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Optimizing conjugate gradient methods for removing noise in digital images

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Abstract

Utilizing strategies based on conjugate gradients and carefully choosing a suitable coefficient conjugate might lead to the production of exceptional outcomes. In this work, a modified version of the conjugate gradient algorithm reported by Hassan [1] is presented in order to establish that the new technique is globally convergent, under the usual assumptions. This study was carried out in order to verify that the new approach. Hideaki and Yasushi are the ones responsible for writing this article. In order to illustrate how successful the new method is, its performance is evaluated in terms of its capacity to eliminate impulsive noise from photographic images.

Subject Classification: *90C30, 65K05, 49M37.*

Keywords: *Derivative coefficient conjugate gradient, Convergence property, Removing noise in digital images.*

1. Introduction

The noisy pixel cleaning model is one of the interesting optimization issues in engineering and management that may be resolved and made simpler by using this problem:

$$f_\alpha(u) = \sum_{(i,j)\in N} \left[\left| u_{i,j} - y_{i,j} \right| + \frac{\beta}{2} (S_{i,j}^1 + S_{i,j}^2) \right] \tag{1}$$

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where $S^1_{i,j} = 2\sum_{(m,n)\in P_{i,j}\cap N^c} \phi_\alpha(u_{i,j} - y_{m,n})$, $S^2_{i,j} = \sum_{(m,n)\in P_{i,j}\cap N} \phi_\alpha(u_{i,j} - y_{m,n})$. In image restoration, $\phi_\alpha = \sqrt{\alpha + x^2}$ is the edge-preserving potential function with parameter α > 0, β is the parameter when the criteria are met, and $y_{i,j}$ is the value of the picture's pixel at position (i,j). A column vector of length |N| that is lexicographically ordered is called $u_{i,j} = [u_{i,j}]_{(i,j)\in N}$. The function that is minimized in formula is an approximate half-quadratic smooth representation of $F_\alpha(u)$ as follows:

$$f_\alpha(u) = \sum_{(i,j)\in N} \left[(2 \times S^1_{i,j} + S^2_{i,j})\right] \tag{2}$$

More details can be found in [2]. Utilizing conjugated gradients, allowed for the value of the formula (3) to be reduced to its lowest possible value:

$$f(x^*) = \min_{x\in R^N} f(u) \tag{3}$$

At the kth iteration of the formula Hassan and Sulaiman [3], the determined using a line search approach as:

$$u_{k+1} = u_k + \alpha_k d_k \tag{4}$$

where the search direction is d_k and the parameter is α_k, which is found using an exact line-search as follows:

$$\alpha_k = -g_k^T d_k / d_k^T Q d_k \tag{5}$$

Iterative methods must be used for generic non-linear functions. Formula Nocedal and Wright [4] has further information. Convergence analysis and conjugate gradient method implementations often employ the Wolfe criteria to assess if the step length is acceptable:

$$\text{“}f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k\text{”} \tag{6}$$

$$\text{“ } d_k^T g(x_k + \alpha_k d_k) \ge \sigma\, d_k^T g_k\text{”} \tag{7}$$

when $0 < \delta < \sigma < 1$, see Hassan [5]. For the purpose of describing the direction of the conjugate gradient for the subsequent iteration, the following form should be used:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{8}$$

where B_k is a scalar. Two well-known methods for selecting β_k are as follows:

$$\beta_k^{FR} = \|g_{k+1}\|^2 / \|g_k\|^2, \beta_{k+1}^{DY} = \|g_{k+1}\|^2 / d_k^T y_k \tag{9}$$

where $y_k = g_{k+1} - g_k$. These results were obtained independently using both the Fletcher and Reeves (FR) approach [6] with the Dai and Yuan (DY) method [7]. The algorithms it uses have essential qualities, notably the property of being globally convergent.

Derivation of the new coefficient conjugate and a fresh perspective on the denominator $d_k^T G v_k$ based on the quadratic model are the major objectives of this work. Our approach is shown to be effective in restoring images and statistically coherent.

2. Deriving new coefficient conjugate for conjugate gradient methods

Assume exact line search and quadratic f. In Hassan [1], introduced conjugate gradient optimization techniques. One is examined in this research as, β_k:

$$\beta_k = g_{k+1}^T Q s_k / d_k^T Q s_k \tag{10}$$

where Q is Hessian matrix and where β_k is satisfies the conjugacy condition:

$$d_{k+1}^T Q d_k = 0 \tag{11}$$

We shall introduce a decent approximation to the $d_k^T Q s_k$ to change this technique. Thus, using Taylor's formula and quadratic model, we have:

$$f_{k+1} = f_k + s_k^T g_k + 1/2 s_k^T Q s_k \tag{12}$$

and:

$$f_k = f_{k+1} - g_{k+1}^T s_k + 1/2 s_k^T Q s_k \tag{13}$$

where Q is the Hessian matrix. Making the gradient:

$$g_{k+1} = g_k + Q(u_k) s_k \tag{14}$$

The results of (12), (13) and (14) are as follows:

$$\begin{aligned} s_k^T Q s_k &= 2r(f_k - f_{k+1} + g_{k+1}^T s_k) + 2(1-r)(f_{k+1} - f_k - s_k^T g_k) \\ &= s_k^T y_k + (2r-1)(2(f_k - f_{k+1}) - s_k^T y_k) \end{aligned} \tag{15}$$

where $0 \le r \le 1$. Now, we can observe from (15), that:

$$d_k^T Q s_k = d_k^T y_k + (2r-1)(2(f_k - f_{k+1}) - s_k^T y_k)/\alpha_k \tag{16}$$

We get the formula for calculating β_k using (10) and (16) as follows:

$$\beta_k = \frac{g_{k+1}^T y_k}{d_k^T y_k + (2r-1)(2(f_k - f_{k+1}) - s_k^T y_k)/\alpha_k} \tag{17}$$

Since f is quadratic function and exact line search is employed, then:

$$\beta_k = \frac{\|g_{k+1}\|^2}{d_k^T y_k + (2r-1)(2(f_k - f_{k+1}) - s_k^T y_k)/\alpha_k} \tag{18}$$

and:

$$\beta_k = \frac{\|g_{k+1}\|^2}{-d_k^T g_k + (2r-1)(2(f_k - f_{k+1}) + s_k^T g_k)/\alpha_k} \tag{19}$$

and:

$$\beta_k = \frac{\|g_{k+1}\|^2}{g_k^T g_k + (2r-1)(2(f_k - f_{k+1}) - \alpha_k g_k^T g_k)/\alpha_k} \tag{20}$$

Presented below is our formula for the modified formula CG, also known as BBC1, BBC2 and BBC3.

Algorithms BBC1, BBC2 and BBC3.

Stage 1. An initial point u_1. Set $d_1 = -g_1$.

Stage 2. Determine the $\alpha_k > 0$ satisfying the Wolfe conditions (7) and (8).

Stage 3. Let $x_{k+1} = x_k + \alpha_k d_k$ and $g_{k+1} = g(x_{k+1})$. If $\|g_{k+1}\| \le 10^{-6}$, then stop.

Stage 4. Compute β_k by the formulae $(18-20)$, then generate d_{k+1} by (8).

Stage 5. Set $k=k+1$ and continue with step 2.

In Hassan [1], reported the general conclusion, which is crucial for proving the convergence results.

3. Numerical results

Our major objective is to minimize four (Lena (Le), House (Ho), Elaine (El) and Cameraman (C512)) test photos collected from Hassan and Alashoor [11] using various numerical experiments in Table 1 based on the Figure 1. To evaluate and contrast the computation impact of the proposed BBC1, BBC2, BBC3, and FR methods in this article with the findings provided by [8]. Other ways exist in addition to those we can see in Hassan and Saad [9], Hassan [10]. The following parameters are used by the BBC1, BBC2, BBC3 and FR techniques in the numerical experiments: $\delta = 0.0001$ and $\sigma = 0.5$. Table 1 contains the numerical findings of the BBC1, BBC2, BBC3, and FR. Here, the terms NI, NF, and PSNR (peak signal to noise ratio) signify the number of iterations, function evaluations, and PSNR, which is defined as:

$$PSNR = 10\,log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \tag{21}$$

where $u_{i,j}^r$ and $u_{i,j}^*$ denote the pixel values of the restored image and the original image, respectively. We stop the iterations [2], if:

$$\frac{|f(u_k)-f(u_{k-1})|}{|f(u_k)|} \le 10^{-4} \text{ and } \|f(u_k)\| \le 10^{-4}(1+|f(u_k)|). \tag{22}$$

Table 1

Numerical results of FR, BBC1, BBC2 and BBC3 algorithms

Image	Nois	FR-Method			BBC1-Method			BBC2-Method			BBC3-Method		
	(%)	NI	NF	PSNR	NI	NF	PSNR	NI	NF	PSNR	NI	NF	PSNR
Le	50	82.0	153	30.5529	29.0	58.0	30.7277	24.0	49.0	30.4342	31.0	64.0	30.8922
	70	81.0	155	27.4824	36.0	73.0	27.3855	30.0	61.0	27.642	36.0	75.0	27.5578
	90	108.0	211	22.8583	52.0	107.0	22.877	44.0	91.0	23.1715	47.0	98.0	22.959
Ho	50	52.0	53.0	30.6845	21.0	40.0	34.9946	29.0	57.0	34.6884	19.0	37.0	35.1732
	70	63.0	116	31.2564	25.0	49.0	31.1489	28.0	55.0	31.4412	29.0	57.0	31.4839
	90	111.0	214	25.287	47.0	99.0	25.4731	49.0	98.0	25.302	39.0	81.0	25.3366
El	50	35.0	36.0	33.9129	19.0	35.0	33.9766	21.0	38.0	33.981	20.0	36.0	33.9902
	70	38.0	39.0	31.864	20.0	37.0	31.949	21.0	40.0	31.9892	19.0	35.0	31.9368
	90	65.0	114	28.2019	30.0	59.0	28.375	33.0	65.0	28.3581	34.0	66.0	28.3742
c512	50	59.0	87.0	35.5359	24.0	49.0	35.6267	25.0	51.0	35.9282	37.0	75.0	35.4632
	70	78.0	142	30.6259	30.0	61.0	31.046	28.0	58.0	30.8829	36.0	73.0	31.0144
	90	121.0	236	24.3962	44.0	91.0	25.0102	46.0	96.0	25.0994	42.0	86.0	25.0414

In iterations and function evaluations, Table 1 demonstrates that our method performs better than the FR methodology.

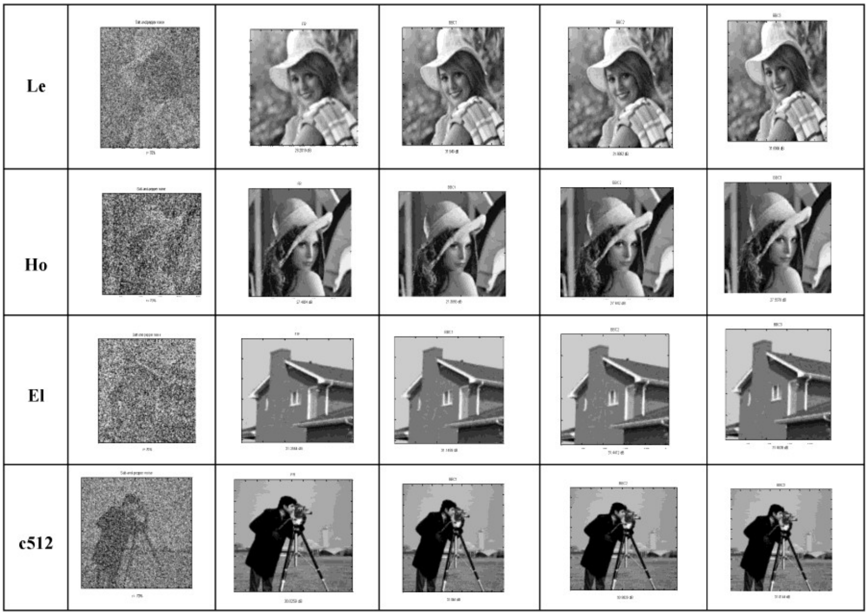

Figure 1
Demonstrates the results of algorithms FR, BBC1, BBC2 and BBC3 of 256 * 256 images for 50%.

4. Conclusion

In order to solve picture restoration issues, we derived a new coefficient conjugate for the conjugate gradient approach, which was based on the Taylor formula. This approach is globally convergent when used to certain very easy situations. The numerical results show that the approach is efficient and improves the number of iterations and function evaluations for impulse denoising while maintaining the same level of picture quality once it has been recovered.

References

[1] B. A. Hassan, "A new formula for conjugate parameter computation based on the quadratic model," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 13, no. 3, pp. 954-961 (2019), doi: 10.11591/ijeecs.v13.i3.

[2] J. F. Cai, R. Chan, and B. Morini, "Minimization of an edge-preserving regularization functional by conjugate gradient type methods," (2007), doi: 10.1007/978-3-540-33267-1_7.

[3] B. A. Hassan and R. M. Sulaiman, “A new class of self-scaling for quasi-Newton method based on the quadratic model,” *Indonesian Journal of Electrical Engineering and Computer Science,* vol. 21, no. 3, pp. 1830-1836 (2021), doi: 10.11591/ijeecs.v21.i3.

[4] J. Nocedal and S. Wright, Numerical Optimization, 2nd ed. New York, NY, USA: Springer (2006). [Online]. Available: https://link.springer.com/book/10.1007/978-0-387-40065-5

[5] B. A. Hassan, “A new type of quasi-Newton updating formulas based on the new quasi-Newton equation,” *Numerical Algebra, Control and Optimization*, vol. 10, no. 2 (2020), doi: 10.3934/naco.2019049.

[6] R. Fletcher and C. M. Reeves, “Function minimization by conjugate gradients,” *The Computer Journal*, vol. 7, no. 2, pp. 149–154 (1964), doi: 10.1093/comjnl/7.2.149.

[7] Y. H. Dai and Y. Yuan, “A nonlinear conjugate gradient method with a strong global convergence property,” *SIAM Journal on Optimization,* vol. 10, no. 1 (1999), doi: 10.1137/S1052623497318992.

[8] B. A. Hassan, K. Muangchoo, F. Alfarag, A. H. Ibrahim, and A. B. Abubakar, “An improved quasi-Newton equation on the quasi-Newton methods for unconstrained optimizations,” *Indonesian J. of Electrical En. and Computer Science*, vol. 22, no. 2, pp. 997-1005 (2021), doi: 10.11591/ijeecs.v22.i2.

[9] B. A. Hassan and A. A. Saad, “Explaining new parameters conjugate analysis based on the quadratic model,” *Journal of Interdisciplinary Mathematics,* vol. 26, no. 6 (2023), doi: 10.47974/JIM-1620.

[10] B. A. Hassan, “A modified quasi-Newton methods for unconstrained optimization,” *Italian Journal of Pure and Applied Mathematics*, no. 42, pp. 504-511 (2019).

[11] B. A. Hassan and H. A. Alashoor, “Pediment new parameters for a conjugate gradient method and using it in restoring distorted images,” in *Proc. 2022 8th Int. Conf. Contemporary Information Technology and Mathematics (ICCITM)* (2022), doi: 10.1109/ICCITM56309.2022.100318.

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