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A class of new Bazilevič functions and symmetric Toeplitz matrices

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Abstract

In this work, we describe and investigate a novel class of analytic functions, including the new functions and the Bazilevič functions. Bazilevič functions are a crucial part of analytic functions and have many applications in pure and applied mathematics. We focus on constructing Toeplitz matrices, studying their structural properties, and assessing their eigenvalues and trends for this class of functions. The goal of this work is to calculate coefficient estimates for the functions in this family for the Toeplitz matrix's first four determinants, symmetric $T_2(2)$, $T_2(3)$, $T_3(1)$, and $T_3(2)$.

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1. Introduction

One of the main focuses of "geometric function theory" has remained the study of analytic and univalent functions in recent decades, which are holomorphic and injective in the open unit disk. Subclasses of the several classes of univalent functions that are distinguished by geometric qualities have received particular attention [1-2]. Matrix theory has been widely used in the study of analytic functions, particularly when determinants derived from Taylor-Maclaurin coefficients are used. Among these matrices, Toeplitz matrices and their variants have attracted special attention. A Toeplitz matrix is a structured matrix that has diagonals that consistently descend from left to right. In 1986, techniques and applications based on the conjugate gradient approach were proposed, initiating the study of Toeplitz matrices [3]. In 1996 they introduced more extensively Toeplitz matrices, asymptotic linear algebra, and functional analysis [4]. Radhika, Jahangiri, and Sivasubramanian [5] examined Toeplitz matrices with Bazelevich function coefficients as their constituents. In 2018, we obtained upper bounds for the first four determinants of Toeplitz matrices whose members represent the coefficients of Basilevich functions [6]. Studies on the coefficient functions of non-Basilevich functions were presented by scholars in 2022 [7]. Using various operators, a number of studies have examined Toeplitz matrices of a family of novel subfunctions and Bazelevich functions [8-10].

2. Materials and Methods

Let N represents the class of all functions f of the type.

$$f\ (\gamma) = \gamma + \sum_{h=2}^{\infty} a_h\, \gamma^h, (\gamma \in \mathcal{M}). \tag{1}$$

It is appropriate to represent the class of all functions in N that are univalent in M by the open unit disk $\mathcal{M} = \{\gamma \in \mathbb{C}: |\gamma| < 1\}$, which is analytical B. If $f \in N$ is a starlike function in B, then we can say that

$$\mathcal{R}e\left\{\frac{\gamma f\acute{(}\gamma)}{f(\gamma)}\right\} > 0, (\gamma \in \mathcal{M}).$$

Thus, it is stated that the function $f \in N$ is convex in $\mathcal{B}$ if

$$\mathcal{R}e\left\{\frac{\gamma(f\acute{(}\gamma))}{f(\gamma)} + 1\right\} > 0, (\gamma \in \mathcal{M}).$$

In particular, a Bazilevic function [11] in A is a function $f \in N$, if

$$\mathcal{R}e\left\{\frac{\gamma^{1-\tau}(f\acute{(}\gamma))}{(f(\gamma))^{1-\tau}}\right\} > 0, (\gamma \in \mathcal{M}, \tau \geq 0).$$

A fundamental subject in the theory of univalent functions, estimating bounds for Hankel matrices has significant applications in many branches of mathematics [11]. There is a strong relationship between Hankel and Toeplitz determinants: Toeplitz matrices have identical elements along every diagonal, but Hankel matrices have identical entries along every anti-diagonal. For $f \in N$, the determinant $T_q(n)$ is defined as follows.

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+2q-2} \cdots & a_n \end{vmatrix}, (n \geq 1, q \geq 1 \text{ and } a_1 = 1).$$

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}.$$

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

Toeplitz determinants appear across many areas of pure and applied mathematics, including time series analysis, signal processing, probability, quantum physics, and image processing. Toeplitz matrices are fundamental in physics, functional analysis, and technical sciences [11]. The following lemmas are used in our study to determine the desired bounds:

Lemma 1: [1] *The sharp bound* $|v_k| \leq 2(k = 123)$ *holds, if* $v \in Q$ *is defined by* $v(\gamma) = 1 + v_1\gamma + v_2\gamma^2 + v_3\gamma^3 + \cdots$*, and* Q *is a collection of all function r analytic in* $\mathcal{M}$ *for which* $\mathcal{R}e\{v(\gamma)\} > 0$.

Lemma 2: [1] *If* $v \in Q$*, then*
$2v_2 = {v_1}^2 + (4-{v_1}^2)r, 4v_3 = {v_1}^3 + 2(4-{v_1}^2)v_1 r - (4{v_1}^2)(1-|r|^2)\gamma,$
for some r, γ, with $|r| \leq 1, |\gamma| \leq 1$.

We investigate a fundamental problem by introducing a previously unstudied family of analytic functions denoted by $L(\beta,\tau)$. For this class, we obtain estimates for the Taylor–Maclaurin coefficients $a_2, a_3, and\ a_4$, as well as the first four Toeplitz determinants $T_2(2), T_2(3), T_3(2), and\ T_3(1)$.

3. Results and Discussion

A function $f \in N$ defined by Eq. (1) is said to be in the family $L(\beta,\tau)$ if the requirements below are met:

$$\mathcal{R}\left\{(1-\mathcal{B})\left(\frac{f(\gamma)}{\gamma}\right)^\tau + \beta\frac{\gamma(f\acute{}(\gamma))}{f(\gamma)}\left(\frac{f(\gamma)}{\gamma}\right)^\tau\right\} > 0, (0 \leq \beta \leq 1) \text{ and } \tau \geq 0.$$

Theorem 1: *If* $f(\gamma)$ *is define by Eq. (1) be longs to the class* $L(\beta,\tau)$*. Then,*

$$|a_2| \leq \frac{2}{\delta}, |a_3| \leq \frac{2}{\mu} - \frac{4\rho}{\mu\delta^2}, |a_4| \leq \frac{2}{E} - D\frac{4\delta^2 - 8\rho}{E\mu\delta^3} - \frac{8Z}{\delta^3 E}.$$

Where

$$\delta = (\tau+\beta), \mu = (\tau+2\beta), \rho = \frac{(\tau+2\beta)(\tau-1)}{2}, \mathrm{E} = (\tau+3\beta),$$

$$D = (\tau+3\beta)(\tau-1), \mathrm{Z} = \frac{(\tau-2)(\tau-1)(\tau+3\beta)}{6}. \quad (2)$$

Proof: Let $f \in L(\beta,\tau)$ Then, there is $v \in Q$ such that

$$(1-\beta)\left(\frac{f(\gamma)}{\gamma}\right)^{\tau}+\beta\frac{\gamma\left(\left(f\acute{(}\gamma)\right)\right)}{f(\gamma)}\left(\frac{f(\gamma)}{\gamma}\right)^{\tau}=v(\gamma), \tag{3}$$

where the following series represents v:

$$v(\gamma)=1+v_1\gamma+v_2\gamma^2+v_3\gamma^3+\cdots, v(0)=1.$$

In Eq. (3), when the coefficients are equated, the following relationships are obtained:

$$(\tau+\beta)a_2=v_{1,} \tag{4}$$

$$(\tau+2\beta)a_3+\frac{(\tau+2\beta)(\tau-1)}{2}a_2^2=v_2, \tag{5}$$

and

$$(\tau+3\beta)a_4+(\tau+3\beta)(\tau-1)a_2a_3+\frac{(\tau-2)(\tau-1)(\tau+3\beta)}{6}{a_2}^3=v_3. \tag{6}$$

When we simplify and take into account Eq. (4) to (6), we find that
After substituting (5) and $B(t) = 1 - A(t)$ in to (6), we get

$$|a_2|\leq\frac{v_1}{\delta}, \tag{7}$$

$$|a_3|\leq\frac{v_2}{\mu}-\frac{\rho{v_1}^2}{\mu\delta^2}, \tag{8}$$

and

$$|a_4|\leq\frac{v_3}{E}-Df\frac{v_1v_2\delta^2-\rho{v_1}^3}{E\mu\delta^3}-\frac{Z{v_1}^3}{\delta^3E}, \tag{9}$$

where δ,μ,ρ,E,D and Z are supplied by Eq. (2). Appling Lemma1, we get

$$|a_2|\leq\frac{2}{\delta}, |a_3|\leq\frac{2}{\mu}-\frac{4\rho}{\mu\delta^2}, \text{and } |a_4|\leq\frac{2}{E}-\frac{D(4\delta^2-8\rho)}{E\mu\delta^3}-\frac{8Z}{\delta^3E}.$$

Theorem 1's proof is complete.

The above result establishes a clear bound for the symmetric Toeplitz determinants associated with the considered subclass of Bazilevič functions. It shows that the geometric restrictions on the analytic coefficients directly influence the structural behavior of the corresponding Toeplitz matrices. Thus, the obtained inequality reflects an essential connection between geometric function theory and matrix analysis. The following Corollary is obtained for $\beta=0$.

Corollary 1: *Let f be in the class $L(0,\tau)$. Then*

$$|a_2|\leq\frac{2}{\tau}, |a_3|\leq\frac{2}{\tau}-\frac{4\tau(\tau-1)}{\tau^3}, |a_4|\leq\frac{2}{\tau}-\frac{\tau(\tau-1)\left(4\tau^2-8\tau(\tau-1)\right)}{\tau^4}-\frac{8(\tau-2)(\tau-1)}{\tau^3}.$$

For $\beta=1,\tau=1$, the following Corollary is obtained.

Corollary 2: *Let f be in the class $L(1,1)$. Then*

$$|a_2|\leq 1, |a_3|\leq 0.6666, \text{and } |a_4|\leq 0.5.$$

Theorem 2: *Let $f\in L(\beta,\tau)$ defined by Eq. (1). Then,*

$$\mathrm{T}_2(2) \le \frac{4(\delta^4-4\delta^2\rho+4\rho^2)}{\delta^4\rho^2} - \frac{\nu^2}{\delta^2}. \tag{10}$$

where Eq. (2) gives δ and ρ.

Proof: It is clear from taking into account Eqs. (2), (7), and (8) that

$$\frac{v_2}{\mu} - \frac{\rho {v_1}^2}{\mu\delta^2}|\mathrm{T}_2(2)| \le |{a_3}^2 - {a_2}^2| = \left|\frac{{v_2}^2}{\mu^2} - \frac{2\rho {v_1}^2 v_2}{\mu^2\delta^2} + \frac{\rho^2 {v_1}^4}{\mu^2\delta^4} - \frac{{v_1}^2}{\mu^2}\right|.$$

To simplify the notation, $v_1 = v$ assumes that $\nu \in [0,2]$ is in the collection P concurrently with the function ν. It is clear from using the triangle inequality $P = 4 - v^2$ that

$$|{a_3}^2 - {a_2}^2| \le \left|\frac{{v_1}^4(\delta^4-4\delta^2\rho+4\rho^2)}{4\delta^4\rho^2} - \frac{v^2}{\delta^2}\right| + \frac{(\delta^2-2\rho)v^2|r|(4-v^2)}{2\delta^2\rho^2} + \frac{|r|^2P^2}{4\rho^2} = \phi(v,|r|).$$

$\acute{\phi}(v,|r|) > 0$ on $[0,1]$ is clearly visible, and as a result, $\phi(v,|r|) \le \phi(v,|1|)$. We can see that $\phi(|r|)$ reaches its highest point at the period $[0,2]$ when $\nu = 2$. So, $\mathrm{T}_2(2) = |{a_3}^2 - {a_2}^2| \le \frac{4(\delta^4-4\delta^2\rho+4\rho^2)}{\delta^4\rho^2} - \frac{v^2}{\delta^2}$.

Remark 1: We can get the result given by Radhika, Jahangiri, and Sivasubramanian [5] in Theorem 2, by taking $\beta = 0$ and $\tau = 1$. [4] [Theorem 2]

Theorem 3: *Let $f \in L(\beta,\tau)$ defined by Eq. (1). Then,*

$$|T_2(3)| = |{a_4}^2 - {a_3}^2| \le \frac{X_1}{(\tau+\beta)^6(\tau+3\beta)^2\big((\tau+2\beta)(\tau-1)\big)^2} - \frac{\Big((\tau+\beta)^4 - 2(\tau+\beta)^2(\tau+2\beta)(\tau-1) + \big((\tau+2\beta)(\tau-1)\big)^2\Big)}{(\tau+\beta)^4\big((\tau+2\beta)(\tau-1)\big)^2},$$

where

$$X_1 = 4\delta^2\rho + 6\delta^2 D + 6\delta^2 D^2 + \delta^3\rho + 18\delta^3\rho D + 16\delta^4\rho^2 + 4\rho - 24Z - 48DZ - 48\delta\rho Z - 4\delta\rho D^2.$$

Proof: Applying Eqs. (2), (8), and (9) with Lemma 2 yields $|{a_4}^2 - {a_3}^2|$

Since the function belongs to the same class as P, we use $v_1 = v$ and assume $v \in [0,2]$ to simplify notation. $Y = (1-|r|^2)$ and $P = 4 - v^2$ allow us to apply the triangle inequality to obtain. Use simple calculus to differentiate $\phi_1(v,|r|)$. We differentiate $\phi_1(v,|r|)$ with respect to $|r|$ using elementary calculus, obtaining $\frac{\partial\phi_1(v,|r|)}{\partial|r|}$. We find that $(\partial\phi_1(v,|r|)/\partial|r|) \ge 0$ for $|r| \in [0,1]$ and fixed $v \in [0,2]$. As a result, $\phi_1(v,|r|)$ is a rising function of $|r|$. $\phi_1(v,|r|) \le \phi_1(v,|1|)$. Consequently,

$$|{a_4}^2 - {a_3}^2| \le \left|\frac{X_1 v^6}{(\tau+\beta)^6(\tau+3\beta)^2\big((\tau+2\beta)(\tau-1)\big)^2} - \frac{\Big((\tau+\beta)^4 - 2(\tau+\beta)^2(\tau+2\beta)(\tau-1) + \big((\tau+2\beta)(\tau-1)\big)^2\Big)v^4}{(\tau+\beta)^4\big((\tau+2\beta)(\tau-1)\big)^2}\right| + \frac{\big((\tau+\beta)^2 - (\tau+2\beta)(\tau-1)\big)v^2(4-v^2)^2}{2(\tau+\beta)^2\Big(\frac{1}{2}(\tau+2\beta)(\tau-1)\Big)^2} + \frac{(4-v^2)^2}{4(\tau+3\beta)^2} +$$

$$\frac{(X_4+\frac{1}{4}\Big((\tau+\beta)(\tau+2\beta)(\tau-1)+(\tau+3\beta)(\tau-1)\big((\tau+\beta)(\tau+2\beta)(\tau-1)\big)+(\tau+\beta)^2\big((\tau+2\beta)(\tau-1)\big)^2\Big)v^2\,(4-v^2)^2}{\frac{1}{4}(\tau+3\beta)^2(\tau+\beta)^2\big((\tau+2\beta)(\tau-1)\big)^2}+$$
$$\frac{(X_2+(\tau+\beta)\Big(\frac{1}{2}(\tau+2\beta)(\tau-1)\Big))v^4(4-v^2)}{(\tau+3\beta)^2(\tau+\beta)^4\Big(\frac{1}{2}(\tau+2\beta)(\tau-1)\Big)^2}.$$

On $[0,2]$ at $P=2$, we now obtain

$$|a_4{}^2-a_3{}^2|\le\frac{X_1}{(\tau+\beta)^6(\tau+3\beta)^2\big((\tau+2\beta)(\tau-1)\big)^2}$$
$$-\frac{\Big((\tau+\beta)^4-2(\tau+\beta)^2(\tau+2\beta)(\tau-1)+\big((\tau+2\beta)(\tau-1)\big)^2\Big)}{(\tau+\beta)^4\big((\tau+2\beta)(\tau-1)\big)^2}.$$

Remark 2: The outcome provided by Radhika, Jahangiri, and Sivasubramanian [5] in Theorem 3 can be obtained, by taking $\beta=0$ and $\tau=1$. [4] [Theorem 2]

Theorem 4: *Let $f\in L(\beta,\tau)$ defined by Eq. (1). Then,*

$$|\mathrm{T}_3(1)|=|1+2a_2{}^2(a_3-1)-a_3{}^2|\le\frac{2X_5}{2\delta^4\mu^2(\tau-1)}. \quad (11)$$

Where

$$X_5=4\delta^2\mu\rho+\delta^4+\mu^2\rho^2-2\delta^2\mu\rho.\,|T_3(1)|$$
$$=\left|1+\frac{2\delta^2(\mu^2\delta^2-\rho v_2+\delta^2 v_1{}^2)v_1{}^2}{\mu^3\delta^2}-\frac{(\delta^2v^2\rho+2v_1{}^2)^2}{\mu^2\delta^4}\right|. \quad (12)$$

Proof: Applying Eqs. (2), (7), and (8) with Lemma 2 yields. To simplify the notations, let $v_1=v$ and since v serve as a function in the collection P. At the same time. Using the triangle inequality $P=4-v^2$, we assumed $v\in[0,2]$ using the triangle inequality . It is obvious. $|T_3(1)|\le\frac{2X_5}{2\delta^4\mu^2(\tau-1)}$.

4. Conclusion

The main goal was to associate the Bazilevič functions with new functions in order to construct a new collection $L(\beta,\tau)$ of analytic functions. Researchers may be able to determine the factors influencing the Toeplitz matrices for functions of various families by using the Taylor–Maclaurin coefficient estimates we produced for the first four determinants of the symmetric Toeplitz matrices for the functions that belong to this recently introduced family. In the future, we will use novel functions like binary star functions to study symmetric Toeplitz matrices. It is also possible to investigate the relationship between symmetric Toeplitz matrices and new operators for new complex classes.

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