

Image inpainting via Burgers' equation

Ahmad Jamal Almansour[#]

Ali Tavakoli^{*}

Department of Applied Mathematics

University of Mazandaran

Babolsar

Iran

Abstract

One of the methods for image inpainting is using parabolic partial differential equations (PDEs). These equations model the image inpainting problem as a diffusion process, where the pixel values diffuse or propagate from the known regions to the missing regions. The diffusion equation is typically solved iteratively to estimate the missing pixels. This paper presents an inpainting model based on Burgers' equation, which is a type of convection nonlinear partial differential equation. We use the Burgers' equation to inpaint a damaged image. The numerical scheme of Burgers' equation is given by forward differences of first derivative in space and time. Some numerical examples are given to confirm the efficiency of our proposed method. We present three damaged image. The convergence of our proposed algorithm is verified by these numerical examples. The efficiency of our proposed method is measured by PSNR and MSE.

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1. Introduction

Image inpainting refers to the process of filling in missing or damaged regions within an image. These regions may be removed intentionally (e.g., for privacy reasons or to remove unwanted objects) or unintentionally (e.g., due to data corruption or sensor noise). Also, image reconstruction is a fundamental task in the field of computer vision and image processing, aiming to recover high-quality images from corrupted, degraded, or incomplete visual data. It plays a crucial role in various applications,

[#] E-mail: almansor86@gmail.com

^{*} E-mail: a.tavakoli@umz.ac.ir (Corresponding Author)

including medical imaging, satellite imagery, security and surveillance, as well as digital photography restoration.

Image inpainting and image reconstruction based on solving partial differential equations (PDEs) has emerged as a powerful approach in the field of computer vision and image processing. By formulating image reconstruction as a PDE-constrained optimization problem, this methodology leverages mathematical principles to recover high-quality images from corrupted or incomplete visual data. The PDEs provide a framework for modeling image formation, capturing the underlying physical processes, and guiding the reconstruction process [1,5,8,9,10].

In 1990, Perona and Malik introduced the concept of anisotropic diffusion, a PDE-based image processing technique. The method significantly contributed to the field of PDE-based image reconstruction and has since been employed in various applications [9]. Rudin et al. proposed total variation (TV) regularization, another influential PDE-based approach for solving inverse problems in image reconstruction. TV regularization is used to recover piecewise-constant images from noisy or degraded observations. The method imposes a constraint on the total variation of the image, promoting piecewise smoothness while preserving edges [10]. In 2003, Chan, Shen and Vese provided an overview of variational methods in image processing, with a focus on PDE-based techniques. It discusses various models, including anisotropic diffusion, TV regularization, and level-set methods, and their applications in denoising, segmentation, and image reconstruction [1]. Lin et al. presented a PDE-based method for image inpainting and the recovery of corrupted low-rank matrices. The authors formulated the image reconstruction problem as a convex optimization problem with a low-rank constraint and used the augmented Lagrange multiplier method to solve it efficiently. The proposed approach has demonstrated promising results in image inpainting and restoration [5]. Min et al. proposed an efficient image smoothing method based on PDEs for high-quality image reconstruction. Their approach formulates image smoothing as a weighted least squares problem, where the weights are computed from local image statistics. The technique achieves global smoothing while preserving fine details and edges, making it useful for various image reconstruction tasks [8]. Sun et al. proposed a PDE-based super-resolution method that leverages multiple low-resolution images to reconstruct a high-resolution image. The approach formulates super-resolution as an energy minimization problem, combining image alignment and fusion using PDEs. The method achieves

fast and robust superresolution by effectively handling motion and misalignment between input frames [11].

Ye et al, presented a novel PDE-based approach for solving inverse problems with total variation (TV) regularization. They introduced the Bregman iteration framework to efficiently solve the TV-regularized optimization problem. The method offers fast convergence and superior performance in image reconstruction tasks, such as denoising and inpainting [12]. Liu et al. proposed a PDEbased image restoration method combining total variation regularization with overlapping group sparsity. The authors introduced an efficient algorithm for solving the resulting convex optimization problem. The approach significantly improved the quality of restored images, especially in scenarios with severe noise and blur [6]. The work of McCann et al. explores the integration of PDE-based techniques with deep learning, particularly convolutional neural networks (CNNs), for solving inverse problems in imaging. The authors discuss various PDE-inspired architectures and their applications in image reconstruction, highlighting the potential of combining traditional PDE methods with deep learning [7]. In 2018, Chen et al. presented a PDE-based deep learning approach for lowlight image enhancement. They introduced a novel image formation model that integrates traditional PDE techniques with a CNN-based denoiser. The method effectively reconstructs visually pleasing images from low-light conditions by combining the advantages of both approaches [2].

In recent years, significant advancements have been made in PDE-based image reconstruction techniques, driven by the combination of mathematical modeling, computational algorithms, and optimization strategies. These methods offer the ability to handle a wide range of image degradation problems, including denoising, inpainting, deblurring, and super-resolution, while preserving important image structures and details. In 2021, Hadri et al. presented an approach based on a non-convex constrained PDE for denoising the ultrasound images [3]. Recently, the reconstruction of missing pixels for remote sensing images based on differential equation has been studied by Huang et al. [4].

2. Proposed method

The objective of this paper is to propose and present a novel method for image inpainting. In this technique, the inpainting model is based on Burgers' equation, which is a specific type of nonlinear partial differential equation. By utilizing this equation, we aim to improve the accuracy and

effectiveness of the inpainting process. By exploring and analysing the properties of Burgers' equation, we strive to develop a more efficient and accurate inpainting model for restoring missing or damaged areas in digital images.

Let $\bar{\mathbf{I}} \in \mathbb{R}^N$ be an image that is denoted as a vector by columnwise stacking. Here, N shows the number of pixels and $\bar{\mathbf{I}}$ is an available image with missing or damaged data that is required to be recovered. A model for image inpainting is given by

$$\min_{\mathbf{I}} \|\Phi(\mathbf{I})\|_1 + \frac{\lambda}{2} \|\mathcal{P}(\mathbf{I} - \bar{\mathbf{I}})\|_2^2, \quad (1)$$

where Φ is a transform operator associated with some regularization matrices, λ is a positive value and $\mathcal{P}$ is a binary matrix of dimension N such that $\mathcal{P}\bar{\mathbf{I}}$ keeps the safe pixels of $\bar{\mathbf{I}}$, i.e., if $\mathcal{D}$ is a set of pixels that the image lost/damaged there, then

$$(\mathcal{P}\mathbf{I})(x) = \begin{cases} \mathbf{I}(x), & x \notin \mathcal{D}, \\ 0, & x \in \mathcal{D}. \end{cases} \quad (2)$$

A usual selection for Φ is $\Phi = \nabla$ (gradient) or $\Phi = \Delta$ (Laplacian). Without loss of generality, we assume that

$$\mathcal{D} = \{x_1, x_2, \dots, x_k\}, \quad (3)$$

where $x_1 < x_2 < \dots < x_k$. Now, our suggestion is given by

$$\Phi(\mathbf{I}) = \left[v_1 \frac{\partial^2 \mathbf{I}}{\partial x_1^2} - \frac{1}{2} \frac{\partial \mathbf{I}^2}{\partial x_1}, v_2 \frac{\partial^2 \mathbf{I}}{\partial x_2^2} - \frac{1}{2} \frac{\partial \mathbf{I}^2}{\partial x_2}, \dots, v_k \frac{\partial^2 \mathbf{I}}{\partial x_k^2} - \frac{1}{2} \frac{\partial \mathbf{I}^2}{\partial x_k} \right], \quad (4)$$

where v_i is a positive constant value. Then, the model (1) is rewritten by:

$$\min_{\mathbf{I}} \sum_{i=1}^N \left| v_i \frac{\partial^2 \mathbf{I}}{\partial x_i^2} - \frac{1}{2} \frac{\partial \mathbf{I}^2}{\partial x_i} \right| + \frac{\lambda}{2} \|\mathcal{P}(\mathbf{I} - \bar{\mathbf{I}})\|_2^2. \quad (5)$$

Some remarks on the proposed model (5):

1. In our proposed model, we first rescale the image into double precision in which the output will be in the range $[0, 1]$.
2. The constant value v should be greater than one, otherwise, the model tries to consider the maximum value for the inpainting points, i.e., $\mathbf{I}(x) = 1$ for $x \in \mathcal{D}$.

3. For each $x \in \mathcal{D}$, v_i plays as a penalty value for considering big $\partial \mathbf{I} / \partial x_i^2$. Hence, the model considers the least $\partial \mathbf{I} / \partial x_i^2$.

Now, in order to solve (5), the inpainting problem can be given by

$$\frac{\partial \mathbf{I}(x)}{\partial t} = v \frac{\partial^2 \mathbf{I}}{\partial x^2} - \frac{1}{2} \frac{\partial \mathbf{I}^2}{\partial x} = v \frac{\partial^2 \mathbf{I}}{\partial x^2} - \mathbf{I} \frac{\partial \mathbf{I}}{\partial x}, \quad x \in \mathcal{D}. \quad (6)$$

It is well known that the partial differential equation (6) is a Burgers' equation that is usually given by

$$\frac{\partial \mathbf{I}}{\partial t} + \mathbf{I} \frac{\partial \mathbf{I}}{\partial x} = v \frac{\partial^2 \mathbf{I}}{\partial x^2}. \quad (7)$$

In order to write an iterative scheme for (6), one can consider $\partial^2 \mathbf{I} / \partial x^2$ as follows:

$$\frac{\partial^2 \mathbf{I}}{\partial x^2}(x, t_n) = \mathbf{I}_n(x+1) - 2\mathbf{I}_n(x) + \mathbf{I}_n(x-1), \quad (8)$$

where $\mathbf{I}_n(x) = \mathbf{I}(x, t_n)$. Also, using forward differences for the first derivative, one can write:

$$\frac{\partial \mathbf{I}}{\partial t_n} = \frac{\mathbf{I}_{n+1}(x) - \mathbf{I}_n(x)}{\Delta t}, \quad (9)$$

where $\Delta t = t_{n+1} - t_n$, $n = 0, 1, \dots$.

Then, for each $x \in \mathcal{D}$, the Eq. (6) can be discretized by

$$\mathbf{I}_{n+1}(x) = \mathbf{I}_n(x) + \Delta t \left[v(\mathbf{I}_n(x+1) - 2\mathbf{I}_n(x) + \mathbf{I}_n(x-1)) - (\mathbf{I}_n(x+1) - \mathbf{I}_n(x))\mathbf{I}_n(x) \right]. \quad (10)$$

The stop criterion can be considered by

$$\|\mathbf{I}_{n+1}(x) - \mathbf{I}_n(x)\| < \varepsilon, \quad (11)$$

where ε is an acceptable tolerance. Algorithm 1 describes the implementation of our proposed method.

Algorithm 1 Image Inpainting

Input: Stop tolerance ε , time step Δt , constant values v_i for $i = 1, 2, \dots, K$, and the damaged/lost image $\mathbf{I}$ (considered as a vector)

1. Rescale the input image into double precision

2. Set $\mathbf{I}_0 := \mathcal{P}\bar{\mathbf{I}}$
3. Set $\mathbf{I}_1 := 1 - \mathbf{I}_0$
4. $n := 0$
5. While $\|\mathbf{I}_{n+1}(x) - \mathbf{I}_n(x)\| < \varepsilon$, do:
 6. For $x \in \mathcal{D}$, do:
 7. $\mathbf{I}_{n+1}(x) := \mathbf{I}_n(x) + \Delta t[v(\mathbf{I}_n(x+1) - 2\mathbf{I}_n(x) + \mathbf{I}_n(x-1)) - (\mathbf{I}_n(x+1) - \mathbf{I}_n(x))\mathbf{I}_n(x)]$
 8. End For
 9. $n := n + 1$
 10. End While

Output: $\mathbf{I} := \mathbf{I}_{n-1}$

3. Numerical experiments

In this section, we present some numerical examples to confirm the efficiency of our proposed method. The stop criterion for all examples is set to $\varepsilon = 10^{-2}$. For comparison between the original image and the inpainted image, we used the MSE and PSNR criteria.

Let I be an original image of size $M \times N$ and u be the inpainted image. The MSE is defined by

$$\text{MSE} = \frac{1}{MN} \sum_{i=1}^M \sum_{j=1}^N (I_{i,j} - u_{i,j})^2, \quad (12)$$

and the PSNR is defined by

$$\text{PSNR} = 10 \log \left(\frac{1}{\text{MSE}} \right). \quad (13)$$

In Figure 1(a), an image with 30 percent of data loss is displayed. By applying Algorithm 1, it can be observed that after 13 iterations, the image is successfully reconstructed with a stop tolerance of $\varepsilon = 10^{-2}$. The reconstructed images at the 2nd, 6th, and 13th iterations are illustrated in Figures 1(b-d), respectively. The value of MSE and PSNR for this image are 6.0712×10^{-5} and 42.1672, respectively.

In the second example, Figure 2(a) presents the image of Lena with some damaged parts, visible as white lines. By applying Algorithm 1, the image is successfully reconstructed after 39 iterations, with a stop tolerance of $\varepsilon = 10^{-2}$. The reconstructed images at the 2nd, 20th, and 39th iterations

are depicted in Figures 2(b-d), respectively. The value of MSE and PSNR for this image are 6.7972×10^{-5} and 41.6767, respectively.

In the third example, Figure 3(a) displays an image with missing data in a specific zone, which has been visually cleared using a circle. Algorithm 1 demonstrates that after 82 iterations, the image is effectively reconstructed with a stop tolerance of $\varepsilon = 10^{-2}$. The reconstructed images at the 2nd, 40th, and 82nd iterations are illustrated in Figures 3(b-d), respectively. The value of MSE and PSNR for this image are 5.3666×10^{-5} and 42.7030, respectively.

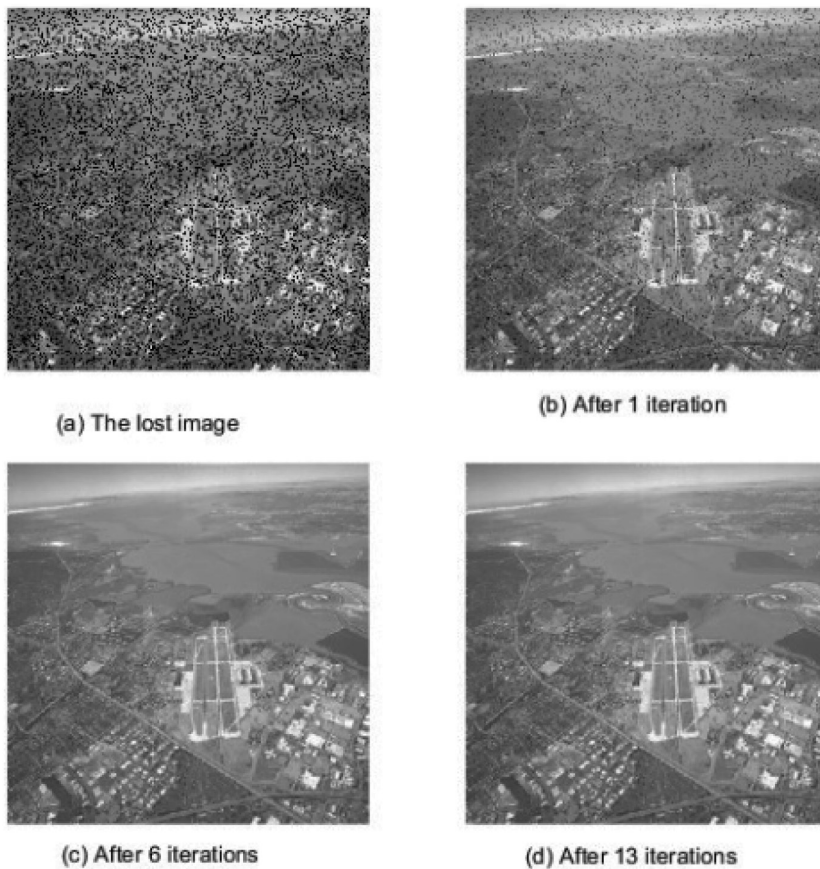


Figure 1

(a) an image with 30 percent lost data, (b-d) implementing Algorithm 1 and convergence after 13 iterations.

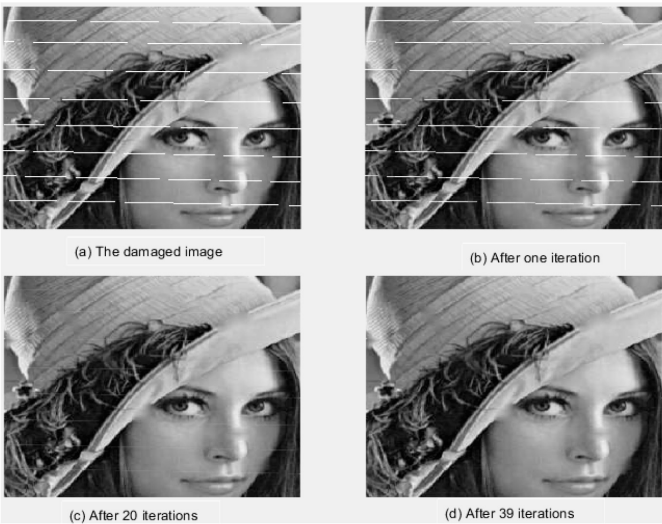


Figure 2

(a) The image of Lena with lost data, (b-d) implementing Algorithm 1and convergence after 39 iterations.

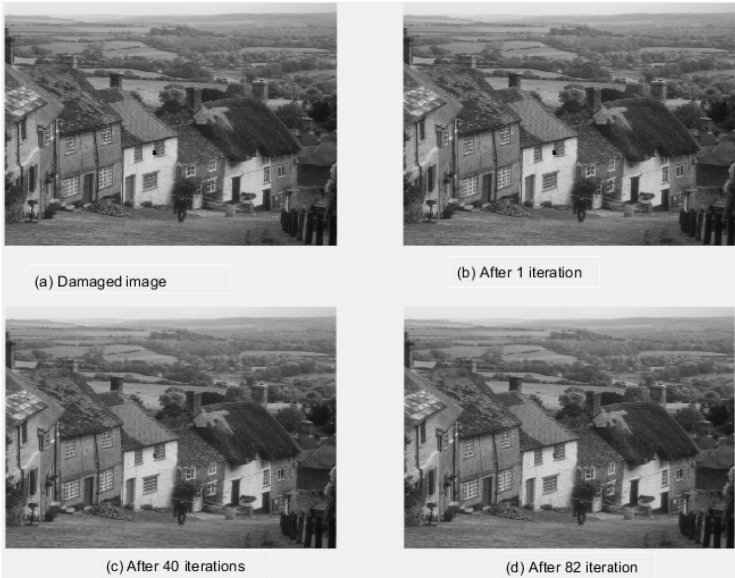


Figure 3

(a) an image with lost data in a zone cleared by a circle, (b-d) implementing Algorithm 1 and convergence after 82 iterations

Conclusion

In this paper, we have used the Burgers' equation to recover lost images. We have proposed an algorithm for this purpose to implement numerical examples on it. It has been shown that our method converges well. The values of PSNR and MSE indicate that our method has effective performance.

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