

An approximate solution for PDEs of third order under mixed and nonlocal boundary conditions

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Abstract

This work aims to find a solution to the problem under investigation and to study non-local boundary-value problems for rectangular domains and two-dimensional third-order partial differential equations (PDEs). A finite-difference method combined with the trapezoidal rule is used to solve problems. The numerical results were determined to be steady and accurate.

Subject Classification: 65K05, 65K10, 65R30, 65R32, 65Y15, 65Y15, 65N12.

Keywords: Finite difference method (FDM), Third-order PDE, Nonlocal boundary conditions.

1. Introduction

Partial differentials equations (PDEs) have been a most important tool available to humans, and some scientist has taken joint step to the explaining and solving important problem, as they were use of a mathematical model for the some real-world application [1], [2]. Various PDEs with nonlocal boundary conditions were studied in [3], whilst a theory with application for the well-posednes in PDEs have been extensively studied by many researchers [4]. In the rectangular domain, the well-posedness for non-local boundary valued problem for the elliptic-types equation were studied in [5]. In Huntul and Tekin [6]

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discussed the third-orders in times PDE. Actually, A method for solutions for non-local boundary valued problem for the various differentials and difference equations of some types of PDEs have been studied extensively by many authors [7], [8], [9], [10], and in [11], [12], [13], [14], [15] authors used the Matlab routine to find solutions to nonlinear problems. Recently, this paper's problem has been shown to exist and be unique by Aliyev, Mehraliyev, and Yusifova [16].

2. Mathematical Formulation

As the form for mathematical models, put D_Y the regions be define as $D_Y = \{(v, y): 0 \leq v \leq 1, 0 \leq y \leq Y\}$ & some fixed numbers, two-dimensional problem $w(v, y)$ for the upcoming equation,

$$w_{yyy}(v, y) = -w_{vv}(v, y) + a(y)w(v, y) + f(v, y) \quad (v, y) \in D_Y, \quad (1)$$

with the nonlocal conditions,

$$w(v, 0) = \psi_0(v), \quad (2)$$

with Neumann boundary condition,

$$w_v(0, y) = 0, y \in [0, Y], \quad (3)$$

and a nonlocal integral condition,

$$\mu w(1, y) + \int_0^1 w(v, y) dv = 0, y \in [0, Y], \quad (4)$$

$$w_y(v, 0) = \psi_1(v), v \in [0, 1], \quad (5)$$

$$w_{yy}(v, Y) = \psi_2(v), v \in [0, 1], \quad (6)$$

where, $\mu > 0$, is given number, $f(v, y), \psi_i(v), (i = 0, 1, 2)$, and $z(y), a(y)$ are given sufficient functions, and $w(v, y)$, is the required function, $v \in [0, 1], y \in [0, Y]$, (v adn y and are spaces component).

3. FDM Schemes of Directed (Forward) Problems

A directed problems is resolve here, i.e., where the unknown $a(y)$ be the assume to give. so to the solving these problems, the FDM schemes has use to find solution for non-local problems give by eq. (1) – (6). When a domains D_Y was divide to $M \times N$ & spatial steps sizes $\Delta v = \frac{1}{M}$ & $\Delta y = \frac{Y}{N}$, when M & N were give positive integer. A grid point is $v_i = i\Delta v, i = \overline{0, M}$. $y_j = j\Delta y, j = \overline{0, N}$. This quantities discretize forms, $w(v_i, y_j) =: w_{i,j}, a(y_j) =: a_j, f(v_i, y_j) =: f_{i,j}, \psi(v_i) =: \psi_i$ for $i = \overline{0, M}, j = \overline{0, N}$. Then a FDM, & trapezoidal rule quadrature for integration, discretize eq. (1), that approximate as,

$$f_{i,j} = \frac{w_{i,j+2} - 2w_{i,j+1} + 2w_{i,j-1} - w_{i,j-2}}{2(\Delta y)^3} + \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{(\Delta v)^2} - a_j w_{i,j}, i = \overline{0, M-1}, j = \overline{1, N}. \quad (7)$$

So we have

$$\zeta = 2(\Delta y)^3 \quad (8)$$

where, $\xi = \frac{2(\Delta y)^3}{(\Delta v)^2}$, and $H_j = (2\xi + \zeta a_j)$. The FDM discretizes equations (2) – (6) as,

$$w_{i,0} = \psi_0(v_i), i = \overline{0, M}, \quad (9)$$

$$\frac{w_{i+1,j} - w_{i-1,j}}{2\Delta v} = 0, i = 0, j = \overline{1, N}, \quad (10)$$

the trapezoidal rule discretizes integral condition (4) as,

$$\mu w_{M,j} + \frac{1}{2M} (w_{0,j} + w_{M,j} + 2 \sum_{i=1}^{M-1} w_{i,j}) = 0, j = \overline{1, N}, \quad (11)$$

since $w_{0,j} = \frac{4w_{1,j} - w_{2,j}}{3}$, hence, the equation (11) become,

$$\left(\mu + \frac{1}{2M}\right) w_{M,j} + \frac{1}{2M} \left(\frac{10}{3} w_{1,j} + \frac{5}{3} w_{2,j} + 2 \sum_{i=3}^{M-1} w_{i,j}\right) = 0, j = \overline{1, N}, \quad (12)$$

$$\frac{w_{i,j+1} - w_{i,j-1}}{2\Delta y} = \psi_1(v_i), i = \overline{0, M}, j = 0, \quad (13)$$

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{(\Delta y)^2} = \psi_2(v_i), i = \overline{0, M}, j = N, \quad (14)$$

So, we get

$$-(1 + H_1)w_{0,1} - 2w_{0,2} + w_{0,3} + 2\xi w_{1,1} = \zeta f_{0,1} - 2\psi_0(v_0) - 2\Delta y \psi_1(v_0), i = 0,$$

$$\xi w_{M-2,1} - (1 + H_1)w_{M-1,1} - 2w_{M-1,2} + w_{M-1,3} + \xi w_{M,1} = \zeta f_{M-1,1} - 2\psi_0(v_{M-1}) - 2\Delta y \psi_1(v_{M-1}), i = \overline{1, M-1},$$

$$\left(\mu + \frac{1}{2M}\right) w_{M,1} + \frac{1}{2M} \left(\frac{10}{3} w_{M-9,1} + \frac{5}{3} w_{M-8,1} + 2 \sum_{i=3}^{M-6} w_{i,1}\right) = 0, i = M,$$

and, the linear system when, $j = 2$, we get,

$$2w_{0,1} - H_2 w_{0,2} - 2w_{0,3} + w_{0,4} + 2\xi w_{1,2} = \zeta f_{0,2} + \psi_0(v_0), i = 0,$$

$$\left(\mu + \frac{1}{2M}\right) w_{M,N} + \frac{1}{2M} \left(\frac{10}{3} w_{M-9,N} + \frac{5}{3} w_{M-8,N} + 2 \sum_{i=3}^{M-6} w_{i,N}\right) = 0, i = M,$$

above systems is write in the matrix forms as $LW = S$;

$$L = \begin{bmatrix} -\hat{I} + Q_1 & -2\hat{I} & \hat{I} & \theta & \theta & \theta & \dots & \theta & \dots & \theta \\ 2\hat{I} & -\hat{I} + Q_2 & -2\hat{I} & \hat{I} & \theta & \theta & \dots & \theta & \dots & \theta \\ -\hat{I} & 2\hat{I} & Q_3 & -2\hat{I} & \hat{I} & \theta & \vdots & \vdots & \vdots & \theta \\ \theta & \ddots & \ddots & \ddots & \ddots & \ddots & \theta & \theta & \theta & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \theta \\ \theta & \theta & \dots & \theta & \theta & -\hat{I} & 2\hat{I} & Q_{M-2} & -2\hat{I} & \hat{I} \\ \theta & \theta & \dots & \theta & \theta & \theta & -\hat{I} & 2\hat{I} & -\hat{I} + Q_{M-1} & \theta \\ \theta & \theta & \dots & \theta & \theta & \theta & \theta & -2\hat{I} & 2\hat{I} & -\hat{I} + Q_M \end{bmatrix}_{(M)^2 \times (N)^2}$$

where, $W = [W_1, W_2, W_3, \dots, W_{N-2}, W_{N-1}, W_N]^T$ with, $W_j = [w_{0,j}, w_{1,j}, w_{2,j}, \dots, w_{M-2,j}, w_{M-1,j}, w_{M,j}]^T$, where, $\zeta = 2(\Delta y)^3$, $\xi = \frac{2(\Delta y)^3}{(\Delta v)^2}$, and $H_j = (2\xi + \zeta a_j)$, $i = \overline{0, M}$, $j = \overline{1, N}$. Here, L be $M^2 \times N^2$ matrix, W is a column with the size $(M+1) \times N$, P is a column with the size $(M+1) \times 1$, $(Q_\iota, \iota = 1, \dots, M)$ is a matrix $(M) \times (N+1)$, θ is zero matrix, and, $\hat{I}$ identity matrix, with the size $(M) \times (N+1)$; $\iota = 1, 2, 3, \dots, M$.

$$Q_l = \begin{bmatrix} -H_2 & 2\xi & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \xi & -H_3 & \xi & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \xi & -H_4 & \xi & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \xi & -H_{M-2} & \xi & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \xi & -H_{M-1} & \xi \\ 0 & 10/6M & 5/6M & 1/M & 1/M & 0 & \dots & 0 & 0 & 0 & (\mu + \frac{1}{2M}) \end{bmatrix}_{(M) \times (N+1)}$$

$$\text{where, } \hat{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(M) \times (N+1)} \quad \text{and, the right-hand side,}$$

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_{N-1} \\ P_N \end{bmatrix}, \text{ s.t., } P_1 = \begin{bmatrix} \zeta f_{0,1} - 2\psi_0(v_0) - 2\Delta y \psi_1(v_0) \\ \zeta f_{1,1} - 2\psi_0(v_1) - 2\Delta y \psi_1(v_1) \\ \zeta f_{2,1} - 2\psi_0(v_2) - 2\Delta y \psi_1(v_2) \\ \vdots \\ \zeta f_{M-1,1} - 2\psi_0(v_{M-1}) - 2\Delta y \psi_1(v_{M-1}) \\ 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} \zeta f_{0,2} + \psi_0(v_0) \\ \zeta f_{1,2} + \psi_0(v_1) \\ \zeta f_{2,2} + \psi_0(v_2) \\ \vdots \\ \zeta f_{M-1,2} + \psi_0(v_{M-1}) \\ 0 \end{bmatrix}, \text{ and, when, } j = 3, \dots, N-2, N-1, N. \text{ we get,}$$

$$P_j = \begin{bmatrix} \zeta f_{0,j} \\ \zeta f_{1,j} \\ \zeta f_{2,j} \\ \vdots \\ \zeta f_{M-1,N-2} \\ 0 \end{bmatrix}, P_{N-1} = \begin{bmatrix} \zeta f_{0,N-1} - (\Delta y)^2 \psi_2(v_0) \\ \zeta f_{1,N-1} - (\Delta y)^2 \psi_2(v_1) \\ \zeta f_{2,N-1} - (\Delta y)^2 \psi_2(v_2) \\ \vdots \\ \zeta f_{M-1,N-1} - (\Delta y)^2 \psi_2(v_{M-1}) \\ 0 \end{bmatrix},$$

$$P_N = \begin{bmatrix} \zeta f_{0,N} - (\Delta y)^2 \psi_2(v_0) \\ \zeta f_{1,N} - (\Delta y)^2 \psi_2(v_1) \\ \zeta f_{2,N} - (\Delta y)^2 \psi_2(v_2) \\ \vdots \\ \zeta f_{M-1,N} - (\Delta y)^2 \psi_2(v_{M-1}) \\ 0 \end{bmatrix}, \text{ such that, } \iota = 1, 2, \dots, M-2, M-1, M.,$$

A unique for solutions to a linear systems resulting from the proposed finite difference-trapezoidal scheme is ensured by the structural properties of the

discretized matrix. Under standard regularity assumptions, the coefficient matrix is either diagonally dominant or positive definite, which guarantees non-singularity. Moreover, the consistent incorporation of boundary and non-local integral conditions ensures that the resulting system be well-posed & admits unique solutions. Therefore, the discrete problem admits a unique solution [17].

3. Numerical Simulation and Discussion

3.1 Example for direct problem

If the illustrations of directed problems give by (1) – (6) & $D_Y : \{v \in [0,1], y \in [0,1]\}$. an input data is $a(y) = \frac{1}{e^{-9-y}}$, $y \in [0,1]$, & analytical solutions be the $w(v, y) = e^{-9+2y}(\frac{1}{2} - v^3)$, $(v, \tau) \in D_Y$, $\mu = \frac{1}{2}$, with, $f(v, y) = 8e^{-9+2y}(\frac{1}{2} - v^3) - 6ve^{-9+2y} - \frac{e^{-9+2y}}{e^{-9-y}}(\frac{1}{2} - v^3)$, $y \in [0,1]$, $\psi_0(v) = e^{-9}(\frac{1}{2} - v^3)$, $\psi_1(v) = 2e^{-9}(\frac{1}{2} - v^3)$, $\psi_2(v) = 4e^{-7}(\frac{1}{2} - v^3)$, $y \in [0,1]$, $v \in [0,1]$.

investigated an accuracy for solutions with diverse meshes grid sizes say $M = N = 10$. In Figure 1, an exceptional match between the exact solution and the numerical approximation is observed, yielding accurate results. As a numbers for the meshes increase, an accuracy for solutions improves, revealing a flexibility for the mesh selection. Example 1, of the size mesh $N = M = 10$. In addition to the Figure 1 results,

Table 1

Comparison between exacts and numerals solutions of example 1.

v	Exact Solutions	Numericals Solutions	Absolute Error
0.0	0.0000	0.0000	0.0000
0.1	0.3911	0.3900	11×10^{-4}
0.2	0.5062	0.5049	13×10^{-4}
0.8	-0.0321	-0.0318	3×10^{-4}
0.9	-0.0603	-0.0600	3×10^{-4}
1.0	-0.0566	-0.0564	11×10^{-4}

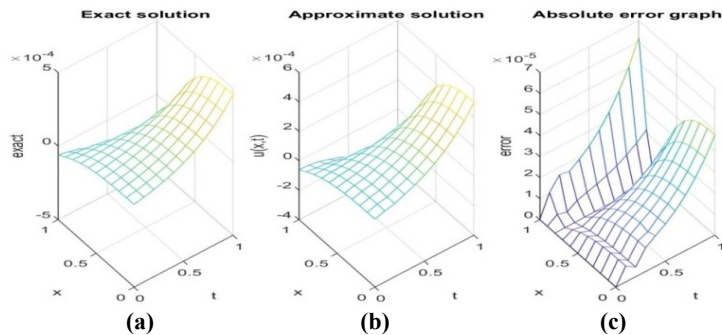


Figure 1

(a). Exact solution, (b). Approximate solution, (c). Absolute error graph

Table 1, provides a quantitative comparison between the exact and numerical solution when $Y = 1$.

4. Conclusions

In this study, third-orders partial differentials equations (PDE) with nonstandard nonlocal and integrals boundary condition was numerically solved using a finite differences methods (FDM) combined with a trapezoidal quadrature rules. The proposed FDM-trapezoidal scheme demonstrated high numerical accuracy, as evidenced by minimal absolute errors across the spatial domain. The method proved to be stable, robust, and capable of accurately capturing solution behavior even under intricate boundary conditions. Additionally, the discretized system maintained well-posedness, ensuring an unique and stabilities for solutions. These results highlight the effectiveness and reliability of the proposed approach for a broad class of elliptic-type problems.

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Received May, 2025