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Codisk–recurrent operators on Hilbert spaces

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Abstract

This work introduces the novel concept, "codisk–recurrence," that provides a precise structural description of recurrence behavior in infinite-dimensional Hilbert spaces in the context of topological dynamics. Many significant attributes of this class of operators have been investigated and analyzed in the second aim. Furthermore, a characterization for codisk recurrence is provided. In particular, codisk–recurrence itself is formulated in terms of the density of the set of their codisk–recurrence vectors.

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Keywords: *Hypercyclity, Codisk-cyclicity, Recurrence, Hypercyclic vector, Codisk–cyclic vector.*

1. Introduction

The study of linear operator dynamics on infinite dimensional spaces began to develop systematically in the 1990s and has continued to grow in recent decades (see [1], [2]). The central idea of this theory is hypercyclicity. A bounded linear operator $\mathrm{T}: H \rightarrow H$ be the hyper cyclic where all points in the dense subsets for the set H have the dense orbit. So for T be the hypercyclic on H & $h \in H$ is s.t $Orb(\mathrm{T}, h)$ be the dense in H, then any neighborhoods Û for the h, the subsequence for orbit be included in Û, indicating h be recurrent of T. More information about hypercyclic (see [3], [4], [5]). According to Rolewicz [6], it is hypercyclic for the absolute value of λ is backward shift Ɓ on ℓ to the p, open paren double-struck cap N close parent if only if absolute value λ λƁ is hypercyclic for a the absolute value of lambda is backward shift Ɓ on $\ell^p(\mathbb{N})$ if only if $|\lambda| > 1$. Therefore, λƁ is not hypercyclic when $|\lambda| \leq 1$. This prompted researchers to investigate disk orbit and codisk orbit concepts. Disc-cyclic and

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codisk-cyclic operators are examined in [7]. $\mathrm{T} \in \mathcal{B}(H)$ is referred to as codisk-cyclic provided that there exists h in H whoes orbit satisfies $\overline{\mathbb{D}^{\mathsf{C}} Orb(\mathrm{T}, h)} = \overline{\{\lambda \mathrm{T}^n h : \mathbb{B}^{\mathsf{C}}, n \geq 1\}} = H$ and h is referred to a codisk-cyclic vector for T. These concepts were also studied in [8], [9], [10], [11].

The study of recurrent linear operators began to gain attention following the early contributions in the literature [12]. The operators T in $\mathcal{B}(H)$ named recurring where any open subsets $\hat{\mathrm{U}}$ for the H, $\exists n \geq 1$ s.t $\mathrm{T}^n(\hat{\mathrm{U}}) \cap \hat{\mathrm{U}} \neq \emptyset$, while vectors $h \in H$ be referred to as recurrent of the T if there is a strictly increases $\{n_{\Bbbk}\} \subset \mathbb{N}$ such that$\mathrm{T}^{n_{\Bbbk}} h$ converges to h as $\Bbbk$ approaches infinity. $Rec(\mathrm{T})$ denote a sets for recurrent vector of T. Multiple recurrence was first studied using the linear dynamics framework by Costakis and Parissis [13]. For further details, see [14], [15], [16], [17-19].

The notion of codisk–recurrence is introduced in this work as a new framework that combines two different classes of operators: codisk-cyclic operators and recurrent operators. To clarify the distinction between these classes, we construct a counterexample demonstrating that while every codisk operator is recurrent, the opposite is not always true. In addition, we provide preliminary results and establish necessary conditions for codisk recurrence. In Section 3, we present a rigorous study of codisk-recurrent behavior through a series of characterizations. One result describes the transition from pointwise recurrence to operator-level recurrence within a precise topological-dynamical framework. Another provides necessary and sufficient criteria for the operator of codisk–recurrent. The conclusions from the findings presented throughout this study are summarized in Section 4.

Throughout this work, we adopt the following notations: For the complex field $\mathbb{C}$, we consider H to be an infinite dimensional Hilbert space over $\mathbb{C}$. Consider T, a linear continuous operator on H ; that is $\mathrm{T} \in \mathcal{B}(H)$. The symbols $\mathbb{B} \coloneqq \{\lambda \in \mathbb{C}: |\lambda| < 1\}$ and $\mathbb{B}^{\mathsf{C}} \coloneqq \{\lambda \in \mathbb{C}: |\lambda| \geq 1\}$ represent the unit disk and codisk, respectively. $\mathbb{N}$ stands for the set of positive integers that do not include zero.

2. Codisk–recurrent Operator and its Properties

In the context of topological dynamics, we introduce and investigate the concept of codisk–recurrent operator in this section.

Definition 2.1: The operators T in $\mathcal{B}(H)$ be the codisk–recurrent where all open subsets $\hat{\mathrm{U}}$ for the H which is $\hat{\mathrm{U}} \neq \emptyset$, $\exists n \geq 1$ & $\lambda \in \mathbb{B}^{\mathsf{C}}$ s.t

$$\hat{\mathrm{U}} \cap \lambda \mathrm{T}^n(\hat{\mathrm{U}}) \neq \emptyset.$$

We say that vectors h in H is a codisk–recurrent of T provided there exist strictly increases sequences $\{n_{\Bbbk}\}_{\Bbbk \in \mathbb{N}} \subset \mathbb{N}$ &the sequences $\{\lambda_{\Bbbk}\}_{\Bbbk \in \mathbb{N}} \subset \mathbb{B}^{\mathsf{C}}$ such that $\lambda_{\Bbbk} \mathrm{T}^{n_{\Bbbk}} h \rightarrow h$ as $\Bbbk \rightarrow \infty$. A sets for each codisk–recurrent vector of T, is presented as$\mathbb{D}^{\mathsf{C}} Rec(\mathrm{T})$.

If $\mathrm{T} \in \mathcal{B}(H)$ is codisk–cyclic, then it must be codisk–recurrent. However, the example stated below illustrates that the reverse is not possible.

Example 2.2: Suppose that $R \geq 1$ and $\lambda_i \in \mathbb{B}^{C}$; $i = 1, \ldots, m$. Let $\mathrm{T} \in \mathcal{B}(H)$. Define a weighted operator with weight $\{\lambda_m : |\lambda_i| = R, i = 1, \ldots, m\}$ by

$$\mathrm{T}: \mathbb{C}^m \to \mathbb{C}^m$$

$$\mathrm{T}(h_1, \ldots, h_m) = (\lambda_1 h_1, \ldots, \lambda_n h_m);$$

According to [7], T is not codisk–cyclic when $n \geq 2$. We demonstrate that T is codisk–recurrent, as shown below.

For any open set $\hat{\mathrm{U}}$ in H which is $\hat{\mathrm{U}} \neq \emptyset$ with $h \in \hat{\mathrm{U}}$, and given that $|\lambda_i| = R$ for all $1 \leq i \leq m$, we have $|(R^{-1}\lambda_i)^n| = 1$ for every n in $\mathbb{N}$, so there is strictly increase sequences $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ with every $1 \leq i \leq n$ and as $k \to \infty$, $\left|R^{-n_k}\lambda_i^{n_k} - 1\right| \to 0$. Thus, $\| \lambda_k \mathrm{T}^{n_k} h - h \| = \sum_{i=1}^{n} \left|R^{-n_k} \lambda_i^{n_k} h_i - h_i\right| \to 0.$

For a k_1 large enough, $\lambda_{k_1} \mathrm{T}^{n_{k_1}} h\, \hat{\mathrm{U}} \cap \hat{\mathrm{U}} \neq \emptyset$. Therefore,T is codisk–recurrent.

The following proposition establishes that codisk–recurrent may not be codisk–cyclicity even in the infinite dimensional Hilbert spaces.

Proposition 2.3: If H the infinite dimension Hilbert spaces. Then, the restriction of an operator T in $\mathcal{B}(H)$ to a nonempty set of its periodic points is codisk–recurrent but not codisk–cyclic.

Proof: Let $\mathrm{T} \in \mathcal{B}(H)$ has at least periodic point. Define $f \coloneqq \mathrm{T}|_E : E \to E$, where E represents the set of all periodic points of T. Fix $e \in E$ arbitrarily, then there is $p \in \mathbb{N}$ with $f^p(e) = e$. Take $n_k = kp$, for all $k \in \mathbb{N}$, that is $\lim_{k \to \infty} f^{n_k}(e) = e$. Suppose that $\hat{\mathrm{U}} \subseteq H$ be an open neighborhood of e. Let $\lambda_k = 1 + \epsilon_k$, where $\epsilon_k \to 0$, thus $\lambda_k \to 1$. Hence $\lambda_k f^{n_k} e \to e$ as $k \to \infty$. Then there exists k_1 large enough with $\lambda_{k_1} f^{n_{k_1}} e \in \hat{\mathrm{U}}$. Therefore, $\lambda_{k_1} f^{n_{k_1}} \hat{\mathrm{U}} \cap \hat{\mathrm{U}} \neq \emptyset$. Therefore, f is codisk–recurrent. Clearly, f is fails to be codisk–cyclic.

Now, our focus is on examining some fundamental features of codisk–recurrent operators.

Proposition 2.4: (Codisk Recurrence Comparison Principle)**:**

Let $\mathrm{T} \in \mathcal{B}(H_1)$ and $S \in \mathcal{B}(H_2)$, where H_1 and H_2 are Hilbert spaces. Suppose that a linear map $\theta: H_1 \to H_2$ with every nonempty open set $\hat{\mathrm{U}}$, $\theta^{-1}(\hat{\mathrm{U}})$ is a nonempty open set and $\theta \mathrm{T} = S\theta$. If T is codisk–recurrent in H_1, then S is codisk–recurrent in H_2.

Proof: Consider an open set $\hat{\mathrm{U}}$ in H_2 which is $\hat{\mathrm{U}} \neq \emptyset$, there is $n \geq 1$, and $\lambda \in \mathbb{B}^{C}$ with $\lambda \mathrm{T}^n \theta^{-1}(\hat{\mathrm{U}}) \cap \theta^{-1}(\hat{\mathrm{U}}) \neq \emptyset$, thus, there is $h \in \theta^{-1}(\hat{\mathrm{U}})$ and $\lambda \mathrm{T}^n h \in \theta^{-1}(\hat{\mathrm{U}})$, hence $\theta(h) \in \hat{\mathrm{U}}$ and $\theta(\lambda \mathrm{T}^n h) \in \hat{\mathrm{U}}$.

But $\theta \mathfrak{T} = S\theta$, then, $\theta(h) \in \hat{\mathrm{U}}$ and $\lambda S^n \theta(h) \in \hat{\mathrm{U}}$, therefore $\lambda S^n \hat{\mathrm{U}} \cap \hat{\mathrm{U}} \neq \emptyset$.

One of an application of the Codisk Recurrence Comparison principle is:

Corollary 2.6: *Let* $\mathrm{T} \in \mathcal{B}(H)$, *where* H *is a Hilbert space. Let* $\widehat{H} \subseteq H$ *be a subspace equipped with the relative topology. for* $\mathrm{T}|_{\widehat{H}}$ *the well-defined codisk–recurrent operators on* $\widehat{H}$, *so is* T *on* H.

3. Characterization of Codisk-recurrent Operators

We devote this section to establishing a characterization of codisk–recurrent operators.

Theorem 3.1: *Let* $\mathrm{T} \in \mathcal{B}(H)$. *The assertions that follow are identical:*

1- T *is codisk – recurrent;*

2- *For every* $h \in H$ *There are* $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{B}^{\mathrm{C}}$ *and* $\{h_k\}_{k\in\mathbb{N}}$ *of* H. *with* $h_k \to h$ *and* $\lambda_k \mathrm{T}^{n_k}(h_k) \to h$ *as* $k \to \infty$*;*

3- *For every* $h \in H$ *and every neighborhood* M *of zero, there are* $l \in H, \alpha \in \mathbb{B}^{\mathrm{C}}$ *and* $m \in N$ *with* $l - h \in M$ *and* $\lambda \mathrm{T}^m l - h \in M$.

Proof: (1) ⇒ (2)*:* Let h ∈ H. For each $k \geq 1$, define an open neighborhood of h $\hat{\mathrm{U}}_k = \mathbb{B}\left(\mathrm{h}, \frac{1}{k}\right)$. Given that T is codisk–recurrent, there is n_k in $\mathbb{N}$ and $\lambda_k \in \mathbb{B}^{\mathrm{C}}$ with $\lambda_k \mathrm{T}^{\mathrm{n}_k}(\hat{\mathrm{U}}_k) \cap \hat{\mathrm{U}}_k \neq \emptyset$. Hence, for all $k \geq 1$ there are $\mathrm{h}_k \in \hat{\mathrm{U}}_k$ with $\|\mathrm{h}_k - \mathrm{h}\| < \frac{1}{k}$ and $\|\lambda_k \mathrm{T}^{\mathrm{n}_k}(\mathrm{h}_k) - \mathrm{h}\| < \frac{1}{k}$. As $k \to \infty, \mathrm{h}_k \to \mathrm{h}$ and $\lambda_k \mathrm{T}^{\mathrm{n}_k} \mathrm{h}_k \to \mathrm{h}$

(2) ⇒ (3): By (2), there are $\{\mathrm{n}_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, \{\mathrm{h}_k\}_{k\in\mathbb{N}} \subset \mathrm{H}$ and $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{B}^{\mathrm{C}}$ for all $\epsilon > 0$ $\exists \mathrm{N} \in \mathbb{N}$ with $\|\mathrm{h}_k - \mathrm{h}\| < \epsilon$ and $\|\lambda_k \mathrm{T}^{\mathrm{n}_k} \mathrm{h}_k - \mathrm{h}\| < \epsilon$ for all $k \geq \mathrm{N}$. Hence for large enough k, $\mathrm{h}_k - \mathrm{h} \in \mathrm{M}$ and $\lambda_k \mathrm{T}^{\mathrm{n}_k} \mathrm{h}_k - \mathrm{h} \in \mathrm{M}$.

(3) ⇒ (1): Put, for any $k \geq 1$, $M_k = \mathbb{B}\left(0, \frac{1}{k}\right)$. Thus by (3), there are $l_k \in H, n_k \in \mathbb{N}$ and $\lambda_k \in \mathbb{B}^{\mathrm{C}}$ such that $\|\lambda_k \mathrm{T}^{n_k} l_k - h\| < \frac{1}{k}$ and $\|h - l_k\| < \frac{1}{k}$. Let $k \to \infty$, then $l_k \to h$ and $\lambda_k \mathrm{T}^{n_k} l_k \to h$. Therefore, for some large enough k, we have $\lambda_k \mathrm{T}^{n_k} \hat{\mathrm{U}} \cap \hat{\mathrm{U}} \neq \emptyset$.

The density for a set for the codisk–recurrent vector of the operators Tgives rise to followed feature.

Theorem 3.2: *Let* $\mathrm{T} \in \mathcal{B}(H)$. T *is a codisk–recurrent if and only if the set of codisk–recurrent vectors is dense.*

Proof: ⇒) Let $\mathcal{B} := \mathbb{B}(h, \epsilon)$ for a given h in H and $0 < \epsilon < 1$, there is $n_1 > 0$ and $\lambda_1 \in \mathbb{B}^{\mathrm{C}}$ with $h_1 \in \frac{1}{\lambda_1} \mathrm{T}^{-n_1} \mathcal{B} \cap \mathcal{B}$. Moreover, a continuity for the T ensure existences for $\epsilon_1 < \frac{1}{2}$ with $\mathbb{B}(h_1, \epsilon_1) \subset \frac{1}{\lambda_1} \mathrm{T}^{-n_1} \mathcal{B} \cap \mathcal{B}$. Proceeding inductively, we create sequences $\{h_k\}_{k\in\mathbb{N}}$ in H, $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{B}^{\mathrm{C}}$, $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and $\{\epsilon_k\} \subset \mathbb{R}^+$, where $\epsilon_k < \frac{1}{2^k}$, with $\mathbb{B}(h_{k+1}, \epsilon_{k+1}) \subset \mathbb{B}(h_k, \epsilon_k)$ and $\lambda_k \mathrm{T}^{n_k} \mathbb{B}(h_{k+1}, \epsilon_{k+1}) \subset \mathbb{B}(h_k, \epsilon_k)$. It follows from Cantor's theorem that there is $l \in H$ with $\cap_{k\in\mathbb{N}} \overline{\mathbb{B}(h_k, \epsilon_k)} = \{l\}$ which is in $\mathbb{D}^{\mathrm{C}} Rec(\mathrm{T})$, in fact, since for all k, $l \in \mathbb{B}(h_k, \epsilon_k)$, i.e. $\|h_k - l\| < \frac{1}{2^k}$. Also, $\lambda_k \mathrm{T}^{n_k} l \in \lambda_k \mathrm{T}^{n_k} \mathbb{B}(h_{k+1}, \epsilon_{k+1}) \subset \mathbb{B}(h_k, \epsilon_k)$, thus $\|\lambda_{k}T^{n_k} l - h_{k}\| < \frac{1}{2^{k}}$, then, $\|\lambda_{k}T^{n_k} l - l\| < \frac{1}{2^{k-1}}$. Hence, $\lambda_{k}T^{n_k} l$ converges to l, which confirms that l is a codisk-recurrent for T.

$\Leftarrow$) Let $\hat{U}$ be an open set in H which is $\hat{U} \neq \emptyset$, and let $\epsilon > 0$. Since a $\text{codisk} - \text{recurrent}$ vector is a dense set, there are $T - \text{recurrent}$ vectors, say, $h_\circ \in \hat{U}$ with $\mathbb{B}(h_\circ, \epsilon) \subset \hat{U}$, the strictly increase sequences $\{n_{k}\}_{k\in\mathbb{N}} \subset \mathbb{N}$, $\{\lambda_{k}\}_{k\in\mathbb{N}} \subset \mathbb{B}^{C}$ & there is $N \in \mathbb{N}$ with $\|\lambda_{k}T^{n_k} h_\circ - h_\circ\| < \epsilon$ for all $k \geq N$. Thus, $\lambda_{k}T^{n_k} h_\circ \in \mathbb{B}(h_\circ, \epsilon) \subset \hat{U}$. Therefore, T is $\text{codisk} - \text{recurrent}$ since $\lambda_{k}T^{n_k}\hat{U} \cap \hat{U} \neq \emptyset$.

5. Conclusion

This study proposes the notion of codisk–cyclic operator, emphasizing the transition from individual orbit behavior to a global operator perspective. It was shown that every codisk–cyclic operator is $\text{codisk} - \text{recurrent}$, whereas the converse fails. Several structural properties were investigated, and a formal characterization was provided.

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