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Inverse problem with an operator that depends on unknown parameter

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Abstract

In this work, we propose a method for solving a linear inverse problem where the operator depends on an unknown parameter. Our approach involves approximating the exact solution using a stochastic iterative Landweber-type scheme, while the optimal parameter value is determined by minimizing a least squares functional.

An exponential inequality will be established to demonstrate the almost complete convergence of the method, specifying both the rate of convergence and the confidence interval within which the solution resides.

Finally, we present a numerical application involving a Fredholm integral equation of the first kind, where the kernel depends on an unknown parameter.

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Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathbb{H}$ a separable Hilbert space.

Let A_α be a bounded linear operator in $\mathbb{H}$ which depends on some unknown parameter $\alpha \in \mathbb{R}_+^*$ and consider the following inverse problem :

$$A_\alpha f = g, \qquad (1)$$

where $g \in \mathbb{H}$ is given and $f \in \mathbb{H}$ the unknown.

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We assume that the map $\alpha \in \mathbb{R}_+^* \mapsto A_\alpha \in \mathcal{L}(\mathbb{H})$ is not necessarily linear.

Furthemore, the second term g in equation (1) is derived from empirical measurements and is therefore not known with high accuracy. It consists of a sequence of approximately known values $(g_k)_{k\in\mathbb{N}^*}$. We presume that the noise impacting the g_k data exhibits a random nature.

Finally, the exact solution f_e of equation (1) and the exact value g_e of its second term satisfy the following equality

$$A_\alpha f_e = g_e. \qquad (2)$$

In the treatment of inverse problems, it has traditionally been assumed that the underlying operator is known, and only the data is affected by noise. However, in practice, the operator is often partially unknown for various reasons.

Several factors can explain this unknown characteristic. For instance, mathematical models do not always capture the complete reality of the phenomenon they aim to describe. Additionally, some properties of the operator might be unknown or challenging to manage due to the complexity of the analytical expressions that define them.

The equations in which they appear cannot be solved using conventional analytical methods. As such, resorting to approximation techniques becomes unavoidable [29].

Defining the concept of unknown operator is a challenge, because of its applicability to different scenarios. In general, an operator is considered unknown if one or more of its characteristics are not known, the most extreme case being that of a totally unknown operator. Here are a few important examples, though not intended to be exhaustive.

One example is integral equations that involving a delay [21], which are commonly used to model phenomena characterized by memory or delayed effects, such as chemical reactions or species dynamics.

The case of an unknown operator for which we have a noisy version [4, 13, 20, 28], or for which only the eigenfunctions are known [8], is common. Another situation arises when the operator depends on an unknown parameter [5, 31, 14]. Additionally, there are cases where the unknown operator is bounded by two known operators [6, 7] or where a good estimator is available [9].

Nevertheless, in all cases, additional observations are available to build estimators for such operators. This variety of situations has led to a range of different approaches.

In the last decade, there have been significant papers dealing with the resolution of inverse problems. For example, on inverse problems governed by Fredholm integral operators, Sarkar, Sen and Saha [29] employ non-polynomial spline functions to approximate the solution of a nonlinear Volterra-Fredholm integral equation of the second kind.

Sarkar et al. [21] employ piecewise linearization in conjunction with the boundary element method to approximate the solution of a nonlinear Fredholm integral equation with constant delay.

Cavalier and Hengartner [8] employ adaptive estimation within a hierarchical Gaussian sequence space model.

Bleyer and Ramlau [5] adopted a deterministic approach that utilizes a two-parameter Tikhonov regularization scheme, specifically called the double regularized total least squares (dbl-RTLS).

Trabs [31] employs a Bayesian approach,aiming to simultaneously develop an estimator for both the unknown variable in the problem and the unknown parameter.

In contrast to prior approaches, we have chosen to adopt a probabilistic perspective(see [26], Chap I, section 2). The assumption that the noise affecting the data is random supports the use of a probabilistic approach.

Let f_e be the exact solution to equation (1), and f^* be an approximate solution in some sense (such as a minimum norm least squares solution, quasisolution, regularized solution,etc.). In a probabilistic approach, the goal is to choose f^* that minimizes the following probability for any arbitrary $\varepsilon > 0$

$$\mathbb{P}(\|f^* - f_e\| > \varepsilon).$$

In this work, we have opted for a stochastic iterative process to approach the solution . If $(f_k)_{k\geq 1}$ is the sequence of iterates approximating the exact solution f_e generated by this scheme, then the probabilistic approach precisely involves examining, for any arbitrary $\varepsilon > 0$, the following probability :

$$\mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon).$$

The modes of convergence for this type of procedure are as follows :

$\lim_{k\to+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) = 0$	:	convergence in probability,
$\sum_{k=1}^{+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) < +\infty$	:	almost complete convergence,
$\lim_{k\to+\infty} \mathbb{E}\|f_{k+1} - f_e\|^2 = 0$	:	convergence in quadratic mean ($\mathbb{E}$ denotes the mathematical expectation).

In the following, we will outline our proposed methodology for addressing problem (1).

The remainder of this paper is structured as follows. Section 1 presents the framework for addressing and solving the inverse problem, with an emphasis on the operator's injectivity condition and the use of Landweber's stochastic iterative scheme as a method for approaching the solution. Section 2 presents and proves essential results that are foundational for the subsequent section. Section 3 focuses on the study of convergence. We begin by establishing an exponential inequality, followed by demonstrating convergence in probability and almost complete convergence of the method. Additionally, we specify the rate of convergence and provide a confidence interval. Finally, in Section 4, we present a numerical example to illustrate the consistency of the method. The example involves a Fredholm integral equation of the first kind,with a kernel that depends on an unknown parameter.

1. Framework and resolution method

First, we'll estimate the second member g in equation (1). To do this, we use the arithmetic mean of the g_k measurements, denoted by $\bar{g}$.

The strong law of large numbers naturally identifies the empirical mean $\bar{g}$ as an exhaustive estimate of the second member g, reducing the problem to solving the equation

$$A_\alpha f = \bar{g}. \qquad (3)$$

It should be noted that $\bar{g}$ is not necessarily in $Range(A_\alpha)$.

Two issues now need to be addressed : the ill-posedness of problem (3) in the Hadamard sense [17] and the uncertainty associated with the parameter α.

Secondly, the operator A_α must be injective. This condition is essential not only for the convergence of the subsequent procedure but also for defining the set to which the parameter α belongs. So, let I_α denote the set defined by :

$$I_\alpha = \{\alpha \in \mathbb{R}_+^* : A_\alpha \text{ is injective}\}. \quad (4)$$

Since the problem is ill-posed, we will construct an approximate solution using a stochastic iterative scheme of the Landweber type [11, 25], where the optimal value of the parameter α, in the least-squares minimization sense, is determined at each iteration. So, we consider the following iterative procedure

$$f_1 \text{ is given and for } k \geq 1, \; f_{k+1} = [I - (A_\alpha)_k^*(A_\alpha)_k]f_k + (A_\alpha)_k^*\bar{g} + w_k\xi_k, \quad (5)$$

where

1. $(A_\alpha)_k$ is the $k-$th operator A_α corresponding to the operator A_α for $\alpha = \alpha_k$.
2. The sequence $(\alpha_k)_{k\in\mathbb{N}^*}$ is a sequence of real numbers belonging to the set I_α and defined as follows

- The initial value α_1 is given.
- For $k \geq 1$, α_{k+1} is determined by

$$\alpha_{k+1} = \min_{\alpha \in I_\alpha} J_k(\alpha),$$

where $J_k(\alpha) = \mathrm{P} A_\alpha f_k - (g + \xi_k)\mathrm{P}^2$.

3. $(w_k)_{k\in\mathbb{N}^*}$ is a sequence of real numbers in the interval $]0,1]$, decreasing towards 0 and satisfying

$$\lim_{k\to+\infty} k w_k = 0, \qquad (6)$$

7. $(\xi_k)_{k\in\mathbb{N}^*}$ is a sequence of independent and identically distributed random variable of zero-mean, with values in $\mathbb{H}$ and satisfying [27]

$$\|\xi_k\| < b \in \mathbb{R}_+^* \text{ for all } k \geq 1, \qquad (1)$$

The sequence of numbers $(\alpha_k)_{k\in\mathbb{N}^*}$ generates the sequence of operators$(A_{\alpha_k})_{k\in\mathbb{N}^*}$.

For the purposes of this paper, we will therefore use the notations

$$(A_\alpha)_k = A_{\alpha_k} \text{ and } (A_\alpha)_k^* = A_{\alpha_k}^*.$$

The sequence $(\alpha_k)_{k\in\mathbb{N}^*}$ consists of iterative minima. At each iteration, it provides the optimal value of the unknown parameter through the least-squares minimization of the functional J_k.

In fact, there are nested iterative processes : one approaching the exact solution f_e through the iterations $(f_k)_{k\geq 1}$, and the other approaching the optimal value of the parameter α through the sequence $(\alpha_k)_{k\geq 1}$.

We assume without loss of generality that

$$\|A_{\alpha_i}\| < 1 \text{ for all } i \geq 1. \qquad (8)$$

Finally, we suppose that there exists a constant $c \in]0,1[$ such that

$$\inf\left\{\|A_{\alpha_i}h\|^2 / \ \|h\| = 1\right\} \geq c \text{ for all } i \geq 1. \qquad (9)$$

Remark 1: A brief explanation of the stochastic version of Landweber's algorithm. Landweber's classic iterative scheme can be written as follows

$$f_{k+1} = [I - A_\alpha^* A_\alpha] f_k + A_\alpha^* g_k, \quad k \geq 1,$$

with f_1 arbitraly chosen and g_k being the result of the k-th empirical measurement of the second term g.

We rewrite the previous algorithm as follows

$$\begin{aligned} f_{k+1} &= [I - A_{\alpha_k}^* A_{\alpha_k}] f_k + A_{\alpha_k}^* g_k + A_{\alpha_k}^* \bar{g} - A_{\alpha_k}^* \bar{g}, \\ &= [I - A_{\alpha_k}^* A_{\alpha_k}] f_k + A_{\alpha_k}^* \bar{g} + A_{\alpha_k}^* (g_k - \bar{g}). \end{aligned}$$

If we consider the deviation between g_e and the g_k measurements as being random, we will have $\mathbb{E}[A_{\alpha_k}^*(g_k - \bar{g})] = 0$ because $\bar{g}$ is a good estimate of $\mathbb{E}[g_k]$.

To ensure the convergence of the method, we model the random term $A_{\alpha_k}^*(g_k - \bar{g})$ by $w_k \xi_k$ where $(w_k)_{k\in\mathbb{N}^*}$ and $(\xi_k)_{k\in\mathbb{N}^*}$ are the two sequences defined in the previous assumptions.

Thus, we construct the stochastic Landweber algorithm by noisying the classical algorithm for the equation $A_\alpha f = \bar{g}$ with the random sequence $w_k \xi_k$:

$$\underbrace{f_{k+1} = [I - A_{\alpha_k}^* A_{\alpha_k}] f_k + A_{\alpha_k}^* \bar{g}}_{classical\ scheme} \quad + \quad \underbrace{w_k \xi_k}_{random\ term} \quad .$$

2. Preliminary Results

Lemma 1: *Let* $(\alpha_i)_{i\in\mathbb{N}^*} \subset I_\alpha$ *and* $(A_{\alpha_i})_{i\in\mathbb{N}^*} \subset \mathcal{L}(\mathbb{H})$. *The deviation between the exact solution and the iterated one satisfies the following recurrent formula*

$$f_{k+1} - f_e = \prod_{i=1}^k D_i (f_1 - f_e) + \sum_{i=1}^k \prod_{j=i+1}^k D_j w_i \xi_i, \qquad (10)$$

where $D_i = I - A_{\alpha_i}^* A_{\alpha_i}$ and the convention $\prod_{j=k+1}^k D_j = I$.

Proof: The iterative formula (5) yields

$$\begin{aligned} f_{k+1} - f_e &= [I - A_{\alpha_k}^* A_{\alpha_k}] f_k - f_e + A_{\alpha_k}^* \bar{g} + w_k \xi_k, \\ &= [I - A_{\alpha_k}^* A_{\alpha_k}] f_k - f_e + A_{\alpha_k}^* A_{\alpha_k} f_e + w_k \xi_k. \end{aligned}$$

Replacing $[I - A_{\alpha_k}^* A_{\alpha_k}]$ by D_k, we write

$$f_{k+1} - f_e = D_k(f_k - f_e) + w_k \xi_k.$$

After successive iterations, we find

$$f_{k+1} - f_e = \prod_{i=1}^{k} D_i(f_1 - f_e) + \sum_{i=1}^{k-1} \left(\prod_{j=i+1}^{k} D_j w_i \xi_i\right) + w_k \xi_k.$$

Using the convention $\prod_{j=k+1}^{k} D_j = I$, we get the result.

Lemma 2: *Let* $(\alpha_i)_{i\in\mathbb{N}^*} \subset I_\alpha$ *and* $(A_{\alpha_i})_{i\in\mathbb{N}^*} \subset \mathcal{L}(\mathbb{H})$. *Then, the following limit holds*

$$\lim_{k\to+\infty} \prod_{i=1}^{k} \|D_i\| = 0, \tag{11}$$

where $D_i = I - A_{\alpha_i}^* A_{\alpha_i}$.

Proof: For $i \in [1,k]$, it is evident that the operator D_i is self-adjoint.

According to the theorem on the norm of self-adjoint operators [10], we have

$$\|D_i\| = \sup\{\langle D_i h, h\rangle : \|h\| = 1\}.$$

Let $h \in \mathbb{H}$ such that $\mathrm{P}h\mathrm{P} = 1$. We have

$$\begin{aligned} \langle D_i h, h\rangle &= \langle (I - A_{\alpha_i}^* A_{\alpha_i})h, h\rangle \\ &= \langle h, h\rangle - \langle A_{\alpha_i}^* A_{\alpha_i} h, h\rangle \\ &= \|h\|^2 - \left\|A_{\alpha_i} h\right\|^2 \\ &= 1 - \left\|A_{\alpha_i} h\right\|^2 \end{aligned} \tag{12}$$

Since $h \neq 0$ and A_{α_i} is injective, it follows that $\left\|A_{\alpha_i} h\right\| \neq 0$. And using assumption (8), we obtain

$$\langle D_i h, h\rangle < 1.$$

Taking the sup, we get

$$\|D_i\| < 1 \text{ for all } i \geq 1.$$

The result follows directly from the previous inequality.

Lemma 3: *Let* $(\alpha_i)_{i\in\mathbb{N}^*} \subset I_\alpha$ *and* $(A_{\alpha_i})_{i\in\mathbb{N}^*} \subset \mathcal{L}(\mathbb{H})$. *For* $i \in [1,k]$, *we have*

$$\prod_{j=i+1}^{k} \mathrm{P}D_j\mathrm{P} \leq \left(\frac{i+1}{k+1}\right)^c, \tag{13}$$

where $D_j = I - A_{\alpha_j}^* A_{\alpha_j}$ *and* $c \in]0,1[$.

Proof: Let $j \in [i+1,k]$ with $i \in [1,k]$.

From the equality (12), we deduce that

$$\left\|D_j\right\| \leq 1 - c \leq 1 - \frac{c}{j} \text{ for all } j \geq 1.$$

Then

$$\prod_{j=k+1}^{k} \left\|D_j\right\| \leq \prod_{j=k+1}^{k} \left(1 - \frac{c}{j}\right).$$

Using the inequality $1+x \leq e^x$ for any real x, we deduce that

$$\prod_{j=i+1}^{k}\left(1-\frac{c}{j}\right) \leq \exp\left(-c\sum_{j=i+1}^{k}\frac{1}{j}\right).$$

Let $j \in [i+1,k]$. For $x \in [j,j+1]$, the function $x \mapsto \frac{1}{x}$ is strictly positive, strictly decreasing and its curve lies inside a rectangle of width $\frac{1}{j}$ and lengh $j+1-j=1$. The area of the rectangle is greater than the area delimited by the curve and the lines $x=j$, $x=j+1$ and $y=0$.

So

$$\frac{1}{j} \geq \int_{j}^{j+1}\frac{1}{x}dx.$$

This implies

$$\sum_{j=i+1}^{k}\frac{c}{j} \geq \sum_{j=i+1}^{k}\int_{j}^{j+1}\frac{c}{x}dx.$$

By computing the integral, we get

$$\sum_{j=i+1}^{k}\frac{c}{j} \geq \sum_{j=i+1}^{k} c(\ln(j+1)-\ln(j)).$$

Thereby

$$\begin{aligned}
-\sum_{j=i+1}^{k}\frac{c}{j} &\leq \sum_{j=i+1}^{k} c(\ln(j)-\ln(j+1)),\\
&\leq \sum_{j=i+1}^{k}\ln\left(\frac{j}{j+1}\right)^{c},\\
&\leq \ln\left(\prod_{j=i+1}^{k}\left(\frac{j}{j+1}\right)^{c}\right).
\end{aligned}$$

Taking the exponential, we get

$$\exp\left(-\sum_{j=i+1}^{k}\frac{c}{j}\right) \leq \prod_{j=i+1}^{k}\left(\frac{j}{j+1}\right)^{c} = \left(\frac{i+1}{k+1}\right)^{c}.$$

3. Convergence Study

Theorem 1*: Let $\varepsilon > 0$, $(\alpha_i)_{i\in\mathbb{N}^*} \subset I_\alpha$, $(A_{\alpha_i})_{i\in\mathbb{N}^*} \subset \mathcal{L}(\mathbb{H})$ and $f_i, \xi_i \in \mathbb{H}$ for $i \in \mathbb{N}^*$.*

Under assumptions (1)-(9), the exact solution of problem (1) satisfies the following exponential inequalities

$$\mathbb{P}(\|f_{k+1}-f_e\| > \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{8b^2\sum_{i=1}^{k}\prod_{j=i+1}^{k}\|D_j\|^2 w_i^2}\right), \tag{14}$$

where $D_i = I - A_{\alpha_i}^* A_{\alpha_i}$, and

$$\mathbb{P}(\|f_{k+1}-f_e\| > \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2(k+1)^{2c}}{8b^2\sum_{i=1}^{k}(i+1)^{2c}w_i^2}\right), \tag{15}$$

where $c \in]0,1[$.

Proof: From formula (10) in Lemma 1, we deduce

$$\mathbb{P}(\|f_{k+1}-f_e\| > \varepsilon) = \mathbb{P}\left(\left\|\prod_{i=1}^{k} D_i(f_1-f_e)+\sum_{i=1}^{k}\prod_{j=i+1}^{k} D_j w_i \xi_i\right\| > \varepsilon\right),$$

$$\leq \mathbb{P}\left(\left\|\prod_{i=1}^{k} D_i(f_1 - f_e)\right\| + \left\|\sum_{i=1}^{k} \prod_{j=i+1}^{k} D_j w_i \xi_i\right\| > \varepsilon\right),$$
$$\leq \mathbb{P}\left(\left\|\sum_{i=1}^{k} \prod_{j=i+1}^{k} D_j w_i \xi_i\right\| > \varepsilon - \left\|\prod_{i=1}^{k} D_i(f_1 - f_e)\right\|\right).$$

On the one hand, we have

$$\left\|\prod_{i=1}^{k} D_i(f_1 - f_e)\right\| \leq \prod_{i=1}^{k} \|D_i\| \ \|f_1 - f_e\|.$$

On the other hand, the result (11) in Lemma 2 implies that for sufficiently large k, we have

$$\prod_{i=1}^{k} \|D_i\| \ \|f_1 - f_e\| \leq \frac{\varepsilon}{2}.$$

As a result

$$\mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) \leq \mathbb{P}\left(\left\|\sum_{i=1}^{k} \prod_{j=i+1}^{k} D_j w_i \xi_i\right\| > \frac{\varepsilon}{2}\right).$$

Noting that $\prod_{j=i+1}^{k} D_j w_i \xi_i = \eta_i$, we have

$$\|\eta_i\| = \left\|\prod_{j=i+1}^{k} D_j w_i \xi_i\right\| \leq w_i \left\|\prod_{j=i+1}^{k} D_j\right\| \|\xi_i\| \leq b \prod_{j=i+1}^{k} \left\|D_j\right\| w_i < +\infty.$$

Appliying Hoeffding's inequality [19] to the sum $\left\|\sum_{i=1}^{k} \eta_i\right\|$, we get the following exponential inequality

$$\mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{8b^2 \sum_{i=1}^{k} \prod_{j=i+1}^{k} \left\|D_j\right\|^2 w_i^2}\right).$$

And by replacing $\prod_{j=i+1}^{k} \left\|D_j\right\|^2$ with the result (13) in lemma 3, we obtain

$$\mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2 (k+1)^{2c}}{8b^2 \sum_{i=1}^{k} (i+1)^{2c} w_i^2}\right).$$

Corollary 1: Under the assumptions of Theorem 1, the iterative procedure (5) converges in probability to the exact solution of problem (1). In other words, we have :

$$\lim_{k \to +\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) = 0. \tag{16}$$

Proof: Firstly, assumption (6) implies the convergence of the series $\sum w_k^2$.

Indeed, we have

$$\lim_{k \to +\infty} k w_k = 0 \ \Rightarrow +\infty \lim_{k \to +\infty} k^2 w_k^2 = 0.$$

which means that

$$w_k^2 = o\left(\frac{1}{k^2}\right).$$

Since the series $\sum \frac{1}{k^2}$ is convergent, this implies that $\sum w_k^2 < +\infty$.

Moreover, by posing $b_i = (i+1)^{2c}$, it is clear that the sequance $(b_i)_{i \geq 1}$ is increasing and verifies

$$\lim_{i \to +\infty} b_i = +\infty \text{ and } \sum b_i^{-1} w_i^2 < +\infty.$$

We can therefore apply Kronecker's lemma [24], which yields

$$\lim_{k\to+\infty} \frac{1}{(k+1)^{2c}} \sum_{i=1}^{k} (i+1)^{2c} w_i^2 = 0.$$

This leads to

$$\lim_{k\to+\infty} 2\exp\left(-\frac{\varepsilon^2}{8b^2 \frac{1}{(k+1)^{2c}} \sum_{i=1}^{k} (i+1)^{2c} w_i^2}\right) = 0.$$

Which directly leads to

$$\lim_{k\to+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) = 0.$$

Remark 2: Under the assumptions of Theorem 1 and for a given level $\rho > 0$, there exists a naturel integer k_ρ such that the solution lies within the closed ball centered at $f_{k_\rho+1}$ with radius ε, with probability greater than or equal to $1-\rho$.

Indeed, by applying the definition of the limit to the result (16), we obtain

$$\lim_{k\to+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) = 0 \iff [\forall\varepsilon > 0, \exists k_\rho \in \mathbb{N}^*,$$
$$k \geq k_\rho \Rightarrow \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) < \rho].$$

Since $\mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) + \mathbb{P}(\|f_{k+1} - f_e\| \leq \varepsilon) = 1$, then

$$\forall\varepsilon > 0, \exists k_\rho \in \mathbb{N}^*, k \geq k_\rho \Rightarrow \mathbb{P}(\|f_{k+1} - f_e\| \leq \varepsilon) \geq 1 - \rho.$$

Theorem 2: *Under the assumptions of Theorem 1, the iterative procedure (5) converges almost completely to the exact solution of problem (1). This is that*

$$\sum_{k=1}^{+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) < +\infty. \tag{17}$$

In addition, we have

$$\|f_{k+1} - f_e\| = o(k^{-2c}), c \in]0,1[. \tag{18}$$

Proof: For $N \in [1, k]$, we can write

$$\frac{1}{(k+1)^{2c}} \sum_{i=1}^{k} (i+1)^{2c} w_i^2 = \frac{1}{(k+1)^{2c}} \sum_{i=1}^{N} (i+1)^{2c} w_i^2 + \frac{1}{(k+1)^{2c}} \sum_{i=N+1}^{k} (i+1)^{2c} w_i^2.$$

Note that the term $M = \sum_{i=1}^{N} (i+1)^{2c} w_i^2$ is finite. Moreover, by Kronecker's lemma [24], we find that

$$\lim_{k\to+\infty} \frac{1}{(k+1)^{2c}} \sum_{i=N+1}^{k} (i+1)^{2c} w_i^2 = 0,$$

So, for sufficiently large k and $\varepsilon' > 0$, we have

$$\frac{1}{(k+1)^{2c}} \sum_{i=N+1}^{k} (i+1)^{2c} w_i^2 < \varepsilon'.$$

This leads to

$$\frac{1}{(k+1)^{2c}} \sum_{i=1}^{k} (i+1)^{2c} w_i^2 \leq \frac{M}{(k+1)^{2c}} + \varepsilon'.$$

It follows that

$$\mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2 (k+1)^{2c}}{8b^2 M}\right).$$

We put $v_k = \frac{\varepsilon^2 (k+1)^{2c}}{8b^2 M}$. By comparative growth, we have

$$\lim_{k\to+\infty} \frac{v_k}{2\ln k} = +\infty.$$

In other words, for sufficiently large k, we have

$$v_k \geq 2\ln k.$$

This gives

$$-v_k \leq 2\ln k = \ln\frac{1}{k^2}.$$

Taking the exponential, which is increasing, we obtain

$$\exp\left(-\frac{\varepsilon^2(k+1)^{2c}}{8b^2M}\right) \leq \frac{1}{k^2}.$$

Since $\frac{1}{k^2}$ is the general term of a convergent series (a Riemann series with an exponent strictly greater than 1), then

$$\sum_{k=1}^{+\infty} \exp\left(-\frac{\varepsilon^2(k+1)^{2c}}{8b^2M}\right) < +\infty. \tag{19}$$

The immediate result is

$$\sum_{k=1}^{+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) < +\infty.$$

This proves the almost complete convergence of the procedure.

To obtain the result on the rate of convergence, simply choose $B = \varepsilon k^{2c}$ in the inequality (17). In fact, we have

$$\sum_{k=1}^{+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > Bk^{-2c}) = \sum_{k=1}^{+\infty} \mathbb{P}(\|f_{k+1} - f_e\| > \varepsilon) < +\infty.$$

4. Numerical Application

It is well known that integral operators have been the subject of extensive research. This interest in integral operators arises from the fact that inverse problems are very often reduced to integral equations [1, 2, 3, 16, 23].

Our numerical application is an integral equation. This is a modified version of an example proposed by Kirsch (see [22] p 45). The simulations were conducted using the MATLAB environment.

We consider the following Fredholm equation of the first kind

$$\frac{1}{2}\int_0^1 (1+\alpha xy)e^{xy}f(y)dy = \frac{e^x(x^3-2x^2+4x-4)}{2x^3} + \frac{2}{x^3}, \tag{20}$$

with the exact solution $f_e(x) = x^2$ and $\alpha \in \mathbb{R}_+^*$ an unknown parameter.

The operator $A_\alpha: \mathbb{L}^2([0,1]) \to \mathbb{L}^2([0,1])$ is given by

$$(A_\alpha f)(x) = \frac{1}{2}\int_0^1 (1+\alpha xy)e^{xy}f(y)dy.$$

We set $K_\alpha(x,y) = \frac{1}{2}(1+\alpha xy)e^{xy}$ the kernel of A_α.

Finally note that the second member of equation (20) belongs to $\mathbb{L}^2([0,1])$.

4.1 *Operator injectivity*

We will show that A_α is injective for all $\alpha \in \mathbb{R}_+^*$, but not surjective.

First, we easily demonstrate by induction that

$$\frac{d^nA_\alpha f(x)}{dx^n} = \int_0^1 (n\alpha + 1 + \alpha xy)y^ne^{xy}f(y)dy, n = 0,1,\ldots$$

Now, let $f \in \mathbb{L}^2([0,1])$ such that $A_\alpha f = 0$.

Let $\{\tilde{P}_i, i \geq 0\}$ be the normalized family of the so-called shifted Legendre polynomials [12, 15, 30] defined on $[0,1]$. This family forms a complete orthonormal basis of $\mathbb{L}^2([0,1])$. Thus, any function $f \in \mathbb{L}^2([0,1])$ expands as follows

$$f(x) = \sum_{i=0}^{+\infty} f_i \tilde{P}_i(x),$$

where $f_i = \langle f(x), \tilde{P}_i(x) \rangle = \int_0^1 f(x) \tilde{P}_i(x) dx$.

For all $x \in [0,1]$ we have

$$A_\alpha f(x) = 0 \Rightarrow \frac{d^n A_\alpha f(x)}{dx^n} = 0 \text{ for all } n = 0,1,\ldots$$

That's to say

$$\int_0^1 (n\alpha + 1 + \alpha xy) y^n e^{xy} f(y) dy = 0, \text{for all } n = 0,1,\ldots$$

In particular for $x = 0$, we have

$$\int_0^1 (n\alpha + 1) y^n f(y) dy = 0, \text{for all } n = 0,1,\ldots$$

As $(n\alpha + 1) \neq 0$, this gives

$$\int_0^1 y^n f(y) dy = 0, \text{for all } n = 0,1,\ldots \tag{21}$$

Since $\tilde{P}_n$ are polynomials, by the linearity of the integral, we can deduce from (21) that

$$\int_0^1 \tilde{P}_n(y) f(y) dy = 0, \text{for all } n = 0,1,\ldots$$

Since, by definition, $\int_0^1 \tilde{P}_n(y) f(y) dy = f_n$, we can deduce that

$$f_n = \langle f(y), \tilde{P}_n(y) \rangle = 0, \text{for all } n = 0,1,\ldots$$

Since the family $\{\tilde{P}_n, n \geq 0\}$ is an orthonormal basis, it immediately follows that

$$f = 0.$$

Thus, A_α is injective for all $\alpha \in \mathbb{R}_+^*$. Furthemore, it can be easily shown that non-zero constant functions are not the image of any function of $\mathbb{L}^2([0,1])$ by the operator A_α.

We can now identify the set to which the parameter a must belong in order to satisfy both the injectivity and boundedness conditions.

We have

$$\|A_\alpha\|^2 = \int_0^1 \int_0^1 (1 + \alpha xy)^2 dx dy \simeq 0.03125(2.97869\alpha^2 + 10.8207\alpha + 14.7355).$$

The two conditions mentioned earlier can be expressed as follows

$$A_\alpha \text{ injective and } \|A_\alpha\| < 1 \Rightarrow 0 < \alpha < 1.19946.$$

It is evident from the above that

$$I_\alpha =]0,1.19946[.$$

4.2 *Discrete formulation of the problem*

Before proceeding with any regularization operation, it is essential to first formulate the problem in a discrete manner. For this, we use the midpoint rule with quadrature and collocations points equidistantly distributed in the interval $[0,1]$ (see [18]). Which give

$$x_i = y_i = \left(i - \frac{1}{2}\right)\frac{1}{N}, i = 1,\ldots,N.$$

The corresponding weights are $h = \frac{1}{N}$.

After approximation of the integral by a finite sum and using collocation, we obtain the following equations

$$f(x_i) - \frac{1}{N}\sum_{j=1}^{N} K(x_i, y_j)f(y_j) = g(x_i), i = 1,\ldots,N.$$

and in matrix writing, we have

$$A_\alpha f = g,$$

where f and g are two vectors of $\mathbb{R}^N$ and A_α a matrix of $\mathcal{M}_N(\mathbb{R})$ whose coefficients are given by

$$a_{ij} = \frac{1}{N}K(x_i, y_j) = \frac{1}{2N}(1 + \alpha x_i y_j)e^{x_i y_j}, i,j = 1,\ldots,N.$$

4.3 *Implementation of the procedure*

The operator A_α satisfies all the hypotheses set forth in the theoretical part, and the conditions of injectivity and boundedness have allowed us to obtain the set I_α, which contains the parameter α. The iterative procedure outlined above can now be applied :

f_1 is given and for $k \geq 1, f_{k+1} = [I - (A_\alpha)_k^*(A_\alpha)_k]f_k + (A_\alpha)_k^*\bar{g} + w_k\xi_k$.

To implement procedure, it suffices to choose the initial value f_1, the sequence of random variables $(\xi_k)_{k\geq 1}$ and the sequence $(w_k)_{k\geq 1}$.

We will test the iterative procedure (5) by considering two types of decay for the sequence $(w_k)_{k\geq 1}$: polynomial and exponential. Specifically, we will first take $w_k = \frac{1}{k^{1+\gamma}}$, followed by $w_k = e^{-\gamma k}$ where $\gamma > 0$.

We take $f_1 = 0.9 \cdot 1$ (1 is the vector where all components are equal to 1), $\alpha_1 = 1$ and we select as noise the sequence of random variables $\xi_k = \delta \cdot \theta_k$ where θ_k follows a normal distribution with zero mean, and δ denotes the noise level.

For $w_k = \frac{1}{k^2}$ and $\delta = 10^{-3}$, we obtain the following result

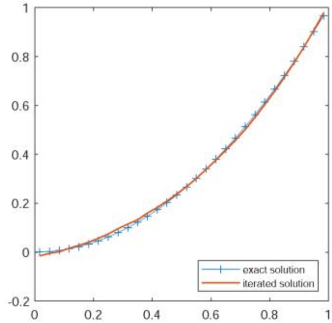

Figure 1
Comparison between the exact solution and the iterated one

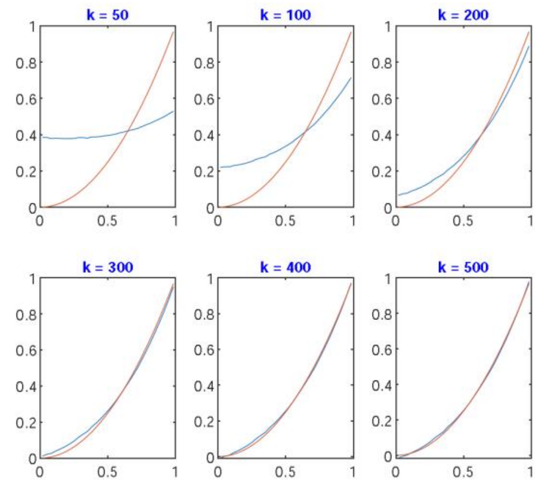

Figure 2
Convergence's evolution

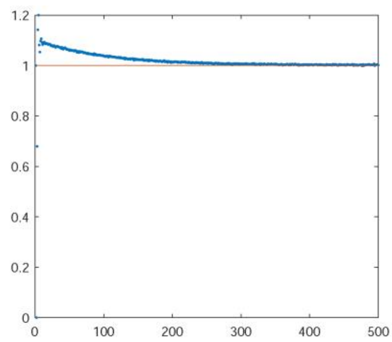

Figure 3
Point cloud of α_k

With the same noise level and $w_k = e^{-k}$, we obtain the following result

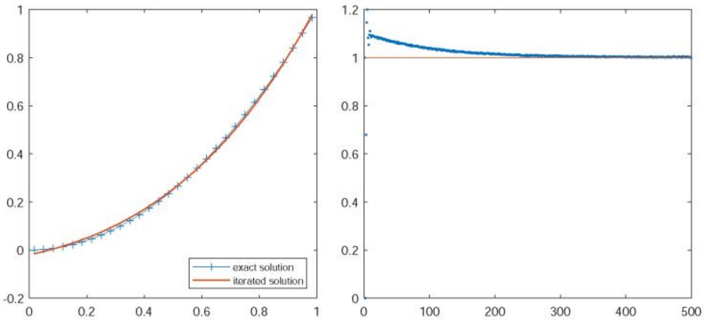

Figure 4

Left : Comparison between the exact and iterated solutions. Right : The α_k points

Figure 1, figure 2, figure 3 and figure 4 show that the iterative procedure converges similarly, regardless of whether the decay is polynomial or exponential. However, as illustrated in Figures 5 and 6, this behavior shifts when more noise is introduced.

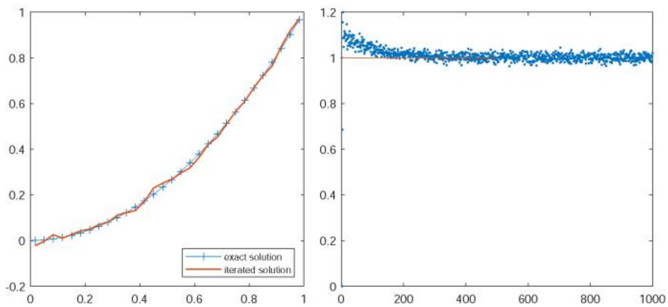

Figure 5

Simulation with $\delta = 10^{-2}$ and $w_k = \frac{1}{k^2}$

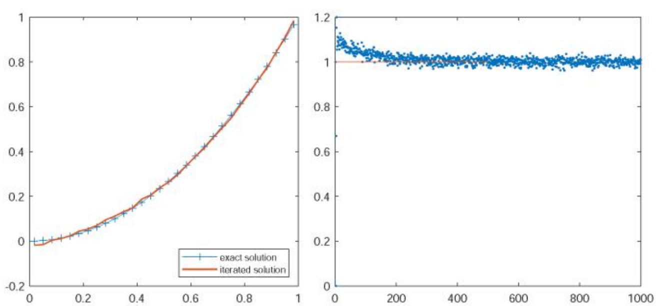

Figure 6

Simulation with $\delta = 10^{-2}$ and $w_k = e^{-k}$

In addition to requiring more iterations to reach convergence, it is evident that the procedure converges significantly faster with exponential decay. The table below shows the squared errors comparison, providing a numerical summary of the previous graphical results.

Table 1

The errors $\|f_k - f_e\|^2$ and $(\alpha_k - 1)^2$ for $\alpha_1 = 1$ with respect to the choice of sequence $(w_k)_{k\geq 1}$ and noise level δ

		$\delta = 0.001$ and $k = 500$	$\delta = 0.01$ and $k = 1000$
$w_k = \frac{1}{k^2}$	$\|f_k - f_e\|^2$	0.0023	0.0042
	$(\alpha_k - 1)^2$	3.5305×10^{-6}	4.4218×10^{-4}
$w_k = e^{-k}$	$\|f_k - f_e\|^2$	0.0023	0.0025
	$(\alpha_k - 1)^2$	1.4967×10^{-6}	3.5420×10^{-5}

Selecting a different initial value α_1 for the sequence (α_k) does not result in any significant changes to the simulation outcomes. For instance, with $\alpha_1 = 0.6$, the following table is obtained :

Table 2

The errors $\|f_k - f_e\|^2$ and $(\alpha_k - 1)^2$ for $\alpha_1 = 0.6$ with respect to the choice of sequence $(w_k)_{k\geq 1}$ and noise level δ

		$\delta = 0.001$ and $k = 500$	$\delta = 0.01$ and $k = 1000$
$w_k = \frac{1}{k^2}$	$\|f_k - f_e\|^2$	0.0023	0.0041
	$(\alpha_k - 1)^2$	3.4311×10^{-6}	4.4221×10^{-4}
$w_k = e^{-k}$	$\|f_k - f_e\|^2$	0.0023	0.0023
	$(\alpha_k - 1)^2$	1.5622×10^{-6}	2.6829×10^{-5}

The numerical results in Table 2 are clearly almost identical to those in Table 1. Simulations conducted with varying initial values of α_1 consistently yield the same result, showing only negligible differences.

4.4 *Optimal parameter value*

It is possible to determine the optimal value for parameter α using an analytical method. The optimal value, denoted as α_*, is defined as follows :

$$J(\alpha_*) = \min_{\alpha \in \mathbb{R}_+^*} J(\alpha),$$

where $J(\alpha) = \|D_\alpha f_e - (g + \xi)\|^2$.

If $\xi = 0$, the functional J is written

$$\begin{aligned} J(\alpha) &= \|D_\alpha f_e - g\|^2, \\ &= \left\| \frac{1}{2}\int_0^1 (1 + \alpha xy) e^{xy} y^2 dy - \left(\frac{e^x(x^3 - 2x^2 + 4x - 4)}{2x^3} + \frac{2}{x^3} \right) \right\|^2, \end{aligned}$$

$$= \int_0^1 \left[\frac{1}{2}\int_0^1 (1+\alpha xy)e^{xy}y^2dy - \left(\frac{e^x(x^3-2x^2+4x-4)}{2x^3} + \frac{2}{x^3}\right)\right]^2 dx.$$

We have

$$\frac{1}{2}\int_0^1 (1+\alpha xy)e^{xy}y^2dy = \frac{e^x[\alpha(x((x-3)x+6)-6)+(x-2)x+2]+6\alpha-2}{2x^3}.$$

After simplification, we find

$$\begin{aligned} J(\alpha) &= (\alpha-1)^2\int_0^1 \left[\frac{e^x(x^3-3x^2+6x-6)+6}{2x^3}\right]^2 dx, \\ &= (\alpha-1)^2 M, \end{aligned}$$

with $M > 0$.

The functional J is convex, its minimum is unique, and it is reached at $\alpha = 1$. It should also be noted that for $\alpha = 1$, $\mathrm{P}DP < 1$.

If $\xi \neq 0$, the functional J will be written

$$\begin{aligned} J(\alpha) = (\alpha-1)^2\int_0^1 &\left[\frac{e^x(x^3-3x^2+6x-6)+6}{2x^3}\right]^2 dx \\ &- \int_0^1 \left(\int_0^1 (1+\alpha xy)e^{xy}y^2dy\right)\xi(x)dx + \int_0^1 \xi^2(x)dx. \end{aligned}$$

In order to examine the effect of noise on the minimum of the J functional under the most critical conditions, we have purposely chosen a noise level far greater than that used in previous simulations.

Thus, we successively take $\xi_{1,2}(x) = 0.1\cos(\pi x)\theta_{1,2}$ where $\theta_1 : \mathcal{N}(0,1)$ and $\theta_2 : \mathcal{N}(0,0.5)$. After running the simulation, we obtain the following result :

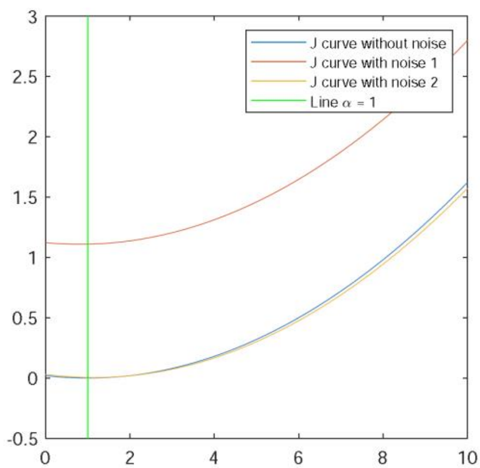

Figure 7

Minimization of the functional J in the absence and presence of noise in the data

Figure 7 illustrates that a noise with zero mean and a sufficiently small norm does not noticeably influence the minimum of the J functional. In short, the optimal value of parameter α is 1 in both cases.

Conclusion

In this study, we present a method to tackle the challenging inverse problem characterized by an operator that depends on an unknown parameter. Our approach involves two nested iterative schemes : a Landweber-type process to approximate the solution of the problem, and a leas-squares minimization technique to update the unknown parameter's optimal value at each iteration.

The numerical results strongly support the theoretical findings, demonstrating the consistency of the proposed method. This work could be further enhanced by extending it to more complex cases, such as nonlinear inverse problems.

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