

A fixed point result in menger probabilistic b-partial metric space with an application

Hadeel Hussein Luaibi *

Salwa Salman Abed [†]

Department of Mathematics

College of Education for Pure Science (Ibn Al-Haitham)

University of Baghdad

Baghdad

Iraq

Abstract

In this article, the concept of Menger probabilistic b-partial metric spaces (or random partial b-metric spaces) are recalled and some concerning triangle functions. In the framework of such spaces, a (b, ψ, λ) -admissible and type of (b, ψ, λ) Contractive mappings are presented in this space to prove a new general fixed point theorem in that special sense. Finally, it has been enhanced by applying it to solve a delay Volterra integral equation in $C[a, b]$, as a complete Menger probabilistic b-partial metric space.

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1. Introduction and Preliminaries

Several researches have extended generalized contraction principle established by Banach by expanding the background spaces [1-3]. These resulting theorems have diversified to include the study of existence of fixed points, common and coincidence points, coupled points and their related subjects, see [4-6]. As a usual generalization of the concept for the metric spaces, Menger 2016, created an idea for the statistical metric spaces, in which the distance between w and κ was substituted by a distribution function $F_{w,\kappa}$ where $F_{w,\kappa}(a)$ Translated as the probability that the distance between w and κ is less than a [6]. The main problem in transferring metric notions in the

* E-mail: hadeel.h.l@ihcoedu.uobaghdad.edu.iq (Corresponding Author)

[†] E-mail: salwa.s.a@ihcoedu.uobaghdad.edu.iq;

salwaalbundi@yahoo.com

probabilistic settings conceived the triangle inequality that is a natural characteristic of the usual metric. Menger passed this difficult defining a procedure Y on the unit interval $[0, 1]$ with the triangular norm feature since this procedure requires only commutativity, monotonicity, $Y(x, 1) > 0$ for $x > 0$, and $Y(1, 1) = 1$ (dispensing with the unit element and associativity). So, Menger's probabilistic shape of the triangle inequality was: $M_{c,h}(g + u) \geq Y(M_{c,s}(g), M_{s,h}(u))$ for all c, s, h and $u, g \in \mathfrak{R}^+$ [7]. Then the results followed in this type of mathematical structures, for example [8]. In the following, recalling basic requirements for this work. Recently, research has emerged with diverse objectives related to applications in probabilistic logic, fuzzy set theory, uncertainty modeling in analysis, fixed-point theory with inherent randomness. The authors [9] worked in the metrization of probabilistic spaces. The authors [10] defined a generalization of inner product space named Menger probabilistic inner product space and then gave some convergence results, probabilistic Pythagorean Theorem and fixed points theorems of the nonlinear contraction in these spaces. In Menger probabilistic metric space authors [11] works to investigated the fixed points theorems of multi-value maps using amorphous binary relations. One can also refer to the sources approved in the above studies to see other results. Now, for the purpose of this paper, we start with the following:

Definition 1.1: [12]. Let $\mathcal{W} \neq \emptyset$ and $\mathcal{C}: \mathcal{W} \times \mathcal{W} \rightarrow \mathfrak{R}^+$ be a function such that for all $v, \xi, \varpi \in \mathcal{W}$ and $g \geq 1$

- i. $\mathcal{C}(\xi, \xi) \leq \mathcal{C}(\xi, \varpi)$;
- ii. if $\mathcal{C}(\xi, \xi) = \mathcal{C}(\xi, \varpi) = \mathcal{C}(\varpi, \varpi)$ then, $\xi = \varpi$;
- iii. $\mathcal{C}(\xi, \varpi) = \mathcal{C}(\varpi, \xi)$;
- iv. $\mathcal{C}(\xi, v) \leq g(\mathcal{C}(\xi, \varpi) + \mathcal{C}(\varpi, v)) - \mathcal{C}(\varpi, \varpi)$.

A pair $(\mathcal{W}, \mathcal{C})$ is called a g -partial metric space.

Several important research have shown that such spaces are not necessarily Hausdorff. However, they have been proved to provide T_0 topological spaces. To demonstrate the generality of this space, the Figure 1 below depict the links between specific places.

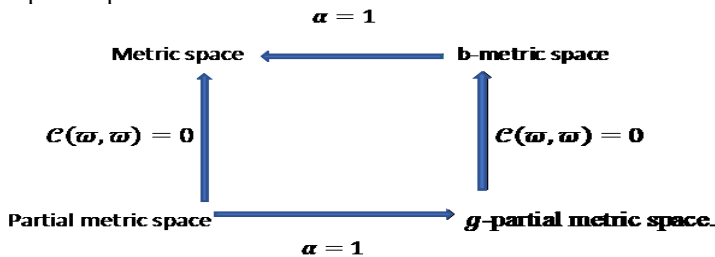


Figure 1
Relation between spaces

Definition 1.2 [13]: Let $(\mathcal{W}, \mathcal{C})$ denote a g-partial metric space. Then,

- i. If $\mathcal{C}(\xi, \xi) = \lim_{n \rightarrow \infty} \mathcal{C}(\xi_n, \xi)$ then the sequence $\{\xi_n\}$ in $\mathcal{W}$ converges to $\xi \in \mathcal{W}$;
- ii. A sequence $\{\xi_n\}$ in $(\mathcal{W}, \mathcal{C})$ is said to be a Cauchy if $\mathcal{C}(\xi, \xi) = \lim_{n, m \rightarrow \infty} \mathcal{C}(\xi_n, \xi_m)$ exists;
- iii. $(\mathcal{W}, \mathcal{C})$ be the complete where each Cauchy sequences $\{\xi_n\}$ in the $\mathcal{W}$ converge to $\xi \in \mathcal{W}$ such that $\mathcal{C}(\xi, \xi) = \lim_{n, m \rightarrow \infty} \mathcal{C}(\xi_n, \xi_m)$.

Definition 1: [13]. Let $Y: [0, 1]^3 \rightarrow [0, 1]$ be a mapping is said to be an 3-th order a -norm if

- a. $Y(\omega, 1, 1) = \omega, 0 = Y(0, 0, 0)$.
- b. $Y(\omega_1, \omega_2, \omega_3) = Y(\omega_2, \omega_3, \omega_1) = Y(\omega_3, \omega_1, \omega_2)$.
- c. $Y(x_1, x_2, x_3) \leq Y(\omega_1, \omega_2, \omega_3)$ if $x_t \leq \omega_t, t = 1, 2, 3$.
- d. $Y(Y(\omega_1, \omega_2, \omega_3), x_2, x_3) = Y(\omega_1, Y(\omega_2, \omega_3, x_2), x_3) = Y(\omega_1, \omega_2, Y(\omega_3, x_2, x_3))$.

Throughout this paper minimum 3-th order a -norm, $Y_m(c, h, s) = \min\{c, h, s\}$ is used and denoted by Y_m .

Definition 1.4: [14]. The nondecreasing left continuous function $M: \mathbb{R} \rightarrow \mathbb{R}^+$ is called distribution if $\sup M(a) = 1$ and $\inf M(a) = 0, a \in \mathbb{R}$.

Let $\mathbb{X} :=$ set of all distribution functions, with one of them defined as

$$K(a) = \begin{cases} 1, & a > 0 \\ 0, & a \leq 0 \end{cases}$$

Definition 1.5: [15]. A function $\Omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a Δ -function if

1. Ω is continuous at 0
2. $\Omega(a) \rightarrow \infty$ where $a \rightarrow \infty$ and $\Omega(a)$ is strictly monotone nondecreasing
3. $\Omega(a) = 0 \Leftrightarrow a = 0$
4. Ω is left continuous in $\mathbb{R}_+$

Denote $\Delta :=$ the class of all Ω -function.

Definition 1.6: [16]. Let $\Theta \neq \emptyset$ and $M: \Theta \times \Theta \rightarrow \mathbb{X}, (c, h) \rightarrow M_{c, h}, \alpha \geq 1$ the pair (Θ, M) is called probabilistic b -partial metric space (short, probabilistic P_b -MS) if:

- 1) $M_{c, h}(a) = M_{h, h}(a) = M_{c, c}(a) \Leftrightarrow c = h$
- 2) $M_{c, h}(a) = M_{h, c}(a)$
- 3) $M_{c, h}(a) \leq M_{h, h}(a)$
- 4) For $a_1, a_2, a_3 \in (1, \infty)$, if $M_{c, s}(a_1) = M_{s, h}(a_2) = M_{s, s}(a_3) = 1$ then $M_{c, h}(a_1, a_2, a_3) = 1$.

Definition 1.7: A probabilistic P_b -MS $(\Theta, \mathbb{M}, Y)$ is called Menger if the inequality

$$\mathbb{M}_{c,h}(\alpha(a_1, a_2, a_3)) \geq Y(\mathbb{M}_{c,s}(a_1), \mathbb{M}_{s,h}(a_2), \mathbb{M}_{s,s}(a_3)) \quad (1)$$

holds, where Y is a continuous 3-th order α -norm with $\alpha \geq 1$.

Note: $(\Theta, \mathbb{M}, Y)$ named Menger probabilistics b -partial metric spaces (shortly, M.P. P_b -MS)

Example 1.8: A b -partial metric space is $(\Theta, \mathcal{B})$ and defined the mapping $\mathbb{M} : \Theta \times \Theta \rightarrow \mathbb{X}$ by $\mathbb{M}_{c,h}(a) = K(a - \mathcal{B}(c, h))$, $\forall h, c \in \Theta$.

Definition 1.9: Let $(\Theta, \mathbb{M}, Y)$ be a M.P. P_b -MS Then

- $\langle c_n \rangle$ be a sequence in Θ is called a converge to c , if for each $a > 0$, $\lim_{n \rightarrow \infty} Y_{c_n, c}(a) = Y_{c, c}(a)$.
- If for each $a > 0$, $\lim_{n \rightarrow \infty} Y_{c_n, c_n}(a)$ exists then $\langle c_n \rangle$ is known as a Cauchy sequence.
- If $\forall \langle c_n \rangle$ Cauchy sequence in Θ , $\exists c \in \Theta \ni \lim_{n \rightarrow \infty} Y_{c_n, c_n}(a) = \lim_{n \rightarrow \infty} Y_{c, c_n}(a) = \lim_{n \rightarrow \infty} Y_{c, c}(a)$ It is claimed to be complete M.P. P_b -MS.

We reform the definition (ψ, λ) -admissible [12] in probabilistic P_b -MS case.

Definition 1.10: [17] Let $\Theta \neq \emptyset$, a mapping $T: \Theta \rightarrow \Theta$ and $\psi, \lambda: \Theta^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are functions. T is called (b, ψ, λ) -admissible if $\forall c, h \in \Theta, \forall a > 0$ we get

$$\psi(c, h, a) \leq 1, \psi(T_c, T_h, a) \leq 1$$

$$\lambda(c, h, a) \geq 1, \lambda(T_c, T_h, a) \geq 1.$$

Lemma 1.11: Let Ω be a Δ function and $(\Theta, \mathbb{M}, Y)$ is M.P. P_b -MS. If $c, h \in \Theta, \alpha \geq 1$ and $\lambda \in (0, 1)$, $\mathbb{M}_{c,h}\Omega(a) \geq \mathbb{M}_{c,h}\Omega\left(\frac{a}{\alpha}\right)$, $\forall a > 0$ then $c = h$.

Definition 1.12: Let $(\Theta, \mathbb{M}, Y)$ be a M.P. P_b -MS, Q is a self-mapping on Θ . Q is called type of (b, ψ, λ) -contractive mapping if the functions $\psi, \lambda: \Theta^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are exists such that for all $c, h \in \Theta, \forall j \in (0, 1), \alpha \geq 1$ and $\Omega \in \Delta$.

$$\psi(c, h, a) \mathbb{M}_{Qc, Qh} \Omega(a) \leq$$

$$\lambda\left(Qc, Qh, \frac{a}{\alpha}\right) \min \left\{ \begin{array}{l} \mathbb{M}_{c,h} \Omega\left(\frac{a}{\alpha}\right), \mathbb{M}_{c, Qc} \Omega\left(\frac{a}{\alpha}\right), \mathbb{M}_{h, Qh} \Omega\left(\frac{a}{\alpha}\right), \mathbb{M}_{h, Qc} \Omega\left(\frac{a}{\alpha}\right), \\ \mathbb{M}_{Q^2c, Qc} \Omega\left(\frac{a}{\alpha}\right), \mathbb{M}_{Q^2c, h} \Omega\left(\frac{a}{\alpha}\right), \mathbb{M}_{Q^2c, Qh} \Omega\left(\frac{a}{\alpha}\right) \end{array} \right\}.$$

Theorem 1.13: Let $(\Theta, \mathbb{M}, Y)$ be a complete M.P. P_b -MS and $Q: \Theta \rightarrow \Theta$ be a type of (b, ψ, λ) -contractive mapping and fulfill the following:

- Q is (b, ψ, λ) -admissible

- ii. Assume $c_0 \in \Theta \ni \lambda(c_0, Q_{c_0}, a) \geq 1$ and $\psi(c_0, Q_{c_0}, a) \leq 1$
- iii. For each $\langle c_n \rangle \in \Theta$ such that $\lambda(c_n, c_{n+1}, a) \geq 1$ and $\psi(c_{n-1}, c_n, a) \leq 1$ and $\lambda(c_z, c_n, a) \geq 1$, $\psi(c_{z-1}, c_{n-1}, a) \leq 1, \forall n, z \in \mathbb{N}, x_0 \in \mathbb{N}, z > n > x_0$
- iv. IF $\langle c_n \rangle \in \Theta$ such that $\lambda(c_n, c_{n+1}, a) \geq 1$ and $\psi(c_{n-1}, c_n, a) \leq 1$ and $c_n \rightarrow c$ afterward, $\lambda(c_n, Q_c, a) \geq 1$, $\psi(c_{n-1}, c_n, a) \leq 1$.

Then Q has a fixed point.

Proof: Assume $c_0 \in \Theta$ be such that (II) holds and $c_{n+1} = Q_{c_n}$ for $n=0,1,\dots$ we suppose that $c_{n+1} \neq c_n$ we get Q has a fixed point. From (I), (II) implicit $\lambda(c_n, c_{n+1}, a) \geq 1$ and $\psi(c_{n-1}, c_n, a) \leq 1$. By continuity of Ω at zero, $\forall a, l \geq 0 \ni a > \Omega(l)$. From (2), definition 1.6 (3) getting

$$\begin{aligned}
 & M_{c_n, c_{n+1}}(a) \geq \psi(c_{n-1}, c_n, l) M_{Q_{c_{n-1}}, Q_{c_n}}(\Omega(l)) \\
 & \geq \lambda\left(c_n, c_{n+1}, \frac{l}{\alpha j}\right) \min \left\{ \begin{aligned} & M_{c_{n-1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_{n-1}, Q_{c_{n-1}}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_n, Q_{c_n}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \\ & , M_{c_n, Q_{c_{n-1}}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{Q^2_{c_{n-1}}, Q_{c_{n-1}}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \\ & , M_{Q^2_{c_{n-1}}, Q_{c_n}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{Q^2_{c_{n-1}}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \end{aligned} \right\} \\
 & \rightarrow \lambda\left(c_n, c_{n+1}, \frac{l}{j}\right) \min \left\{ \begin{aligned} & M_{c_{n-1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_{n-1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_n, c_{n+1}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_n, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \\ & M_{c_{n+1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_{n+1}, c_{n+1}}\left(\Omega\left(\frac{l}{\alpha j}\right)\right), M_{c_{n+1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \end{aligned} \right\} \\
 & \rightarrow M_{c_n, c_{n+1}}(a) \geq M_{c_{n-1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \\
 & M_{c_n, c_{n+1}}(a) \geq M_{c_{n-1}, c_n}\left(\Omega\left(\frac{l}{\alpha j}\right)\right) \geq \dots \geq M_{c_0, c_1}\left(\Omega\left(\frac{l}{(\alpha j)^n}\right)\right)
 \end{aligned}$$

We have $M_{c_n, c_{n+1}}(a) \rightarrow 1$, when $n \rightarrow \infty$

Now, we prove $\langle c_n \rangle$ is Cauchy sequence. Assume $\langle c_n \rangle$ is not Cauchy sequence for any $\delta < 0, r \in (0, 1), \exists \langle c_{n(t)} \rangle$ and $\langle c_{m(t)} \rangle$ of $\langle c_n \rangle \ni m(t) > n > m(t)$ $M_{c_{m(t)}, c_{n(t)}}(\delta) \leq 1 - r$ and $M_{c_{m(t)}, c_{n(t)-1}}(\delta) > 1 - r$

From definition 1.5 of $\Omega(a)$ (1, 2), $\exists \delta_1 > 0$ then $\Omega(\delta_1) < \delta$

We conclude that $M_{c_{m(t)}, c_{n(t)}}\Omega(\delta_1) \leq 1 - r$ so $\langle c_n \rangle$ is not Cauchy sequence.

Then, because of this increasing sequence of integers $m(t)$ and $n(t)$ can be obtained using the a forementioned arguments with $t > m(t) > n(t)$, so that

$$M_{c_{m(t)}, c_{n(t)}}\Omega(\delta_1) \leq 1 - r \text{ and } M_{c_{m(t)}, c_{n(t)-1}}\Omega(\delta_1) > 1 - r$$

Let $k > 0, k \in R$ then $0 < k < \Omega\left(\frac{\delta_1}{\alpha_j}\right) - \Omega(\delta_1)$ and by using (5) consequently

$$M_{c_{m(t)}, c_{n(t)-1}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right) - \Omega(\delta_1)\right) > M_{c_{m(t)}, c_{n(t)-1}}\Omega(\delta_1) > 1 - r$$

For any $0 < r_1 < r < 1$, (1.4) can discover a positive number $N_1 \ni \forall t > N_1$

Our possessions

$$M_{c_{m(t)-1}, c_{m(t)}}\Omega(k) > 1 - r_1, M_{c_{n(t)}, c_{n(t)-1}}\Omega(k) > 1 - r_1$$

$$\text{Implicit that } M_{c_{m(t)-1}, c_{m(t)}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right) \geq M_{c_{m(t)-1}, c_{m(t)}}\Omega(k) > 1 - r_1 >$$

$1 - r$

$$\begin{aligned} 1 - r &\geq M_{c_{m(t)}, c_{n(t)}}(\Omega(\delta_1)) = M_{Q_{c_{m(t)-1}}, Q_{c_{n(t)-1}}}(\Omega(\delta_1)) \\ &\geq \psi(c_{m(t)-1}, c_{n(t)-1}, \delta_1) M_{Q_{c_{m(t)-1}}, Q_{c_{n(t)-1}}}(\Omega(\delta_1)) \\ &\geq \lambda\left(Q_{c_{m(t)-1}}, Q_{c_{n(t)-1}}, \frac{\delta_1}{\alpha_j}\right) \\ &\min \left\{ \begin{aligned} &M_{c_{m(t)-1}, c_{n(t)-1}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right), M_{c_{m(t)-1}, Q_{c_{m(t)-1}}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right) \\ &, M_{c_{n(t)-1}, Q_{c_{n(t)-1}}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right), M_{c_{n(t)-1}, Q_{c_{m(t)-1}}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right) \\ &M_{Q^2_{c_{m(t)-1}, Q_{c_{m(t)-1}}}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right), M_{Q^2_{c_{m(t)-1}, Q_{c_{n(t)-1}}}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right) \\ &, M_{Q^2_{c_{m(t)-1}, c_{n(t)-1}}}\left(\Omega\left(\frac{\delta_1}{\alpha_j}\right)\right) \end{aligned} \right\} \\ &> \min\{1 - r, \dots, 1 - r\} > 1 - r. \end{aligned}$$

Which is contradiction, then $\langle c_n \rangle$ is Cauchy sequence. The complete M.P. P_b -MS implies that $c_n \rightarrow c$

Now, to show that c is a fixed point, then

$$M_{Q_c, c}(2a) \geq Y\left(M_{Q_c, c}\Omega(l), M_{c_n, c_n}\Omega(l), M_{c_n, c}(2a - 2\Omega(l))\right)$$

Since Ω is continuous, $l > 0, \exists a > \Omega(l)$, then for $0 < b < 1 \exists d_0 \in N$ such that $\forall d_0 \leq n \rightarrow M_{c_n, c}(2a - 2\Omega(l)) \geq 1 - b$ (14)

so we conclude

$$M_{Q_c, c}(2a) \geq Y(M_{Q_c, c}\Omega(l), 1 - b, 1 - b). \text{ Since } Y \text{ is continuous and } 0 < b \rightarrow M_{Q_c, c}(2a) \geq M_{Q_c, c}\Omega(l).$$

From (IV, 2),

$$M_{c, Q_c}(2a) \geq M_{c_n, Q_c}\Omega(l) \geq \psi(c_{n-1}, c, l) M_{Q_{c_{n-1}}, Q_c}(\Omega(l))$$

$$\geq \lambda \left(Q_{c_{n-1}}, Q_c, \frac{l}{\alpha j} \right) \min \left\{ \begin{array}{l} M_{c_{n-1}, c} \left(\Omega \left(\frac{l}{j} \right) \right), M_{c_{n-1}, Q_{c_{n-1}}} \left(\Omega \left(\frac{l}{\alpha j} \right) \right), M_{c, Q_c} \left(\Omega \left(\frac{l}{\alpha j} \right) \right) \\ , M_{c, Q_{c_{n-1}}} \left(\Omega \left(\frac{l}{\alpha j} \right) \right), M_{Q^2_{c_{n-1}}, Q_{c_{n-1}}} \left(\Omega \left(\frac{l}{\alpha j} \right) \right) \\ , M_{Q^2_{c_{n-1}}, c} \left(\Omega \left(\frac{l}{\alpha j} \right) \right), M_{Q^2_{c_{n-1}}, Q_c} \left(\Omega \left(\frac{l}{\alpha j} \right) \right) \end{array} \right\}$$

Then $M_{c, Q_c}(\Omega(l)) \geq M_{c, Q_c} \left(\Omega \left(\frac{l}{\alpha j} \right) \right)$. From lemma 1.9, implicit that $c = Q_c$.

As special case when $\alpha = 1$ we give the following example.

Example 1.14: Let $\Theta = [0, 1]$ and (Θ, M, Y) is a complete M.P. P_b -MS. Let $M_{c, h}(a) = K(a - \mathcal{B}(c, h))$, such that $\mathcal{B}(c, h) = \max\{c, h\}$. Define $T: \Theta \rightarrow \Theta$ by

$$T(c) = \begin{cases} \frac{c}{8} & \text{if } c \in [0, \frac{1}{2}] \cup (\frac{1}{2}, 2] \\ 0 & \text{if } c = \frac{1}{2} \end{cases}$$

And $\psi, \lambda: \Theta^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\psi(c, h, a) = \frac{a + \max\{c, h\}}{t + \min\{c, h\}}$, $\lambda(c, h, a) = \frac{a}{h + c + a}$. Assume $j = \frac{1}{2}$, and consider $\Omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Define $\Omega(a) = a$. So by assumption for the theorem (1.11) be satisfies, therefore T have the fixed points.

2. Application:

Many studies including applications in equation, like in [15], [16] as well as here. Consider the delay Volterra integral

$$c(a) = w(a) + T \int_w^a F(a, z, c(z), c(z - a)) dz$$

$\exists a \in [w, e], T \geq 0$, with $c(a) = \delta(a)$, $a \in [w - a, w]$, $w, e \in \mathbb{R}$. Since $w: [w, e] \rightarrow \mathbb{R}$ is continuous and $F \in C([w, e]^2 \times \mathbb{R}^2, \mathbb{R})$.

Define a b -partial metric on $C([w, e], \mathbb{R})$,

$$\mathcal{B}(c, h) = \max\{|c|, |h|\} + A, c, h \in C([w, e], \mathbb{R})$$

The mapping $M: C([w, e], \mathbb{R}) \times C([w, e], \mathbb{R}) \rightarrow \mathbb{R}$ define by

$$M_{c, h}(a) = K(a - \mathcal{B}(c, h)).$$

Where $\Theta = C([w, e], \mathbb{R})$, then (Θ, M, Y_m) is complete M.P. P_b .M.S.

Theorem 2.1: Let $w \in C([w, e], \mathbb{R})$ & $F \in C([w, e]^2 \times \mathbb{R}^2, \mathbb{R})$ satisfying the following

1. $\exists g \in (0, 1), \forall a \in C([w, e], \mathbb{R}), |F(a, z, c(z), c(z - a))| \leq g(|c(z) - c(z - a)|)$
2. $v = T.g < 1$

$$3. \quad \|F\| = \sup |F(a, z, c(z), c(z-a))| < \infty$$

$$4. \quad |c(z) - c(z-a)| < |c(z)|$$

Then equation 2.1 has solution $c^* \in C([w, e], \mathbb{R})$.

Proof: $\forall c, h \in C([w, e], \mathbb{R})$ and by using (16), (18) then

$$\begin{aligned} & \mathcal{B}(Q_c, Q_h) = \max\{|Q_c|, |Q_h|\} + A = \\ & \max\left\{\left|w(a) + T \int_w^a F\left(\frac{a, z, c(z)}{c(z-a)}\right) dz\right|, \left|w(a) + T \int_w^a F\left(\frac{a, z, h(z)}{h(z-a)}\right) dz\right|\right\} + A \\ & \leq v \max\{\mathcal{B}(c, h), \mathcal{B}(c, Q_c), \mathcal{B}(h, Q_h), \mathcal{B}(h, Q_c), \mathcal{B}(h, Q_c^2), \mathcal{B}(Q_c, Q_c^2), \mathcal{B}(Q_h, Q_c^2)\} \\ & \quad \text{From 2.3, we get } M_{Q_c, Q_h}(a) = K(a - \mathcal{B}(Q_c, Q_h)) \\ & \geq K\left(\frac{a}{v} - \max\{\mathcal{B}(c, h), \mathcal{B}(c, Q_c), \mathcal{B}(h, Q_h), \mathcal{B}(h, Q_c), \mathcal{B}(h, Q_c^2), \mathcal{B}(Q_c, Q_c^2), \mathcal{B}(Q_h, Q_c^2)\}\right) \\ & \geq \min\left\{M_{c, h} \Omega\left(\frac{a}{av}\right), M_{c, Q_c} \Omega\left(\frac{a}{av}\right), M_{h, Q_h} \Omega\left(\frac{a}{av}\right), M_{h, Q_c} \Omega\left(\frac{a}{av}\right), M_{Q_c^2, Q_c} \Omega\left(\frac{a}{av}\right), \right. \\ & \quad \left. M_{Q_c^2, h} \Omega\left(\frac{a}{av}\right), M_{Q_c^2, Q_h} \Omega\left(\frac{a}{av}\right)\right\} \end{aligned}$$

Theorem 1.13 prerequisites are implicitly satisfied when $\Omega(a) = a$, $a > 0$ and $\psi(c, h, a) = \lambda(c, h, a) = 1$. Then $c^* = Q_c(a)$.

Conclusions

This search focuses on the accomplishment of presenting the concept of probabilistic P_b -MS as an extension of both probabilistic metric space and partial b-metric space, while also exploring its topological characteristics. Additionally, we introduce a ϕ -contraction real-valued function within this framework. Moreover, the fixed points are examined for mappings that fulfill these conditions in the newly established setting. To provide a supporting for the result, we present application in delay volterra integral equation in $C[a, b]$ space as a probabilistic P_b -MS.

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