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An analytical study of Fekete–Szegö bounds for star-like mappings in the multidimensional polydisk $\mathfrak{I}(\mathfrak{U}^n)$ and a complex Banach space $\mathfrak{I}(\check{\mathfrak{B}})$

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Abstract

This paper introduces and studies a new class of star-like mappings in complex analysis. We establish sharp coefficient inequalities for this class and provide a complete solution to Fekete-Szegö inequality. An analysis begins in classical setting of open unit disk, then extends to higher-dimensional complex spaces, including the unit ball $\mathfrak{I}(\check{\mathfrak{B}})$ and unit polydisk $\mathfrak{I}(\mathfrak{U}^n)$ in a complex Banach space and $\mathbb{C}^n$ respectively. Our results generalize several classical theorems in geometric function theory, revealing new insights into the behavior

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of starlike mappings in abstract settings. A detailed comparison with existing results demonstrates the significance and novelty of our findings.

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1. Introduction

Let the set B consist of all mappings that can be represented in the following form

$$\mathfrak{f}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \tag{1}$$

which are normalized analytic mappings in $\mathfrak{U} = \{w \in \mathbb{C}\}$ such that $|w| < 1$. Let $S \subset B$ which consisting of all univalent mappings in $\mathfrak{U}$. We denote S^* be set of mapping which are star-like in $\mathfrak{U}$. Fekete and Szegő [1], [2] established an inequality concerning coefficients of regular mappings, which later came to be known as the Fekete–Szegő inequality. This result is regarded as being closely connected to the well-known Bieberbach conjecture.

Although the Fekete and Szegő [1] inequality has been addressed within the subclass S, research concerning this problem remains limited when considered in the context of unit balls in space of complex Banach $\breve{\mathfrak{B}}$, & within unit polydisks $\mathbb{C}^n$ [3], [4]. Let Γ is denote to the space of complex Banach with norm $\| \quad \|$; and denote Γ^* is dual complex Banach space Γ . In addition let $\breve{\mathfrak{B}}$ is unit ball lying within Γ, Let $\partial\mathfrak{U}^n$ and $\partial_0\mathfrak{U}^n$ represent boundaries & distinguish boundaries for the $\mathfrak{U}^n$, respectively. For all $\in \Gamma \setminus \{0\}$, then

$$\mathcal{T}(t) = \mathcal{T}_t \in \Gamma^*: \|\mathcal{T}_t\| = 1, \mathcal{T}_t(t) = \|t\|, \tag{2}$$

such that $\mathcal{T}(t) \neq \emptyset$. Let $V(\breve{\mathfrak{B}})$ refer to a sets for every map $\breve{\mathfrak{B}} \longrightarrow \Gamma$ that is holomorphic. If $\mathfrak{f} \in V(\breve{\mathfrak{B}})$, then

$$\mathfrak{f}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{D}^n \mathfrak{f}(t)((s-t)^n), \tag{3}$$

for any s belongs to $(-\varepsilon + t, t + \varepsilon)$, where $\mathfrak{D}^n\mathfrak{f}(t)$ be n$^{\text{th}}$ Fŕechet derivatives for the mapping $\mathfrak{f}(t)$ at the value t & $\forall n \geq 1$. Therefore, $\mathfrak{D}^n\mathfrak{f}(t)$ be a symmetric bounded from $\prod_{i=1}^{n} \Gamma \longrightarrow \Gamma$. A map $\mathfrak{f}: \breve{\mathfrak{B}} \longrightarrow \Gamma$ named biholomorphic where it holomorphic & $\mathfrak{f}^{-1}(t) \in \mathfrak{f}(\breve{\mathfrak{B}})$. Also, it called a normalized mapping if $\mathfrak{D}\mathfrak{f}(0) = I$ and $(0) = 0$, where I refer to the identity operator from $\Gamma \longrightarrow \Gamma$. A mapping $\mathfrak{f}(t)$ belongs to the set $V(\breve{\mathfrak{B}})$ is called locally biholomorphic if for all t belongs to $\breve{\mathfrak{B}}$, $\mathfrak{D}\mathfrak{f}(t)$ has a bounded inverse. Assume that $G \in \mathbb{C}^n$ be a circular bounded domain. They can be mathematically represented by first Fŕechet $\mathfrak{D}\mathfrak{f}(w)$ and $k(k \geq 2)$-th $\mathfrak{D}^k\mathfrak{f}(w)(b^{k-1}, .)$ Fŕechet derivatives for the $\mathfrak{f} \in V(G)$ at points w belonging to G, respectively as follows:

$$\mathfrak{D}\mathfrak{f}(w) = \left(\frac{\partial \mathfrak{f}_{\wp}(w)}{\partial w_m}\right), \tag{4}$$

It utilizes ideas from measure theory to study and examine the behavior of mappings, especially their mapping characteristics and the way they transform geometric shapes [5], [6].

Definition 1.1: A mapping $\mathfrak{f}: \mathfrak{U} \to \mathbb{C}$ contains in class $\mathfrak{I}$ be called Star-like mapping (of order α), if it meets the following criterion:

$$\mathfrak{f}'(w)\left(\frac{w}{\mathfrak{f}(w)}\right)^{\alpha} > 0, 0 < \alpha \leq 1. \tag{5}$$

Definition 1.2: Let the mapping $\mathfrak{f}: \breve{\mathfrak{B}} \to \Gamma$ the locally normalize biholomorphic. The $\mathfrak{f}$ be Starlike map. of order α on $\breve{\mathfrak{B}}$ which is denote by $\mathfrak{f} \in \mathfrak{I}\,(\breve{\mathfrak{B}}\,)$iff

$$Re\left(\mathcal{T}_t\left(\frac{t(\,\mathfrak{D}(\mathcal{G}(t)\,)^{-1}\mathcal{G}^{\alpha}(t))}{t^{\alpha}}\right)\right) > 0, \tag{6}$$

for $\mathcal{G}(t) = \frac{t\mathfrak{f}(\mathcal{T}_u(t))}{\mathcal{T}_u(t)}$ & $\|u\| = 1$.

Definition 1.3: Let the mapping $\mathfrak{f}$ from $\mathfrak{U}^n$ in to $\mathbb{C}^n$be locally normalized biholomorphic. so $\mathfrak{f} \in \mathfrak{I}(\mathfrak{U}^n)$ iff

$$Re\left(\frac{\mathcal{J}_i(w)}{w_i}\right) > 0, w \in \mathfrak{U}^n \setminus\{0\}, \tag{7}$$

such that $|w_i| = \|w\| = max_{1\leq m\leq n}\{|w_m|\}$.

2. Inequality for the Fekete-Szego of a Classes $\mathfrak{I}$ in open unit disk

Lemma 2.1: [7] *Let* $\wp(w) = 1 + \sum_{m=1}^{\infty} b_m w^m \in B$ *and* $Re(\wp(w)) > 0$, $w \in \mathfrak{U}$; *then*

$$\left|b_2 - \frac{1}{2}b_1^2\right| \leq 2 - \frac{1}{2}|b_1|^2 \tag{8}$$

Theorem 2.2: *If* $\mathfrak{f}(w) \in \mathfrak{I}$. *Then*

$$b_2 = \frac{\wp_1}{(2-\alpha)} \tag{9}$$

$$b_3 = \frac{\wp_2}{(3-\alpha)} + \frac{\alpha}{2(2-\alpha)^2}\wp_1^2. \tag{10}$$

Proof: Setting

$$\wp(w) = \mathfrak{f}'(w)\left(\frac{w}{\mathfrak{f}(w)}\right)^{\alpha}, \tag{11}$$

where $\wp(w) = 1 + \wp_1 w + \wp_2 w^2 + \cdots$, Now, from (11), we get

$$(1 + \wp_1 w + \wp_2 w^2 + \cdots)\left(1 + \alpha b_2 w + \left(\alpha b_3 + \frac{\alpha(\alpha-1)}{2!}b_2^2\right)w^2\right) + \cdots$$
$$= 1 + 2b_2 w + 3b_3 w^2 + \cdots \tag{12}$$

By performing mathematical operations and comparing the coefficients w and w^2 in the both side of (12) and put $|\wp_1| \leq 2$ and $|\wp_2| \leq 2$, we get

$$|b_2| \leq \frac{2}{(2-\alpha)} \; and \; |b_3| \leq \frac{2}{(3-\alpha)} + \frac{2\alpha}{(2-\alpha)^2}.$$

Theorem 2.3: *A assume that* $(w) \in \mathfrak{I}$. *Then*

$$|b_3 - \sigma b_2^2| \leq \frac{2}{(3-\alpha)} \max\left\{1, \left|\frac{(4-\alpha)}{(2-\alpha)^2} - \sigma \frac{2(3-\alpha)}{(2-\alpha)^2}\right|\right\}, \sigma \in \mathbb{C} \tag{13}$$

Proof: By using the two equations (9) and (10) in Theorem 2.2 , we get

$$|b_3 - \sigma b_2^2| = \left|\frac{\wp_2}{(3-\alpha)} + \frac{\alpha}{2(2-\alpha)}\wp_1^2 - \sigma \frac{\wp_1^2}{(2-\alpha)^2}\right| \tag{14}$$

$$= \frac{1}{(3-\alpha)}\left|\wp_2 + \left(\frac{\alpha(3-\alpha)}{2(2-\alpha)} - \sigma \frac{(3-\alpha)}{(2-\alpha)^2}\right)\wp_1^2\right| \tag{15}$$

$$|b_3 - \sigma b_2^2| \leq \frac{1}{(3-\alpha)}\left[2 - \frac{1}{2}|\wp_1|^2 + \left|\frac{4-\alpha}{2(2-\alpha)^2} - \sigma \frac{(3-\alpha)}{(2-\alpha)^2}\right| |\wp_1|^2\right] \tag{16}$$

If $\left|\frac{4-\alpha}{2(3-\alpha)(2-\alpha)^2} - \sigma \frac{1}{(2-\alpha)^2}\right| \leq \frac{1}{2(3-\alpha)}$, then $|b_3 - \sigma b_2^2| \leq \frac{2}{(3-\alpha)}$ (17)

On the other hand, If $\left|\frac{4-\alpha}{2(3-\alpha)(2-\alpha)^2} - \sigma \frac{1}{(2-\alpha)^2}\right| \geq \frac{1}{2(3-\alpha)}$, then we use $|\wp_1| \leq 2$ and get

$$|b_3 - \sigma b_2^2| \leq \left|\frac{2(4-\alpha)}{(3-\alpha)(2-\alpha)^2} - \sigma \frac{4}{(2-\alpha)^2}\right| \tag{18}$$

Therefore from inequalities (17) and together with (18). The proof is complete.

If we put $\alpha = 1$ in Theorem 2.2, we get the Corollary [8].

3. Fekete and Szego [1] Inequality for the Class $\mathfrak{I}$ in Several Variables.

Theorem 3.1: *A assume that the mapping* $\mathfrak{f}(t)$ *belongs to the space* S . *Then the mapping* $\mathcal{G}$ *defined by* $\mathcal{G}(t) = \frac{t\mathfrak{f}(\mathcal{T}_u(t))}{\mathcal{T}_u(t)}$, *where* $\|u\| = 1$ *is contained in* $\mathfrak{I}$ *(*$\breve{\mathfrak{B}}$ *) iff* $\mathfrak{f}(t) \in \mathfrak{I}$.

Proof: Define $\mathcal{J}(t) = \frac{\mathfrak{f}(\mathcal{T}_u(t))}{\mathcal{T}_u(t)}$. Given that $\mathcal{G}(t) = t\,\mathcal{J}(t)$, it follows that

$$\zeta\mathfrak{D}(\mathcal{G}(t)) = t(\zeta\mathfrak{D}\mathcal{J}(t)) + \zeta\,\mathcal{J}(t)\,, \zeta \in \Gamma \tag{19}$$

After minimal calculation, it emerges that

$$\zeta\mathfrak{D}(\mathcal{G}(t)) = \mathcal{J}(t)\left[\zeta + \frac{t(\zeta\mathfrak{D}\mathcal{J}(t))}{\mathcal{J}(t)}\right] \tag{20}$$

We conclude that

$$\zeta\left(\mathfrak{D}(\mathcal{G}(t))\right)^{-1} = \frac{1}{\mathcal{J}(t)}\left[\zeta + \frac{t\left(\frac{\zeta\mathfrak{D}\mathcal{J}(t)}{\mathcal{J}(t)}\right)}{1+\frac{t\mathfrak{D}\mathcal{J}(t)}{\mathcal{J}(t)}}\right] = \frac{1}{\mathcal{J}(t)}\left[\zeta + \frac{t(\zeta\mathfrak{D}\mathcal{J}(t))}{\mathfrak{D}(\mathcal{G}(t))}\right] \tag{21}$$

$$\frac{t\left(\mathfrak{D}(\mathcal{G}(t))\right)^{-1}\mathcal{G}^{\alpha}(t)}{t^{\alpha}} = \frac{t\,\mathfrak{f}^{\alpha}(\mathcal{T}_u(t))}{\mathfrak{f}'(\mathcal{T}_u(t))\mathcal{T}_u^{\alpha}(t)}. \tag{22}$$

Hence, by Definition 1.2, we conclude that $\mathcal{G} \in \mathfrak{I}(\breve{\mathfrak{B}})$ if and only if $\mathfrak{f} \in \mathfrak{I}$.

Theorem 3.2: *A assume that* $\mathfrak{f}(t) \in \mathfrak{I}(\breve{\mathfrak{B}})$ *and*

$$\left(\frac{\alpha-2}{\alpha-3}\right)\mathcal{T}_t\left(\mathfrak{D}^2\mathfrak{f}(0)\left(t,\frac{\mathfrak{D}^2\mathfrak{f}(0)(t^2)}{2!}\right)\right)\|t\| = \frac{\alpha}{2}\left(\frac{\mathcal{T}_t(\mathfrak{D}^2\mathfrak{f}(0)(t^2))}{2!}\right)^2, t \in \breve{\mathfrak{B}}, \mathcal{T}_t \in \mathcal{T}(t) \quad (23)$$

Then

$$\left|\frac{\mathcal{T}_t(\mathfrak{D}^3\mathfrak{f}(0)(t^3))}{3!\,\|t\|^3} - \sigma\left(\frac{\mathcal{T}_t(\mathfrak{D}^2\mathfrak{f}(0)(t^2))}{2!\,\|t\|^2}\right)^2\right| \le \frac{2}{(3-\alpha)}\max\left\{1, \left|\frac{(4-\alpha)}{(2-\alpha)^2} - \sigma\,\frac{2(3-\alpha)}{(2-\alpha)^2}\right|\right\}.$$

Proof: Let t belongs to the set $\breve{\mathfrak{B}}\setminus\{0\}$ refer $t_0 = \frac{t}{\|t\|}$ and $\wp: \mathfrak{U} \to \mathbb{C}$ which is given by

$$\wp(\eta) = \begin{cases} \frac{T_{x_0}(\mathcal{J}(\eta t_0))}{\eta} & \eta \ne 0, \\ 1 & \eta = 0 \end{cases} \quad (24)$$

where $\mathfrak{I} = t^{1-\alpha}\left(\mathfrak{D}\big(\mathcal{G}(t)\big)\right)^{-1}\mathcal{G}^{\alpha}(t)$. Then $\wp \in B, \wp(0) = 1$ and

$$\wp(\eta) = 1 + \frac{T_{t_0}(\mathfrak{D}^2\mathcal{J}(0)(t_0^2))\eta}{2!} + \cdots + \frac{T_{t_0}(\mathfrak{D}^k\mathcal{J}(0)(t_0^k))\eta^{k-1}}{k!} + \cdots \quad (25)$$

Since $\mathfrak{f} \in \mathfrak{I}(\breve{\mathfrak{B}})$, we have from Definition 1.1. $Re\big(\wp(\eta)\big) > 0$, and Lemma 2.1

$$\left|\frac{\|t\|\mathcal{T}_t(\mathfrak{D}^3\mathcal{J}(0)(t^3))}{3!} - \frac{1}{2}\left(\frac{\mathcal{T}_t(\mathfrak{D}^2\mathcal{J}(0)(t^2))}{2!}\right)^2\right| \le 2\|t\|^4 - \frac{1}{2}\left|\frac{\mathcal{T}_t\big(\mathfrak{D}^2\mathcal{J}(0)(t^2)\big)}{2!}\right|^2 \quad (26)$$

From another perspective, since

$$\mathcal{J}(t) = \frac{t^{1-\alpha}\mathcal{G}^{\alpha}(t)}{\mathfrak{D}(\mathcal{G}(t))}, \quad (27)$$

Through homogeneous expansion of the dual-side in (27), we extract the polyvalent coefficients

$$(\alpha-2)\ \frac{\mathfrak{D}^2\mathfrak{f}(0)(t^2)}{2!} = \frac{\mathfrak{D}^2\mathcal{J}(0)(t^2)}{2!} \quad (28)$$

$$\alpha\frac{\mathfrak{D}^3\mathfrak{f}(0)(t^3)}{3!} = \frac{\mathfrak{D}^3\mathcal{J}(0)(t^3)}{3!} + \frac{\mathfrak{D}^3\mathfrak{f}(0)(t^3)}{2!} + \mathfrak{D}^2\mathfrak{f}(0)\left(t,\frac{\mathfrak{D}^2\mathcal{J}(0)(t^2)}{2!}\right). \quad (29)$$

We can rewrite equation (29) as follows

$$(\alpha-3)\frac{\mathfrak{D}^3\mathfrak{f}(0)(t^3)}{3!} = \frac{\mathfrak{D}^3\mathcal{J}(0)(t^3)}{3!} + (\alpha-2)\,\mathfrak{D}^2\mathfrak{f}(0)\left(t,\frac{\mathfrak{D}^2\mathfrak{f}(0)(t^2)}{2!}\right). \quad (30)$$

Thus from (28) and (30), we get $\left|\frac{\mathcal{T}_t(\mathfrak{D}^3\mathfrak{f}(0)(t^3))\|t\|}{3!} - \sigma\left(\frac{\mathcal{T}_t(\mathfrak{D}^2\mathfrak{f}(0)(t^2))}{2!}\right)^2\right|$

$$= \frac{1}{(\alpha-3)}\left|-\ \frac{\mathcal{T}_t\big(\mathfrak{D}^3\mathcal{J}(0)(t^3)\big)}{3!} - \frac{1}{2}\left(\frac{\mathcal{T}_t\big(\mathfrak{D}^2\mathcal{J}(0)(t^2)\big)}{2!}\right)^2 + \left(\frac{4-\alpha}{2(2-\alpha)^2} - \sigma\frac{(3-\alpha)}{(2-\alpha)^2}\right)\left(\frac{\mathcal{T}_t\big(\mathfrak{D}^2\mathcal{J}(0)(t^2)\big)}{2!}\right)^2\right|.$$

Now, from equation (26), we have

$$= \frac{1}{(\alpha-3)}\left(2\|t\|^4 - \frac{1}{2}\left|\frac{\mathcal{T}_t\big(\mathfrak{D}^2\mathcal{J}(0)(t^2)\big)}{2!}\right|^2 + \left|\frac{4-\alpha}{2(2-\alpha)^2} - \sigma\frac{(3-\alpha)}{(2-\alpha)^2}\right|\left|\frac{\mathcal{T}_t\big(\mathfrak{D}^2\mathcal{J}(0)(t^2)\big)}{2!}\right|^2\right).$$

If $\left|\frac{4-\alpha}{2(3-\alpha)(2-\alpha)^2}-\sigma\frac{1}{(2-\alpha)^2}\right|\leq\frac{1}{2(3-\alpha)}$, then $\left|\frac{\mathcal{T}_t(\mathfrak{D}^3\mathfrak{f}(0)(t^3))\|t\|}{3!}-\sigma\left(\frac{\mathcal{T}_t(\mathfrak{D}^2\mathfrak{f}(0)(t^2))}{2!}\right)^2\right|\leq$

$$\frac{1}{(3-\alpha)}\left(2\|t\|^4-\frac{1}{2}\left|\frac{\mathcal{T}_t\left(\mathfrak{D}^2\mathcal{J}(0)(t^2)\right)}{2!}\right|^2+\frac{1}{2}\left|\frac{\mathcal{T}_t\left(\mathfrak{D}^2\mathcal{J}(0)(t^2)\right)}{2!}\right|^2\right)=\frac{2}{(3-\alpha)}\|t\|^4. \tag{31}$$

From another perspective. If $\left|\frac{4-\alpha}{2(3-\alpha)(2-\alpha)^2}-\sigma\frac{1}{(2-\alpha)^2}\right|\geq\frac{1}{2(3-\alpha)}$, then use $\frac{\mathcal{T}_t\left(\mathfrak{D}^2\mathcal{J}(0)(t^2)\right)}{2!}\leq 2\|t\|^2$. Then

$$\left|\frac{\mathcal{T}_t(\mathfrak{D}^3\mathfrak{f}(0)(t^3))\|t\|}{3!}-\sigma\left(\frac{\mathcal{T}_t(\mathfrak{D}^2\mathfrak{f}(0)(t^2))}{2!}\right)^2\right|\leq$$

$$\frac{2}{(3-\alpha)}\|t\|^4+\frac{1}{(3-\alpha)}\left(\left|\frac{4-\alpha}{2(2-\alpha)^2}-\sigma\frac{(3-\alpha)}{(2-\alpha)^2}\right|-\frac{1}{2}\right)\left|\frac{\mathcal{T}_t\left(\mathfrak{D}^2\mathcal{J}(0)(t^2)\right)}{2!}\right|^2$$

$$=\left|\frac{2(4-\alpha)}{(3-\alpha)(2-\alpha)^2}-\sigma\frac{4}{(2-\alpha)^2}\right|\|t\|^4\,. \tag{32}$$

Therefore from inequalities (31) and (32), the proof is complete.

If we put $\alpha=1$ in Th 3.2, and get Coro. [9].

Theorem 3.3: *Assume* $\mathfrak{f}(w)\in\mathfrak{J}(\mathfrak{U}^n)$ *&*

$$\left(\frac{\alpha-2}{\alpha-3}\right)\mathfrak{D}^2\mathfrak{f}_m(0)\left(w_0,\frac{\mathfrak{D}^2\mathfrak{f}(0)(w_0^2)}{2!}\right)\frac{w_m}{\|w\|}=\frac{\alpha}{2}\left(\frac{\mathfrak{D}^2\mathfrak{f}_m(0)(w_0^2)}{2!}\right)^2,w\in\mathfrak{U}^n \tag{33}$$

for $w\in\mathfrak{U}^n\setminus\{0\}$, where $m=1,2,3,\dots,n\,,w_0=\frac{w}{\|w\|}$. Then

$$\left\|\frac{\mathfrak{D}^3\mathfrak{f}(0)(w^3)}{3!}-\sigma\left(\frac{\mathfrak{D}^2\mathfrak{f}(0)(w^2)}{2!}\right)^2\right\|\leq\frac{2\|w\|^3}{(3-\alpha)}\max\left\{1,\left|\frac{(4-\alpha)}{(2-\alpha)^2}-\sigma\frac{2(3-\alpha)}{(2-\alpha)^2}\right|\right\},\ \sigma\in\mathbb{C}.$$

Proof: For any $w\in\mathfrak{U}^n\setminus\{0\}$ and denote $w_0=\frac{w}{\|w\|}$ and $\wp_i:\mathfrak{U}\to\mathbb{C}$ which is given by

$$\wp_i(\eta)=\begin{cases}\frac{\mathcal{J}_i(\eta w_0)\|w\|}{\eta w_i} & \eta\neq 0,\\ 1 & \eta=0\end{cases} \tag{34}$$

Then $\wp_i\in B,\wp_i(0)=1$ and

$$\wp_i(\eta)=1+\frac{\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)\|w\|\eta}{2!w_i}+\frac{\mathfrak{D}^3\mathcal{J}_i(0)(w_0^3))\|w\|\,\eta^2}{3!w_i}+\cdots.$$

Since $\in\mathfrak{J}(\mathfrak{U}^n)$, by 1.3, deduce that $Re\left(\wp_i(\eta)\right)>0,\eta\in\mathfrak{U}$.

Also we obtain from view Lemma 2.1.

$$\left|\frac{\mathfrak{D}^3\mathcal{J}_i(0)(w_0^3)\|w\|}{3!\,w_i}-\frac{1}{2}\left(\frac{\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right)^2\right|\leq 2-\frac{1}{2}\left|\frac{\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right|^2.$$

Hence, in view of (28) and (30) of Theorem 3.2 , we obtained

$$\left|\frac{\mathfrak{D}^3\mathfrak{f}_i(0)(w_0^3)\|w\|}{3!\,w_i}-\sigma\left(\frac{\mathfrak{D}^2\mathfrak{f}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right)^2\right|=$$

$$\left|\frac{1}{(\alpha-3)}\frac{\mathfrak{D}^3\mathcal{J}_i(0)(w_0^3)\|w\|}{3!\,w_i}+\left(\frac{\alpha-2}{\alpha-3}\right)\mathfrak{D}^2\mathfrak{f}_i(0)\left(w,\frac{\mathfrak{D}^2\mathfrak{f}(0)(w_0^2)}{2!}\right)\frac{\|w\|}{w_i}-\sigma\left(\frac{\mathfrak{D}^2\mathfrak{f}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right)^2\right|.$$

We applied the hypothesis in equation (33)

$$=\frac{1}{(\alpha-3)}\left(2-\frac{1}{2}\left|\frac{\|w\|\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)}{2!\,w_i}\right|^2+\left|\frac{4-\alpha}{2(2-\alpha)^2}-\sigma\frac{(3-\alpha)}{(2-\alpha)^2}\right|\left|\frac{\|w\|\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)}{2!\,w_i}\right|^2\right)$$

If $\left|\frac{4-\alpha}{2(3-\alpha)(2-\alpha)^2}-\sigma\frac{1}{(2-\alpha)^2}\right|\leq\frac{1}{2(3-\alpha)}$, then $\left|\frac{\|w\|\mathfrak{D}^3\mathfrak{f}_i(0)(w_0^3)}{3!\,w_i}-\sigma\left(\frac{\|w\|\mathfrak{D}^2\mathfrak{f}_i(0)(w_0^2)}{2!\,w_i}\right)^2\right|$

$$\leq\frac{2}{(3-\alpha)} \tag{35}$$

From another perspective. If $\left|\frac{4-\alpha}{2(3-\alpha)(2-\alpha)^2}-\sigma\frac{1}{(2-\alpha)^2}\right|\geq\frac{1}{2(3-\alpha)}$, then we put $\left|\frac{\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right|\leq 2$. Therefore $\left|\frac{\mathfrak{D}^3\mathfrak{f}_i(0)(w_0^3)\|w\|}{3!\,w_i}-\sigma\left(\frac{\mathfrak{D}^2\mathfrak{f}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right)^2\right|$

$$=\frac{2}{3-\alpha}+\frac{1}{(3-\alpha)}\left(\left|\frac{4-\alpha}{2(2-\alpha)^2}-\sigma\frac{(3-\alpha)}{(2-\alpha)^2}\right|-\frac{1}{2}\right)\left|\frac{\mathfrak{D}^2\mathcal{J}_i(0)(w_0^2)\|w\|}{2!\,w_i}\right|^2$$

$$\leq\left|\frac{2(4-\alpha)}{(3-\alpha)(2-\alpha)^2}-\sigma\frac{4}{(2-\alpha)^2}\right|. \tag{36}$$

Therefore from equations (35) and (36), the proof is complete.

4. Conclusion

This research has fundamentally advanced the theory of strongly starlike mappings through the introduction and comprehensive analysis of a new class in complex space. By employing an innovative synthesis of variational methods and geometric function theory, we have successfully characterized the precise coefficient bounds for this class, revealing intricate patterns in their Taylor expansions that distinguish them from conventional starlike mappings. Our work establishes the sharp Fekete and Szegö [1] inequality across three distinct spatial configurations: the classical open unit disk $\mathfrak{U}$.

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