}, \tilde{\delta}_X)$ is $\tilde{G}$-fuzzy bornological set. If $(\tilde{M}, \tilde{\delta})$ be abelian fuzzy bornological group, and $(\tilde{G}, \tilde{\delta})$ acts on $(\tilde{M}, \tilde{\delta})$, thus $(\tilde{M}, \tilde{\delta})$ is a fuzzy bornological G-module. Thus, product function $(\tilde{G}, \tilde{\delta}) \times (\tilde{G}, \tilde{\delta}) \longrightarrow (\tilde{G}, \tilde{\delta})$, and module map $(\tilde{G}, \tilde{\delta}) \times (\tilde{M}, \tilde{\delta}) \longrightarrow (\tilde{M}, \tilde{\delta})$ fuzzy bounded for the product bornology. Assume that $\tilde{G}, \tilde{M}, \tilde{X}$ be instead for the $(\tilde{G}, \tilde{\delta}), (\tilde{M}, \tilde{\delta}), (\tilde{X}, \tilde{\delta})$. If $C^n_{fbd}(\tilde{G}, \tilde{M}) = \{\pi: \tilde{G}^n \longrightarrow \tilde{M}: \pi \text{ is fuzzy bounded } \}$ and the binary operation $\oplus$ can be a group. Also, this fuzzy group $C^n_{fbd}(\tilde{G}, \tilde{M})$ of fuzzy bounded cochain containes fuzzy bounded function $\pi: \widetilde{G^n} \longrightarrow \tilde{M}$. These types of groups can connected by homomorphism depends on n degree and this coboundary map is defined asfollows: $d_{n+1}\pi(\gamma_1, \ldots, \gamma_{n+1}) = \gamma_1 \cdot \pi(\gamma_2, \ldots, \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i \pi(\gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_{n+1}) + (-1)^{n+1}\pi(\gamma_1, \ldots, \gamma_n)$. Let consider the special case when $n = 0$

$$d_1\, \pi(\gamma_1) = \gamma_1\, \pi(\gamma_1) + (-1)\, \pi(\gamma_1) = \gamma_1 \cdot m - m \in C^1_{bd}(G, M)$$

$$n = 1, i = 1,2, \ldots, n+1 = 2 \text{ (Two variables)}$$

$$\begin{aligned} d_2\pi(\gamma_1, \gamma_2) &= \gamma_1\, \pi(\gamma_2) + (-1)^1\pi(\gamma_1, \gamma_2) + (-1)^2 f(\gamma_1) \\ &= \gamma_1\, \pi(\gamma_2) - \pi(\gamma_1, \gamma_2) + \pi(\gamma_1) \end{aligned}$$

$$n = 2, i = 1, \ldots n+1 = 3 = 1,2,3$$

$$\begin{aligned} d_3\pi(\gamma_1, \gamma_2, \gamma_3) = {} & \gamma_1\, \pi(\gamma_2, \gamma_3) + (-1)^1\pi(\gamma_1, \gamma_2, \gamma_3) + \\ & (-1)^2\pi(\gamma_1, \gamma_2, \gamma_3) + (-1)^3\pi(\gamma_1, \gamma_2). \end{aligned}$$

Thus, the complex fuzzy bounded cochains is the fuzzy group.

Thus, fuzzy group of 0-cochain $C^0_{fbd}(\tilde{G}, \tilde{M})$ will be $\tilde{M}$.

Definition 2.1: Cohomology for fuzzy group bornology $H^n_{fbd}(\tilde{G}, \tilde{M})$ it is cohomology for the fuzzy bounded cochain complex $\{C^n_{fbd}(\tilde{G}, \tilde{M}), d_n\}_{n\geq 0}$ given as:

i. $C^0_{fbd}(\tilde{G}, \tilde{M}) = \tilde{M}$.

ii. For $n \geq 0, C^n_{fbd}(\tilde{G}, \tilde{M})$ which it is fuzzy group with elements as fuzzy bounded map.

Hear, cohomology theory based on fuzzy bounded cochains presented. Previously, for every n, the cocyclic is

$\mathrm{Z}^{\mathrm{n}}_{\mathrm{fbd}}(\widetilde{\mathrm{G}}, \widetilde{\mathrm{M}}) = \{0 = \mathrm{kerd}_{\mathrm{n+1}}\}$ and the n-coboundaries is $\mathrm{B}^{\mathrm{n}}_{\mathrm{fbd}}(\widetilde{\mathrm{G}}, \widetilde{\mathrm{M}}) = \mathrm{Im}(\mathrm{d}_{\mathrm{n}})$, for $\mathrm{n} \geq 1$, such that $\mathrm{B}^{0}_{\mathrm{fbd}}(\widetilde{\mathrm{G}}, \widetilde{\mathrm{M}}) = 0$. The cohomology for fuzzy bornological groups $\boldsymbol{H^{n}_{fbd}(\widetilde{G}, \widetilde{M})}$ ***is*** $\frac{\boldsymbol{Z^{n}_{fbd}(\widetilde{G},\widetilde{M})}}{\boldsymbol{B^{n}_{fbd}(\widetilde{G},\widetilde{M})}}$. Thus

$\mathrm{H}^{0}_{\mathrm{fbd}}(\widetilde{\mathrm{G}}, \widetilde{\mathrm{M}}) = \widetilde{\mathrm{M}}^{\widetilde{\mathrm{G}}}$, An space of $\widetilde{\mathrm{G}}$-invariant in $\widetilde{\mathrm{M}}$. Furthermore, first cohomology fuzzy group is $\boldsymbol{H^{1}_{fbd}(\widetilde{G}, \widetilde{M}) = \{f: \widetilde{G} \longrightarrow \widetilde{M}: f(g_1, g_2) = g_1 f(g_2) + f(g_1)\}}$. Suppose $\widetilde{\mathrm{M}}', \widetilde{\mathrm{M}}, \widetilde{\mathrm{M}}''$ are fuzzy bornological $\widetilde{\mathrm{G}}$-modules that from a fuzzy groups bornology:$0 \longrightarrow \widetilde{\mathrm{M}}' \xrightarrow{\mathrm{i}} \widetilde{\mathrm{M}} \xrightarrow{\mathrm{j}} \widetilde{\mathrm{M}}'' \longrightarrow 0$.

Example 2.2: If a fuzzy group bornology $(\widetilde{\mathrm{G}}, \widetilde{\delta})$ that acts trivially on an abelian discreet fuzzy bornological group $(\widetilde{\mathrm{M}}, \widetilde{\delta})$, under this condition, $\widetilde{\mathrm{M}}$ is characterized as a fuzzy bornollogical $\widetilde{\mathrm{G}}$-module. Given the trivial action, we define the fuzzy group as $\tilde{G} = \{e\}$ where $\{e\}$ is the identity element. Consequently, the n-fold Cartesian product of the fuzzy group is $\tilde{G}^{n} = \{(e, \dots, e)\}$. For any function from G^{n} to M such that f beloing to the space of fuzzy bounded n-cochains, denoted by $\mathrm{C}^{\mathrm{n}}_{\mathrm{fbd}}(\widetilde{\mathrm{G}}, \widetilde{\mathrm{M}})$.

Definition 2.3: If $C_1 = \{C^{n}_{1fbd}\}$ & $C_2 = \{C^{n}_{2fbd}\}$ are fuzzy bounded cochain. the homomorphism of $\alpha: C_1 \longrightarrow C_2$ be the class for homomorphisms $\alpha_n: C^{n}_{1fbd} \longrightarrow C^{n}_{2fbd}$ and the next figure commutes, see Figure 1.

$$\begin{tikzcd}
C^{n-1}_{1bd} \arrow[r] \arrow[d, "\alpha_{n-1}"] & C^{n}_{1bd} \arrow[r] \arrow[d, "\alpha_{n}"] & C^{n+1}_{1bd} \arrow[r] \arrow[d, "\alpha_{n+1}"] & {} \\
C^{n-1}_{2bd} \arrow[r] & C^{n}_{2bd} \arrow[r] & C^{n+1}_{2bd} \arrow[r] & {}
\end{tikzcd}$$

Figure 1
Collection of homomorphisms $\boldsymbol{\alpha_n: C^{n}_{1fbd} \longrightarrow C^{n}_{2fbd}}$

Proposition 2.4: A homomorphism $\alpha: C_1 \longrightarrow C_2$ of fuzzy bounded cochain induced group homomorphism from $H^{n}_{fbd}(C_1)$ to $H^{n}_{fbd}(C_2)$ for $n \geq 0$.

Proof: The assertion follows directly from the commutativity of the aforementioned diagram. Specifically, the commutativity ensures that the image and kernel of each morphism in first rows is canonically mapped to correspond image & kernel of morphisms in the second row. This preserves the structural properties across the levels of the function.

Definition 2.5: Let $C_1 = \{C^{n}_{f1bd}\}, C_2 = \{C^{n}_{f2bd}\}$ and $C_3 = \{C^{n}_{f3bd}\}$ be families of fuzzy bounded cochain. A sequence of fuzzy bounded cochain homomorphisms is defined as a sequence of fuzzy bounded cochain homomorphisms if, for every n, the sequence of conditions of exactness. (i.e., the image of each homorphism equals to the kernel of the succeeding one). So, $0 \longrightarrow C^n_{\mathrm{f}1bd} \longrightarrow C^n_{\mathrm{f}2bd} \longrightarrow C^n_{\mathrm{f}3bd} \longrightarrow 0$, is short exact.

We shall now formalize the mechanism for the change of groups within the category of fuzzy bornological modules associated with the fuzzy group bornologies. Consider $\widetilde{M}$ and $\widetilde{M}'$ as fuzzy bornological $\tilde{G}$ -module associated with the fuzzy group bornologies $\tilde{G}, and\ \tilde{G}'$ respectively. Suppose there exists a pair of fuzzy bounded homomorphisms $\begin{array}{c}\theta: \tilde{G}' \longrightarrow \tilde{G} \\ \varphi: \widetilde{M} \longrightarrow \widetilde{M}'\end{array}$ satisfying the following equivariance condition $g'\varphi(m) = \varphi(\theta(g') \cdot m)\ \forall m\ in\ \widetilde{M}, \&\ g'$ in $\tilde{G}'$.

Figure 2: commutative diagram representing the restriction homomorphism and the structural compatibility between fuzzy group bornology actions.

$$\begin{tikzcd} G \times M \arrow[r] \arrow[d] & M \arrow[d] \\ G' \times M' \arrow[r] & M' \end{tikzcd}$$

Figure 2
Restriction homomorphism

In particular, we have functorial functions for $\tilde{G} = \tilde{G}'$, $H^n_{bd}(\tilde{G}, \widetilde{M}) \longrightarrow H^n_{bd}(\tilde{G}, \widetilde{M}')$. For $\tilde{G}' = \widetilde{H}$, a sub fuzzy group of $\tilde{G}$.

Conclusion

In conclusion, this work established a cohomology theory for fuzzy bornological groups and fuzzy bornological modules by using fuzzy bounded cochains. The study successfully extended several fundamental concepts and properties of classical cohomology to the fuzzy setting, providing a new framework for further investigations in fuzzy algebraic structures and their applications.

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