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Closed-neighborhood soft topology on soft graphs

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Abstract

In this paper, we introduce a new soft topological structure on soft graphs based on the concept of closed neighborhoods of vertices. By associating each vertex with its closed neighborhood and its complement, we construct a family of soft sets that serves as a subbase for a soft topology on the vertex set of a given soft graph. This construction naturally induces a closed-neighborhood soft topology whose properties are investigated through several illustrative graph classes. The proposed framework establishes a systematic connection between graph-theoretic neighborhood structures and soft topological spaces, providing a flexible setting for further theoretical developments in soft graph theory.

Subject Classification: *Primary 54A05, Secondary 05C75, 54C05.*

Keywords: *Soft topology, Soft graphs, Closed neighborhood, Soft subbase, Graph theory.*

1. Introduction

Graph theory studies relational structures formed by vertices and edges and provides a mathematical framework for modeling discrete interactions. Since Euler's pioneering work on the Königsberg bridge problem, the subject has evolved into a central area of combinatorics with wide applications in computer science, engineering, and network analysis [1-4].

Soft set theory was initiated by Molodtsov [5] as a parameter-based framework for handling uncertainty. Unlike probability or fuzzy approaches, soft sets rely on parameterization, which allows flexible modeling of incomplete or attribute-dependent information. Applications in decision-making problems were later developed in [6]. The concept of soft graphs was introduced as an

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extension of classical graph theory within the soft set framework in [7], and further structural investigations were carried out in [8-11].

In parallel, various soft topological notions, including separation axioms and Alexandroff-type structures, have been studied in soft topological spaces [12]. These developments motivate the study of graph-induced soft topologies and their structural properties.

2. Preliminaries

This section recalls basic definitions from graph theory, soft set theory, and soft graphs.

Definition 2.1: [1] The graph Υ= ($\mho$,$\mathfrak{E}$) has no loops and no parallel is named a simple graph.

Definition 2.2: [7] Let Υ= ($\mho$,$\mathfrak{E}$) be a simple-graph and let Λ any non-empty set. Consider an arbitrary binary relevance $\mathcal{R}$ among Λ and vertices of $\mho$. That is $\mathcal{R} \sqsubseteq \Lambda \times \mho$. Define a set-valued function $\mathcal{F}$: $\Lambda \rightarrow \wp$ ($\mho$) by $\mathcal{F}$ (x) = {y$\in \mho$ |x$\mathcal{R}$y}. The set $\mathcal{F}_{\Lambda}$ is a soft set over $\mho$.

Definition 2.3: [7], [10] Let Λ be any set that is not empty and Υ= ($\mho$, $\mathfrak{E}$) be a simple graph. Assume that the relation from Λ to $\mho$ is random and that $\mathcal{R} \sqsubseteq \Lambda \times \mho$ is a subset of that relation. You can define a mapping $\mathcal{F}^{\mathfrak{f}}$: $\Lambda \rightarrow \mathcal{P}_{\wp}(\mho)$ by $\mathcal{F}^{\mathfrak{f}}(\varkappa) = \{\gimel \in \mho \mid \varkappa \, \mathcal{R} \, \gimel\}$ and a mapping $\Pi^{\mathfrak{f}}$: $\Lambda \rightarrow \mathcal{P}_{\wp}(\mathfrak{E})$ by $\Pi^{\mathfrak{f}}(\varkappa) = \{\partial\ell \in \mathfrak{E}/\{\partial, \ell\} \sqsubseteq \mathcal{F}^{\mathfrak{f}}(\varkappa)\}$.

A 4-tuple Υ= (Υ, $\mathcal{F}^{\mathfrak{f}}$, $\Pi^{\mathfrak{f}}$, Λ) is named a soft graph of Υ if it has these features:

1. Υ= ($\mho$, $\mathfrak{E}$) is a simple graph.
2. Λ is a set of parameters and $\Lambda \neq \emptyset$.
3. $\mathcal{F}^{\mathfrak{f}}_{\Lambda}$ is a soft set over $\mho$.
4. ${\Pi^{\mathfrak{f}}}_{\Lambda}$is a soft set over $\mathfrak{E}$.
5. ($\mathcal{F}^{\mathfrak{f}}(\mathfrak{w})$, $\Pi^{\mathfrak{f}}(\mathfrak{w})$) is a subgraph of Υ for all $\mathfrak{w}\in \Lambda$.

The symbol $\boldsymbol{\mathcal{S}(\Psi)}$ represents the set of all soft graph of Υ.

Definition 2.4: [5] Let $\mathcal{X}_{\varkappa}$ be a universal set and Σ a collection of parameters. Define $\mathcal{P}_{\wp}(\mathcal{X}_{\varkappa})$ as the power set of $\mathcal{X}_{\varkappa}$, with Λ being a non-empty subset of $\mathfrak{E}$. A soft set, denoted as ${\mathcal{F}^{\mathfrak{f}}}_{\Lambda}$, is a pair where $\mathcal{F}^{\mathfrak{f}}$ is a function $\mathcal{F}^{\mathfrak{f}}$: $\Lambda \rightarrow \mathcal{P}_{\wp}(\mathcal{X}_{\varkappa})$. In essence, a soft set over $\mathcal{X}_{\varkappa}$ is a parameterized collection of subsets of $\mathcal{X}_{\varkappa}$. For each ð $\in \Lambda$, $\mathcal{F}^{\mathfrak{f}}$(ð) represents the set of elements approximately corresponding to the parameter ð. Importantly, a soft set is distinct from a conventional set.

Definition 2.5: [2], [3], [4] Let Υ= ($\mho$,$\mathfrak{E}$) be a simple-graph, where $\mho$ is the set of vertices and $\mathfrak{E}$ is the set of edges. The closed neighborhood of a vertex $\eta \in \mho$ is defined as $\mathcal{CN}(\eta)$ = {{ η }$\cup${$\sigma\in \mho$:(σ, η)$\in \mathfrak{E}$}, for all vertices}. The complement of the closed neighborhood is $\mathcal{CNC}(\eta) = \{\ \mho \setminus \mathcal{CN}\}$ for all vertices.

3. Closed-Neighborhood Soft Topology on Soft Graphs

In this section, a soft topological structure induced by closed neighborhoods on soft graphs is introduced.

Definition 3.1: Let $\Psi = (\mho, \mathfrak{E})$ be a soft graph and let $\mho$ be the corresponding set of parameters. For each vertex ${}^{g}_{s}\eta \in \mho$, let $\mathcal{CN}({}^{g}_{s}\eta)$ and $\mathcal{CNC}({}^{g}_{s}\eta)$ denote the closed neighborhood of η and its complement, respectively, as defined in the previous section. We define two soft sets $\mathcal{R}^{\mathcal{CN}}$ and $\mathcal{R}^{\mathcal{C}}$ over $\mho$ by associating each parameter ${}^{g}_{s}\eta \in \mho$ with the subsets $\mathcal{CN}({}^{g}_{s}\eta)$ and $\mathcal{CNC}({}^{g}_{s}\eta)$, respectively. The collection $\mathcal{CS}_{\mathcal{N}} = \{\mathcal{R}^{\mathcal{CN}}, \mathcal{R}^{\mathcal{C}}\}$ is called the closed-neighborhood soft subbase of Ψ.

Definition 3.2: Let $\Psi = (\mho, \mathfrak{E})$ be a soft graph and let $\mathcal{CS}_{\mathcal{N}}$ be the closed-neighborhood soft subbase defined in Definition 3.1. The family $\mathcal{CT}_{\mathcal{N}}(\Psi)$ of all soft sets obtained as arbitrary unions of finite intersections of members of $\mathcal{CS}_{\mathcal{N}}$ is called *the closed-neighborhood soft topology* on $\mho$ induced by Ψ.

Remark 3.3: Since the closed-neighborhood soft subbase $\mathcal{CS}_{\mathcal{N}}$ consists of only two soft sets, the resulting closed-neighborhood soft topology $\mathcal{CT}_{\mathcal{N}}(\Psi)$ is generated by their finite intersections and arbitrary unions. Consequently, the structure of $\mathcal{CT}_{\mathcal{N}}(\Psi)$ strongly depends on the neighborhood properties of the underlying soft graph Ψ.

Soft graphs admit various structural forms. The following examples illustrate the closed-neighborhood soft topology on different classes of soft graphs.

Throughout the examples, the closed-neighborhood relation and its complement are considered in the sense of Definition 3.1, and the corresponding closed-neighborhood soft topology is constructed as described in Definition 3.2.

Example 3.4: Let Ψ be a soft graph whose underlying graph is the path $\mathbb{sP}_3$ with vertex set $\{{}^{g}_{s}\eta_1, {}^{g}_{s}\eta_2, {}^{g}_{s}\eta_3, {}^{g}_{s}\eta_4\}$. The parameter set is chosen to coincide with the vertex set. The soft path graph $\mathbb{sP}_3$ is illustrated in Figure 1.

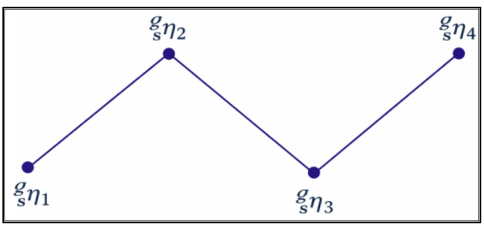

Figure 1

Following Definition 3.1, the closed neighborhoods $\mathcal{CN}\left({}^{g}_{s}\eta_i\right)$ and their complements are computed directly from the adjacency relations of the path.

Accordingly, we define

$$\mathcal{R}_{\mho}^{\mathcal{CN}} = \{({}^{g}_{s}\eta_i, \mathcal{CN}\left({}^{g}_{s}\eta_i\right) : i = 1,\ldots,4\}.$$

$$\mathcal{R}_{\mho}^{\mathcal{C}} = \{({}^{g}_{s}\eta_i, \mho \setminus \mathcal{CN}\left({}^{g}_{s}\eta_i\right) : i = 1,\ldots,4\}.$$

The closed-neighborhood soft topology induced by Ψ is therefore

$$\mathcal{CT}_{\mathcal{N}}(\Psi) = \{ \breve{\mho}, \breve{\varphi}, \mathcal{R}_{\mho}^{\mathcal{CN}}, \mathcal{R}_{\mho}^{\mathcal{C}}\}.$$

Remark 3.5: Example 3.4 explicitly illustrates the construction of the closed-neighborhood soft topology from its generating soft subbase. In the following examples, this construction will be applied without repeating intermediate steps, and only the resulting closed-neighborhood soft topology will be presented.

Example 3.6: Let Ψ be a soft graph whose underlying graph is the cycle $\mathbb{sC}_3$, with vertex set $\{{}^{g}_{s}\eta_1, {}^{g}_{s}\eta_2, {}^{g}_{s}\eta_3\}$. The parameter set coincides with the vertex set.

The soft cycle graph $\mathbb{sC}_3$ is illustrated in Figure 2.

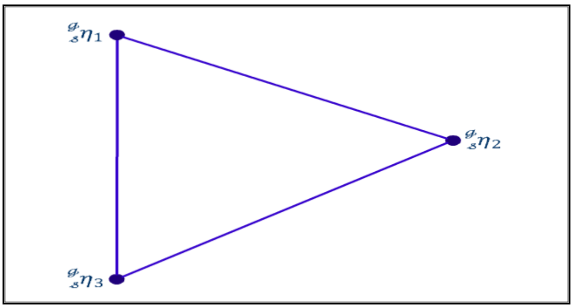

Figure 2
The soft cycle graph sC3.

In this case, each closed neighborhood coincides with the entire vertex set $\mho$. Hence, $\mathcal{R}_{\mho}^{\mathcal{CN}} = \breve{\mho}$, $\mathcal{R}_{\mho}^{\mathcal{C}} = \breve{\varphi}$.

Therefore, the induced closed-neighborhood soft topology reduces to the indiscrete topology $\mathcal{CT}_{\mathcal{N}}(\Psi) = \{ \breve{\mho}, \breve{\varphi}\}$.

Remark 3.7. In the case of the cycle graph $\mathbb{sC}_3$, every vertex has the entire vertex set as its closed neighborhood. Consequently, the induced closed-neighborhood soft topology $\mathcal{CT}_{\mathcal{N}}(\Psi)$ reduces to the indiscrete soft topology $\{\breve{\mho}, \breve{\varphi}\}$.

Remark 3.8. A similar situation occurs for complete soft graphs, where the closed neighborhood of each vertex coincides with the entire vertex set. In this case, the induced closed-neighborhood soft topology is also indiscrete.

In contrast, for soft graphs whose closed neighborhoods are not identical for all vertices, the induced closed-neighborhood soft topology may contain nontrivial soft open sets.

Proposition 3.9: Let Ψ be a soft graph. If the closed neighborhood of every vertex of Ψ coincides with the entire vertex set, then the induced closed-neighborhood soft topology $\mathcal{CT}_{\mathcal{N}}(\Psi)$ is indiscrete.

Proof: Assume that $\mathcal{CN}({}^{g}_{s}\eta) = \mho(\Psi)$ for every vertex ${}^{g}_{s}\eta \in \mho(\Psi)$.

Then $\mathcal{CNC}({}^{g}_{s}\eta) = \mho(\Psi) \setminus \mathcal{CN}({}^{g}_{s}\eta) = \emptyset$ for all ${}^{g}_{s}\eta \in \mho(\Psi)$.

By Definition 3.1, the soft set $\mathcal{R}_{\mho}^{\mathcal{CN}}$ assigns $\mho(\Psi)$ to each parameter η, hence $\mathcal{R}_{\mho}^{\mathcal{CN}} = \breve{\mho}$, while $\mathcal{R}_{\mho}^{\mathcal{C}}$ assigns $\emptyset$ to each parameter ${}^{g}_{s}\eta$, hence $\mathcal{R}_{\mho}^{\mathcal{C}} = \breve{\varphi}$.

Therefore, the closed-neighborhood soft subbase $\mathcal{CS}_{\mathcal{N}} = \{ \mathcal{R}_{\mho}^{\mathcal{CN}}, \mathcal{R}_{\mho}^{\mathcal{C}}\}$, reduces to $\{\breve{\mho}, \breve{\varphi}\}$. By Definition 3.2, the topology generated by this subbase is exactly $\{\breve{\mho}, \breve{\varphi}\}$, i.e., the indiscrete closed-neighborhood soft topology.

Remark 3.10: The induced closed-neighborhood soft topology is necessarily Alexandroff, since it is generated from a finite soft subbase and arbitrary intersections of open soft sets remain open. Furthermore, the topology satisfies the Soft-$\mathbb{T}_0$ axiom if and only if distinct vertices of Ψ have distinct closed neighborhoods. Hence, separation properties of the induced topology are completely determined by neighborhood equivalence classes.

Example 3.11: Let Ψ be a soft graph whose underlying graph is a bipartite graph with vertex set $\{{}^{g}_{s}\eta_1, {}^{g}_{s}\eta_2, {}^{g}_{s}\eta_3, {}^{g}_{s}\eta_4, {}^{g}_{s}\eta_5, {}^{g}_{s}\eta_6, {}^{g}_{s}\eta_7\}$. The parameter set is chosen to coincide with the vertex set. The soft bipartite graph is illustrated in Figure 3.

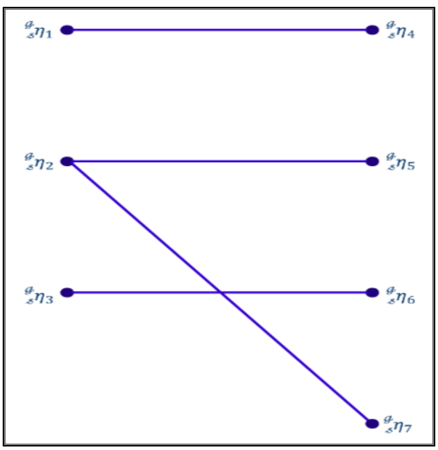

Figure 3

A soft bipartite graph

Following the same construction as in Example 3.4, the closed neighborhoods of the vertices are computed from the adjacency structure. Accordingly, we define

$$\mathcal{R}_{\mho}^{\mathcal{CN}} = \{({}_{s}^{g}\eta_i, \mathcal{CN}({}_{s}^{g}\eta_i)): i = 1,\dots,7\}.$$
$$\mathcal{R}_{\mho}^{\mathcal{C}} = \{({}_{s}^{g}\eta_i, \mho \setminus \mathcal{CN}({}_{s}^{g}\eta_i)): i = 1,\dots,7\}.$$

Since the closed neighborhoods are not identical for all vertices, the induced closed-neighborhood soft topology is given by $\mathcal{CT}_{\mathcal{N}}(\Psi)$ ={ $\breve{\mho}, \breve{\varphi}$, $\mathcal{R}_{\mho}^{\mathcal{CN}}$, $\mathcal{R}_{\mho}^{\mathcal{C}}$} and therefore, the topology is not indiscrete.

Proposition 3.12: Let Ψ be a soft graph whose underlying graph is a star $\mathbb{sS}_n$ with vertex set $\mho(\Psi) = \{{}_{s}^{g}\xi, {}_{s}^{g}\eta_1, \dots, {}_{s}^{g}\eta_n\}$, where ${}_{s}^{g}\xi$ is the center and ${}_{s}^{g}\eta_1$, …, ${}_{s}^{g}\eta_n$ are the leaves. Let $\mathcal{CS}_{\mathcal{N}}$ = { $\mathcal{R}_{\mho}^{\mathcal{CN}}$, $\mathcal{R}_{\mho}^{\mathcal{C}}$} be the closed-neighborhood soft subbase of Ψ. Then $\mathcal{R}_{\mho}^{\mathcal{CN}} \sqcap \mathcal{R}_{\mho}^{\mathcal{C}} = \breve{\varphi}$. Consequently, the induced closed-neighborhood soft topology is $\mathcal{CT}_{\mathcal{N}}(\Psi)$ ={ $\breve{\mho}, \breve{\varphi}, \mathcal{R}_{\mho}^{\mathcal{CN}}, \mathcal{R}_{\mho}^{\mathcal{C}}$}.

Proof: In a star $\mathbb{sS}_n$, the closed neighborhoods satisfy:

$$\mathcal{CN}({}_{s}^{g}\xi) = \mho \text{ and } \mathcal{CN}({}_{s}^{g}\eta_i) = \{{}_{s}^{g}\xi, {}_{s}^{g}\eta_i\} \text{ for } i = 1, \dots, n.$$

Hence, $\mathcal{CNC}({}_{s}^{g}\xi) = \mho \setminus \mathcal{CN}({}_{s}^{g}\xi) = \emptyset$, $\mathcal{CNC}({}_{s}^{g}\eta_i) = \mho \setminus \{{}_{s}^{g}\xi, {}_{s}^{g}\eta_i\}$

By Definition 3.1, $\mathcal{R}_{\mho}^{\mathcal{CN}}({}_{s}^{g}\eta) = \mathcal{CN}({}_{s}^{g}\eta)$, $\mathcal{R}_{\mho}^{\mathcal{C}}({}_{s}^{g}\eta) = \mathcal{CNC}({}_{s}^{g}\eta)$ for each ${}_{s}^{g}\eta \in \mho$. Thus, $\mathcal{R}_{\mho}^{\mathcal{CN}}({}_{s}^{g}\xi) \cap \mathcal{R}_{\mho}^{\mathcal{C}}({}_{s}^{g}\xi) = \mho \cap \emptyset = \emptyset$, and for every leaf ${}_{s}^{g}\eta_i$,

$$\mathcal{R}_{\mho}^{\mathcal{CN}}({}_{s}^{g}\eta_i) \cap \mathcal{R}_{\mho}^{\mathcal{C}}({}_{s}^{g}\eta_i) = \{{}_{s}^{g}\xi, {}_{s}^{g}\eta_i\} \cap (\mho \setminus \{{}_{s}^{g}\xi, {}_{s}^{g}\eta_i\}) = \emptyset.$$

Therefore, $(\mathcal{R}_{\mho}^{\mathcal{CN}} \sqcap \mathcal{R}_{\mho}^{\mathcal{C}})({}_{s}^{g}\eta) = \emptyset$ for all ${}_{s}^{g}\eta \in \mho$, which implies $\mathcal{R}_{\mho}^{\mathcal{CN}} \sqcap \mathcal{R}_{\mho}^{\mathcal{C}} = \breve{\varphi}$.

Since $\mathcal{CS}_{\mathcal{N}}$ consists of $\mathcal{R}_{\mho}^{\mathcal{CN}}$ and $\mathcal{R}_{\mho}^{\mathcal{C}}$, the family of finite intersections generated by $\mathcal{CS}_{\mathcal{N}}$ is $\{\mathcal{R}_{\mho}^{\mathcal{CN}}, \mathcal{R}_{\mho}^{\mathcal{C}}, \breve{\varphi}\}$. Taking arbitrary unions yields exactly $\{\breve{\mho}, \breve{\varphi}, \mathcal{R}_{\mho}^{\mathcal{CN}}, \mathcal{R}_{\mho}^{\mathcal{C}}\}$. Hence, $\mathcal{CT}_{\mathcal{N}}(\Psi)$ ={ $\breve{\mho}, \breve{\varphi}, \mathcal{R}_{\mho}^{\mathcal{CN}}, \mathcal{R}_{\mho}^{\mathcal{C}}$}.

5. Discussion and Conclusion

A closed-neighborhood soft topology on soft graphs has been developed. The construction associates each parameter with the closed neighborhood of the corresponding vertex and its complement, forming a soft subbase that generates the topology in a natural way. The results show that the induced topology is fully determined by the structure of closed neighborhoods. When all closed neighborhoods coincide with the entire vertex set, the topology becomes indiscrete. In contrast, variations in neighborhood structure produce nontrivial soft open sets. In particular, the induced topology is necessarily Alexandroff, and its separation behavior depends entirely on whether distinct vertices have distinct closed neighborhoods. This perspective clarifies how local graph-theoretic properties influence global soft topological behavior and provides a structured framework for further investigation of separation axioms and continuity in soft graph settings.

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