Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 4, pp. 875–890
DOI : 10.47974/JIM-2338

Global exponential stability for generalized Cohen-Grossberg neural networks with variable delay

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Abstract

New, easily verifiable criteria for the global exponential stability (GES) and the global practical exponential stability (PGES) of generalized Cohen–Grossberg neural networks with time-varying delays are presented. The analysis leverages a generalized Halanay inequality and the theory of Dini derivatives to establish results that are less restrictive than previous approaches. The derived conditions are validated through three illustrative examples, confirming the theoretical advancements and their practical utility.

Subject Classification: *92B20, 34K20, 93D09, 34D20.*

Keywords: *Generalized CGNNs, Global exponential stability, Halanay inequality, Dini derivatives, Time-varying delays.*

1. Introduction

Artificial neural networks (ANNs), inspired by the biological brain, provide a powerful bionic framework for modeling cognitive processes and solving complex engineering problems. Over the decades, a diverse family of dynamical neural networks—including competitive NNs [1], cellular NNs [2], memristor-based NNs [3], Hopfield networks [4], Cohen–Grossberg networks (CGNNs) [5, 6], and bidirectional associative memory networks [7, 8]—has been

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developed and successfully applied across numerous domains. These applications range from pattern recognition and associative memory to optimization, parallel computing, and control systems (see [9]–[14]).

A fundamental requirement for the effective deployment of NNs in these applications is the guarantee of stable dynamics. Specifically, a network must possess a unique equilibrium point that is globally asymptotically stable, ensuring convergent behavior from any initial condition. Consequently, stability analysis has become a cornerstone of neural network theory (e.g., [15]–[20]).

A significant challenge in practical implementations arises from inherent time delays, caused by the finite switching speed of amplifiers and finite signal propagation times between neurons. These delays can drastically alter a network's dynamics, potentially inducing instability, oscillations, bifurcations, or chaos in a system that is otherwise stable in their absence. This makes the analysis of delayed NNs a critical area of study. Extensive research has been dedicated to establishing sufficient conditions for global asymptotic stability in such systems, often employing Lyapunov functionals, Razumikhin techniques, and M-matrix theory (see [21]–[26]).

However, stability analysis must further evolve to address a more complex class of problems: neutral-type delays, where time lags affect not only the neuron states but also their time derivatives. This gives rise to neutral-type delayed neural networks, described by systems of nonlinear neutral delay differential equations. A prominent and general model within this class is the neutral-type delayed Cohen–Grossberg neural network, governed by:

$$\begin{aligned}\frac{du_i(s)}{ds} &= -a_i(u_i(s))\Big(b_i(u_i(s)) - \sum_{j=1}^{n} c_{ij}(s)W_j(u_j(s)) \\ &\quad - \sum_{j=1}^{n} d_{ij}(s)W_j(u_j(s-\tau_j(s))) + I_i(s)\Big), \quad i = 1,2,\dots,n.\end{aligned} \tag{1.1}$$

While early work on CGNNs focused on systems with constant delays [27, 28, 29], the more general and practically relevant case of time-varying delays has since been investigated [30, 31, 6, 32, 33, 34, 23, 35]. Building upon this foundation, the present paper aims to derive novel and less conservative criteria for the global exponential stability (GES) of the neutral-type delayed CGNN described by system (1.1). Our approach leverages the generalized Halanay inequality, which allows us to establish sufficient conditions that are both rigorous and straightforward to verify in applications.

The remainder of this paper is organized as follows. Section 2 details the model description and essential preliminaries. Section 3 presents our main stability results. The effectiveness of these theoretical findings is demonstrated through three numerical examples in Section 4. Finally, Section 5 offers concluding remarks.

Model Description and Preliminary Assumptions

We consider the neural network model described by the system of differential equations (1.1). The system comprises $n \geqslant 2$ neurons, and the variable $s \geqslant s_0 \geqslant 0$ denotes time. The components of the model are defined as follows:

- $u_i(s)$ represents the state of the i-th neuron.
- $a_i(u_i(s))$ is a state-dependent amplification function.
- $b_i(u_i(s))$ is a nonlinear function that encapsulates the neuron's intrinsic dynamics.
- $C(s) = (c_{ij}(s))_{n\times n}$ is the connection weight matrix, defining the strength of synaptic connections at time s.
- $D(s) = (d_{ij}(s))_{n\times n}$ is the delayed connection weight matrix.
- $\tau_j(s)$ is a time-dependent transmission delay.
- $W_j(\cdot)$ is the activation function, which models the neuronal output in response to its input. It is typically chosen as a smooth, monotonic sigmoid function.
- $I_i(s)$ is the external input signal received by the i-th unit at time s.

To establish the analytical framework for our study, we impose the following assumptions on system (1.1):

Assumption $(\mathcal{A}_1)$ (Amplification Function)

For each $i = 1,2,\dots,n$, the function $a_i\colon \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous. That is, there exist positive constants $\overline{a}_i$ and k_i such that for all $u, x, y \in \mathbb{R}$:

$$\begin{aligned} |a_i(u)| &\leqslant \overline{a}_i, \\ |a_i(x) - a_i(y)| &\leqslant k_i|x-y|. \end{aligned}$$

Assumption $(\mathcal{A}_2)$ (Dynamics Function)

For each $i = 1,2,\dots,n$, the function $G_i(u) = a_i(u)b_i(u)$ is continuously differentiable on $\mathbb{R}$. Furthermore, it is strictly increasing, satisfying:

$$\frac{dG_i(u)}{du} \geqslant \alpha_i > 0 \quad \text{for all } u \in \mathbb{R}.$$

Assumption $(\mathcal{A}_3)$ (Bounded Time-Varying Parameters)

For each $i, j = 1,2,\dots,n$, the functions $c_{ij}(s)$, $d_{ij}(s)$, and $\tau_j(s)$ are continuous and bounded on $[0,\infty)$. Specifically, there exist constants $\overline{c}_{ij}$, $\overline{d}_{ij}$, and $\overline{\tau} > 0$ such that for all $s \geqslant 0$:

$$\begin{aligned} |c_{ij}(s)| &\leqslant \overline{c}_{ij}, \\ |d_{ij}(s)| &\leqslant \overline{d}_{ij}, \\ 0 < \tau_i(s) &\leqslant \overline{\tau}. \end{aligned}$$

Assumption $(\mathcal{A}_4)$ (Activation Function)

For each $i = 1,2,\dots,n$, the activation function $W_i\colon \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous. That is, there exist positive constants M_i and L_i such that for all $u, x, y \in \mathbb{R}$:

$$\begin{aligned} |W_i(u)| &\leqslant M_i, \\ |W_i(x) - W_i(y)| &\leqslant L_i|x - y|. \end{aligned}$$

The system (1.1) is analyzed under the following initial conditions:

$$u_i(\varepsilon) = \varphi_i(\varepsilon), \quad \text{for} \quad \varepsilon \in [-\tau, 0], \quad i = 1,2, \dots, n,$$

where each $\varphi_i: [-\tau, 0] \to \mathbb{R}$ is a continuous function.

Remark 2.1: The boundedness and Lipschitz continuity properties stipulated in Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ ensure that the vector field defined by the right-hand side of (1.1) satisfies a local Lipschitz condition with respect to the state variable. This guarantees that the system (1.1), supplemented with the given initial conditions, has a unique continuous solution $u(s)$ that exists for all $s \in [-\tau, \infty)$, as established in the classical literature [37].

Definition 2.2: ([32]). The network (1.1) is globally exponentially stable (GES) if there exist constants $M \geqslant 1$ and $\gamma > 0$ such that for any two solutions $u(s)$ and $v(s)$ with initial conditions $\psi, \varphi \in C^1[-\tau, 0]$, respectively, the following inequality holds for all $s \geqslant s_0$:

$$|u(s) - v(s)| \leqslant M \|\boldsymbol{\psi} - \boldsymbol{\varphi}\|_\tau e^{-\gamma(s-s_0)}.$$

Here, $\|\boldsymbol{\psi} - \boldsymbol{\varphi}\|_\tau = \sup_{\varepsilon \in [s_0 - \tau, s_0]} \sum_{j=1}^{n} |\psi_j(\varepsilon) - \varphi_j(\varepsilon)|$ defines the norm on the initial condition space.

Definition 2.3 ([36]). Let $r \geqslant 0$ and define the ball $\mathcal{B}_r = \{u \in \mathrm{R}^n \ / \ \|u\| r\}$.

(i) The ball $\mathcal{B}r$ is GES if there exist constants $\gamma > 0$ and $c \geqslant 0$ such that for every initial condition $\phi \in C^1[-\tau, 0]$, the corresponding solution $u(s)$ satisfies

$$|u(s)| \leqslant c\|\phi\|_\tau \, e^{-\gamma(s-s_0)} + r \quad \forall \ s \geqslant s_0$$

(i) The system (1.1) is globally practically exponentially stable (GPES) if there exists an $r > 0$ such that the ball $\mathcal{B}_r$ is globally exponentially stable.

Definition 2.4: [37] Let $f(s): \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the upper right Dini derivative is defined as

$$D^+ f(s) = \overline{\lim_{h \to 0^+}} \frac{1}{h}(f(s+h) - f(s)).$$

Lemma 2.5: [32] (Generalized Halanay Inequality*). Assume $p(s)$ and $q(s)$ be continuous with $p(s) > q(s) \geqslant 0$ and $inf_{s \geqslant s_0} \frac{p(s) - q(s)}{1 + 2\tau q(s)} \geqslant \sigma > 0$ for all $s \geqslant 0$, and $v(s)$ is a continuous function on $s \geqslant s_0$ satisfying the following inequality for all $s \geqslant s_0$:*

$$D^+ v(s) \leqslant -p(s)v(s) + q(s)\overline{v}(s),$$

where $\overline{v}(s) = sup_{s-\tau \leqslant \varepsilon \leqslant s}\{|v(\varepsilon)|\}$. Then for $s \geqslant s_0$, we have

$$v(s) \leqslant \overline{v}(s_0) e^{-\lambda^*(s-s_0)}$$

in which λ^* *is defined as*

$$\lambda^* = \inf_{s \geqslant s_0} \{\lambda(s):\ \lambda(s) - p(s) + q(s)e^{\lambda(s)\tau} = 0\}.$$

3. Main results

The aim of this section is to analyze the stability properties of network (1.1). We employ a generalized Halanay inequality to obtain sufficient criteria that ensure its GES as well as its GPES.

Theorem 3.1: *Let* $u(s)$ *and* $v(s)$ *be two solutions of the system (1.1) with initial functions* ψ *and* φ *in* $C^1[-\tau, 0]$ *, respectively. Suppose assumptions* $(\mathcal{A}_1)$- $(\mathcal{A}_4)$ *hold. If, in addition, the following conditions are satisfied:*

1. $p(s) := \alpha_i - k_i \sum_{j=1}^{n} M_j |c_{ij}(s)| - \bar{I}_i k_i > 0$,
2. $q(s) := \bar{a}_i \sum_{j=1}^{n} L_j \big(|c_{ij}(s)| + |d_{ij}(s)|\big) \geqslant 0$,
3. $\inf_{s \geqslant s_0} \left\{\frac{p(s) - q(s)}{1 + 2\bar{\tau} q(s)}\right\} \geqslant \sigma > 0$, *for some constant* σ,

for all $i = 1,2, \ldots, n$ *and* $s \geqslant s_0$*, then the neural network (1.1) is GES.*

Proof: Consider the difference between the two solutions $u(s)$ and $v(s)$. For each neuron i, we define the error term $z_i(s) = u_i(s) - v_i(s)$. From system (1.1), the dynamics of this error are given by:

$$\begin{aligned}
\frac{dz_i(s)}{ds} = &-[a_i(u_i(s))b_i(u_i(s)) - a_i(v_i(s))b_i(v_i(s))] \\
&+[a_i(u_i(s)) - a_i(v_i(s))] \textstyle\sum_{j=1}^{n} c_{ij}(s) W_j(v_j(s)) \\
&+a_i(u_i(s)) \textstyle\sum_{j=1}^{n} c_{ij}(s)\big[W_j(u_j(s)) - W_j(v_j(s))\big] \\
&+[a_i(u_i(s)) - a_i(v_i(s))] \textstyle\sum_{j=1}^{n} d_{ij}(s) W_j(v_j(s - \tau_j(s))) \\
&+a_i(u_i(s)) \textstyle\sum_{j=1}^{n} d_{ij}(s)\big[W_j(u_j(s - \tau_j(s))) - W_j(v_j(s - \tau_j(s)))\big] \\
&-[a_i(u_i(s)) - a_i(v_i(s))] I_i(s).
\end{aligned} \tag{3.1}$$

For the purpose of estimation, we define the absolute error $\zeta_i(s) = |z_i(s)| = |u_i(s) - v_i(s)|$ and analyze the Dini derivative of $\zeta_i(s)$. Under assumptions $(\mathcal{A}_1)$- $(\mathcal{A}_4)$, we derive the following upper bounds for the terms in (3.1):

Term 1: $\quad -[G_i(u_i(s)) - G_i(v_i(s))] \leqslant -\alpha_i \zeta_i(s),$ (3.2)

Term 2: $\quad \big|[a_i(u_i(s)) - a_i(v_i(s))] \sum_{j=1}^{n} c_{ij}(s) W_j(v_j(s))\big|$

$\leqslant k_i \zeta_i(s) \sum_{j=1}^{n} M_j |c_{ij}(s)|,$ (3.3)

Term 3: $\quad \big|a_i(u_i(s)) \sum_{j=1}^{n} c_{ij}(s)\big[W_j(u_j(s)) - W_j(v_j(s))\big]\big|$

$\leqslant \bar{a}_i \sum_{j=1}^{n} L_j |c_{ij}(s)| \zeta_j(s),$ (3.4)

Term 4: $\quad \big|[a_i(u_i(s)) - a_i(v_i(s))] \sum_{j=1}^{n} d_{ij}(s) W_j(v_j(s - \tau_j(s)))\big|$

$\leqslant k_i \zeta_i(s) \sum_{j=1}^{n} M_j |d_{ij}(s)|,$ (3.5)

Term 5: $\left|a_i(u_i(s))\sum_{j=1}^{n} d_{ij}(s)\left[W_j(u_j(s-\tau_j(s)))-W_j(v_j(s-\tau_j(s)))\right]\right|$

$$\leqslant \overline{a}_i \sum_{j=1}^{n} L_j|d_{ij}(s)|\zeta_j(s-\tau_j(s)), \tag{3.6}$$

Term 6: $|-[a_i(u_i(s))-a_i(v_i(s))]I_i(s)|\overline{I}_i k_i\zeta_i(s)$. (3.7)

Substituting the bounds (3.2)–(3.7) into the derivative of $\zeta_i(s)$ yields the following differential inequality:

$$\begin{aligned}\frac{d\zeta_i(s)}{ds} \leqslant & -\alpha_i\zeta_i(s) + k_i\zeta_i(s)\sum_{j=1}^{n} M_j|c_{ij}(s)| + \overline{I}_i k_i\zeta_i(s)\\ & +\overline{a}_i\sum_{j=1}^{n} L_j|c_{ij}(s)|\zeta_j(s) + k_i\zeta_i(s)\sum_{j=1}^{n} M_j|d_{ij}(s)|\\ & +\overline{a}_i\sum_{j=1}^{n} L_j|d_{ij}(s)|\zeta_j(s-\tau_j(s)).\end{aligned}$$

Grouping the terms affecting $\zeta_i(s)$ and those affecting delayed errors $\zeta_j(s-\tau_j(s))$, we obtain:

$$\begin{aligned}\frac{d\zeta_i(s)}{ds} \leqslant & -\left(\alpha_i - k_i\sum_{j=1}^{n} M_j(|c_{ij}(s)|+|d_{ij}(s)|) - \overline{I}_i k_i\right)\zeta_i(s)\\ & +\overline{a}_i\sum_{j=1}^{n} L_j|c_{ij}(s)|\zeta_j(s) + \overline{a}_i\sum_{j=1}^{n} L_j|d_{ij}(s)|\zeta_j(s-\tau_j(s)).\end{aligned}$$

A conservative estimate for the right-hand side, using the definition of $p(s)$ and $q(s)$ from the theorem statement, leads to:

$$\frac{d\zeta_i(s)}{ds} \leqslant -p(s)\zeta_i(s) + q(s)\sup_{s-\overline{\tau}\leqslant\xi\leqslant s}\max_j \zeta_j(\xi) \tag{3.8}$$

$$= -p(s)\zeta_i(s) + q(s)\,\overline{\zeta}(s), \tag{3.9}$$

where

1. $\overline{\zeta}(s) = \sup_{s-\overline{\tau}\leqslant\xi\leqslant s}\max_j\zeta_j(\xi)$,
2. $p(s) := \alpha_i - k_i\sum_{j=1}^{n} M_j|c_{ij}(s)| - \overline{I}_i k_i$,
3. $q(s) := \overline{a}_i\sum_{j=1}^{n} L_j\left(|c_{ij}(s)|+|d_{ij}(s)|\right)$.

Given the conditions $p(s)>0$, $q(s)\geqslant 0$, and $\inf_{s\geqslant s_0}\left\{\frac{p(s)-q(s)}{1+2\overline{\tau}q(s)}\right\}\geqslant\sigma>0$, the assumptions of Lemma 2.5 are satisfied. Therefore, there exists a positive constant λ^* such that:

$$\zeta_i(s) \leqslant \overline{\zeta}(s_0)e^{-\lambda^*(s-s_0)}, \quad \text{for all } s\geqslant s_0,$$

or equivalently,

$$|u_i(s)-v_i(s)| \leqslant \|\boldsymbol{\psi}-\boldsymbol{\varphi}\|_\tau e^{-\lambda^*(s-s_0)}, \quad i=1,2,\dots,n,$$

where $\|\cdot\|$ is the supremum norm on $[-\tau,0]$. This confirms that the system (1.1) is GES. This completes the proof.

Theorem 3.2: *Suppose assumptions $(\mathcal{A}_2)$–$(\mathcal{A}_4)$ hold. Furthermore, assume the amplification functions are constant and positive, i.e.,* $a_i(u)=a_i>0$ *for* $i=1,2,\dots,n$. *If the following conditions are also satisfied:*

1. *The functions* $p(s)=\alpha_i$ *and* $q(s)=\overline{a}_i\sum_{j=1}^{n} L_j\left(|c_{ij}(s)|+|d_{ij}(s)|\right)$ *are such that* $p(s)>q(s)\geqslant 0$ *for all* $s\geqslant s_0$.

2. $\inf_{s\geqslant s_0}\left\{\frac{p(s)-q(s)}{1+2\overline{\tau}q(s)}\right\}\geqslant\sigma>0$, *where* σ *is a constant,*

then the neural network (1.1) *is GES.*

Proof: Consider two solutions $u(s)$ and $v(s)$ of system (1.1) with distinct initial conditions. Let $z_i(s)=u_i(s)-v_i(s)$ be the difference between these solutions for the i-th neuron. From (1.1), and using the simplification $a_i(u)=a_i$, the derivative of $z_i(s)$ is:

$$\begin{aligned}\frac{dz_i(s)}{ds}&=-a_i[b_i(u_i(s))-b_i(v_i(s))]\\&\quad+a_i\sum_{j=1}^{n}c_{ij}(s)\big[W_j(u_j(s))-W_j(v_j(s))\big]\\&\quad+a_i\sum_{j=1}^{n}d_{ij}(s)\big[W_j(u_j(s-\tau_j(s)))-W_j(v_j(s-\tau_j(s)))\big].\end{aligned}$$

Now, define the absolute error $\zeta_i(s)=|z_i(s)|$. Employing standard comparison techniques for differential inequalities (e.g., the Dini derivative) and invoking assumptions $(\mathcal{A}_2)$–$(\mathcal{A}_4)$, we estimate the growth of $\zeta_i(s)$:

$$\begin{aligned}\frac{d\zeta_i(s)}{ds}&\leqslant-\alpha_i\zeta_i(s)\quad\text{(from }(\mathcal{A}_2))\\&\quad+\overline{a}_i\sum_{j=1}^{n}|c_{ij}(s)|L_j\zeta_j(s)\quad\text{(Lipschitz property of }\mathrm{W}_j)\\&\quad+\overline{a}_i\sum_{j=1}^{n}|d_{ij}(s)|L_j\zeta_j(s-\tau_j(s)).\quad\text{(Lipschitz property of }W_j)\end{aligned}$$

This simplifies to the inequality:

$$\frac{d\zeta_i(s)}{ds}\leqslant-\alpha_i\zeta_i(s)+\overline{a}_i\sum_{j=1}^{n}L_j\big|c_{ij}(s)\big|\zeta_j(s)+\overline{a}_i\sum_{j=1}^{n}L_j\big|d_{ij}(s)\big|\zeta_j(s-\tau_j(s)).\tag{3.10}$$

We can construct a more conservative but simpler inequality by bounding the right-hand side. Let $\overline{\zeta}(s)=\sup_{s-\overline{\tau}\leqslant\xi\leqslant s}\max_{1\leqslant j\leqslant n}\zeta_j(\xi)$ be the maximum error over the recent time interval of length $\overline{\tau}$. Then, inequality (3.10) implies:

$$\begin{aligned}\frac{d\zeta_i(s)}{ds}&\leqslant-\alpha_i\zeta_i(s)+\overline{a}_i\Big(\sum_{j=1}^{n}L_j|c_{ij}(s)|\Big)\overline{\zeta}(s)+\overline{a}_i\Big(\sum_{j=1}^{n}L_j|d_{ij}(s)|\Big)\overline{\zeta}(s)\\&=-\underbrace{\alpha_i}_{p(s)}\zeta_i(s)+\underbrace{\overline{a}_i\sum_{j=1}^{n}L_j\big(|c_{ij}(s)|+|d_{ij}(s)|\big)}_{q(s)}\overline{\zeta}(s).\end{aligned}$$

Thus, we arrive at the key differential inequality for each i:

$$\frac{d\zeta_i(s)}{ds}\leqslant-p(s)\zeta_i(s)+q(s)\overline{\zeta}(s).\tag{3.11}$$

Given the conditions $p(s)>q(s)\geqslant 0$ and $\inf_{s\geqslant s_0}\left\{\frac{p(s)-q(s)}{1+2\overline{\tau}q(s)}\right\}\geqslant\sigma>0$, the assumptions of Lemma 2.5 are satisfied. Applying the lemma, we conclude that there exists a decay rate $\lambda^*>0$ such that:

$$\zeta_i(s)\leqslant\|\boldsymbol{\psi}-\boldsymbol{\varphi}\|_\tau e^{-\lambda^*(s-s_0)},\quad\text{for all } s\geqslant s_0,$$

where $\|\boldsymbol{\psi}-\boldsymbol{\varphi}\|=\sup_{-\overline{\tau}\leqslant\xi\leqslant0}\max_i|\psi_i(\xi)-\varphi_i(\xi)|$ is the initial error in the supremum norm. This establishes that the system (1.1) is globally exponentially stable.

Remark 3.3: Since Theorem 3.1 and Theorem 3.2 guarantee that all solutions of the network (1.1) are GES, it follows that the network must possess a unique equilibrium point.

Theorem 3.4: *Assume that assumptions* $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ *hold. Then the neural network (1.1) is GPES.*

Proof: To establish global practical exponential stability, we analyze the dynamics of $u_i^2(s)$ for each neuron $i = 1,2,\dots,n$. Multiplying both sides of the i-th equation in (1.1) by $2u_i(s)$ yields:

$$2u_i(s)\frac{du_i(s)}{ds} = -2u_i(s)a_i(u_i(s))(b_i(u_i(s)) - \sum_{j=1}^n c_{ij}(s)W_j(u_j(s)) - \sum_{j=1}^n d_{ij}(s)W_j(u_j(s-\tau_j(s))) + I_i(s)).$$

Under assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$, we can derive the following differential inequality:

$$\frac{d}{ds}(u_i^2(s)) \leqslant -2\alpha_i u_i^2(s) + 2\gamma_i|u_i(s)|, \tag{3.12}$$

where $\gamma_i = |a_i(0)b_i(0)| + \sum_{j=1}^n (|\overline{c}_{ij}| + |\overline{d}_{ij}|)M_j + |\overline{a_i}||\overline{I}_i|$ is a positive constant.

To simplify (3.12), we complete the estimate. For any $u_i(s)$, the following holds:

$$\begin{aligned}-2\alpha_i u_i^2(s) + 2\gamma_i|u_i(s)| &= -\alpha_i u_i^2(s) - \alpha_i u_i^2(s) + 2\gamma_i|u_i(s)|\\ &\leqslant -\alpha_i u_i^2(s) + \sup_{x\in\mathbb{R}}(-\alpha_i x^2 + 2\gamma_i|x|).\end{aligned}$$

The supremum of the function $f(x) = -\alpha_i x^2 + 2\gamma_i|x|$ is attained at $|x| = \gamma_i/\alpha_i$ and its maximum value is γ_i^2/α_i. Denoting $\beta_i = \gamma_i^2/\alpha_i$, we obtain the simplified inequality:

$$\frac{d}{ds}(u_i^2(s)) \leqslant -\alpha_i u_i^2(s) + \beta_i. \tag{3.13}$$

Now, let $v_i(s) = u_i^2(s)$. Inequality (3.13) becomes:

$$\frac{dv_i(s)}{ds} \leqslant -\alpha_i v_i(s) + \beta_i.$$

Consider the comparison equation:

$$\frac{dz_i(s)}{ds} = -\alpha_i z_i(s) + \beta_i, \tag{3.14}$$

with initial condition $z_i(\varepsilon) = v_i(\varepsilon)$ for some $\varepsilon \in [-\tau, 0]$. The solution to (3.14) is given by:

$$z_i(s) = \left(v_i(\varepsilon) - \frac{\beta_i}{\alpha_i}\right)e^{-\alpha_i(s-\varepsilon)} + \frac{\beta_i}{\alpha_i}, \quad s \geqslant \varepsilon.$$

By the comparison principle for differential inequalities, it follows that:

$$v_i(s) \leqslant z_i(s) = \left(v_i(\varepsilon) - \frac{\beta_i}{\alpha_i}\right)e^{-\alpha_i(s-\varepsilon)} + \frac{\beta_i}{\alpha_i}, \quad s \geqslant \varepsilon.$$

Since $v_i(\varepsilon) = u_i^2(\varepsilon)$ and recalling that $u_i(\varepsilon) = \phi_i(\varepsilon)$ for $\varepsilon \in [-\tau, 0]$, we have:

$$u_i^2(s) \leqslant \left(\phi_i^2(\varepsilon) - \frac{\beta_i}{\alpha_i}\right) e^{-\alpha_i s} + \frac{\beta_i}{\alpha_i},$$

where we have absorbed the constant $e^{\alpha_i \varepsilon}$ into the initial condition term without loss of generality for the stability bound. Taking square roots and applying the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for $a, b \geqslant 0$, we obtain:

$$\begin{aligned} |u_i(s)| &\leqslant \left[\left(\phi_i^2(\varepsilon) - \frac{\beta_i}{\alpha_i}\right) e^{-\alpha_i s} + \frac{\beta_i}{\alpha_i}\right]^{1/2} \\ &\leqslant |\phi_i(\varepsilon)| e^{-\frac{\alpha_i}{2} s} + \sqrt{\frac{\beta_i}{\alpha_i}}. \end{aligned}$$

This inequality holds for all $i = 1,2, \dots, n$. Therefore, the state trajectories decay exponentially to a ball of radius $\max_i \sqrt{\beta_i/\alpha_i}$ around the origin, which is the definition of global practical exponential stability. This completes the proof.

4. Numerical examples

In the following section, we present numerical examples to support our theoretical analysis. All simulations were performed using MATLAB.

Example 1: We illustrate the applicability of Theorem 3.1 with the following numerical example of a delayed Cohen–Grossberg neural network:

$$\begin{cases} \frac{du_1}{ds} &= -a_1(u_1(s))[b_1(u_1(s)) - \sum_{j=1}^{2} c_{1j} W_j(u_j(s)) \\ & - \sum_{j=1}^{2} d_{1j} W_j(u_j(s-5)) + I_1(s)], \\ \frac{du_2}{ds} &= -a_2(u_2(s))[b_2(u_2(s)) - \sum_{j=1}^{2} c_{2j} W_j(u_j(s)) \\ & - \sum_{j=1}^{2} d_{2j} W_j(u_j(s-5)) + I_2(s)], \end{cases} \tag{4.1}$$

where

$$a_i(u_i) = 7 + \frac{1}{1+u_i^2}, \quad b_i(u_i) = u_i, \quad W_j(u_j) = \tanh(u_j), \quad \tau = 5,$$

$$I_1(s) = -(4 + e^{-s}), \quad I_2(s) = -\cos(s),$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} -\frac{1}{15} & -0.1 \\ \frac{1}{4} & -\frac{3}{7} \end{bmatrix}.$$

The network parameters satisfy assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ with:

$$L_i = 1, \quad M_i = 1, \quad \alpha_i = 6.5, \quad \overline{a}_i = 8, \quad k_i = 1 \quad (i = 1,2),$$
$$\overline{I}_1 = 5, \quad \overline{I}_2 = 1.$$

A direct computation of the terms in Theorem 3.1 yields:

1. $p(s) > q(s) > 0$,
2. $\inf_{ss_0} \left\{\frac{p(s)-q(s)}{1+2\tau q(s)}\right\} 0.8 > 0$.

Since all conditions of Theorem 3.1 are satisfied, the network (1.1) is GES.

Numerical simulations confirm this result. Figure 1 shows the state trajectories for two distinct initial conditions:

Case 1: $\phi_1(\varepsilon) = 25 - \frac{1}{1+\varepsilon^2}$, $\phi_2(\varepsilon) = -35 + \frac{1}{1+2\varepsilon^2}$,

Case 2: $\phi_1(\varepsilon) = -20$, $\phi_2(\varepsilon) = 38$, for$\varepsilon \in [-5,0]$.

In both cases, the states $u_1(s)$ and $u_2(s)$ converge exponentially to the unique equilibrium, demonstrating the robustness of the stability property guaranteed by Theorem 3.1.

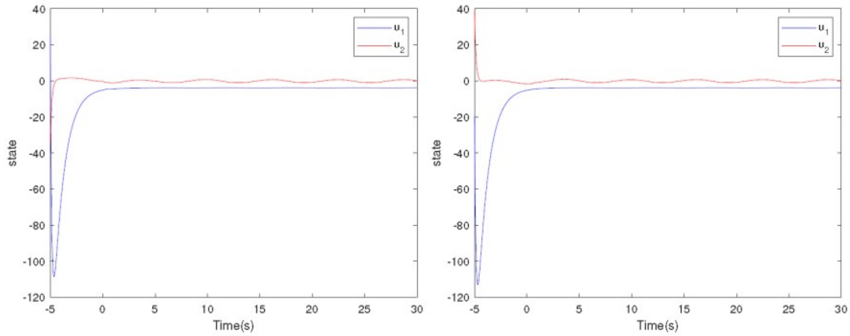

Figure 1
Time response of the states in (4.1) for two different initial conditions, confirming global exponential stability.

Example 2: To further demonstrate the applicability of our results, consider the system:

$$\begin{cases} \frac{du_1}{ds} = -a_1(u_1)[b_1(u_1) - c_{11}(s)W_1(u_1) - d_{12}(s)W_2(u_2(s-4)) + I_1(s)], \\ \frac{du_2}{ds} = -a_2(u_2)[b_2(u_2) - c_{22}(s)W_2(u_2) - d_{21}(s)W_1(u_1(s-4)) + I_2(s)], \end{cases} \tag{4.2}$$

where for $i = 1,2$:

$$a_i(u_i) = 5 + \frac{1}{2+u_i^2}, \quad b_i(u_i) = u_i, \quad W_i(u_i) = \sin(u_i), \quad \tau = 4,$$

$$c_{11}(s) = -\frac{\sin(s)}{13}, \quad c_{22}(s) = 0.2, \quad d_{12}(s) = -0.1, \quad d_{21}(s) = -\frac{\sin(s)}{6},$$

$$I_1(s) = -(4 + e^{-s}), \quad I_2(s) = -\frac{10}{1+s^2}.$$

Assumptions $(\mathcal{A}_2)$-$(\mathcal{A}_4)$ are satisfied with:

$$L_i = 1, \quad M_i = 1, \quad \alpha_i = 4.5, \quad \overline{a}_i = 5.5.$$

The criteria of Theorem 3.2 are met:

$$(1)\ p(s) \geqslant q(s) > 0, \quad (2)\ \inf_{s \geqslant s_0} \left\{ \frac{p(s) - q(s)}{1 + 2\tau q(s)} \right\} \geqslant 1.789 > 0.$$

Hence, Theorem 3.2 guarantees GPES for (4.2). Figure 2 confirms this, showing state trajectories for initial histories

Case 1: $\phi_1(\varepsilon) = -10 - \cos(\varepsilon), \quad \phi_2(\varepsilon) = 15 + 10\sin(\varepsilon),$

Case 2: $\phi_1(\varepsilon) = \frac{-5}{1+2\varepsilon^2}, \quad \phi_2(\varepsilon) = \frac{-5}{1+\varepsilon^2}, \quad \varepsilon \in [-4,0],$

converging exponentially despite significant initial perturbations.

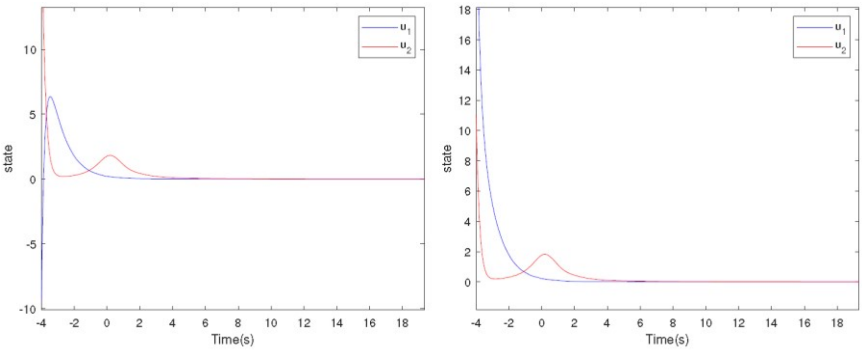

Figure 2
Exponential convergence of states in (4.2) for two different initial conditions, validating Theorem 3.2.

Example 3: This example demonstrates GPES for the following system:

$$\begin{cases} \frac{du_1(s)}{ds} = -u_1(s) + 0.5\tanh\big(u_1(s)\big) + 0.3\tanh\big(u_2(s)\big) + 0.1\tanh(u_1(s-5)) \\ \qquad +0.1\tanh(u_2(s-5)) - e^{-s}, \\ \frac{du_2(s)}{ds} = -u_2(s) + 0.4\tanh\big(u_1(s)\big) + 0.6\tanh\big(u_2(s)\big) + 0.1\tanh(u_1(s-5)) \\ \qquad +0.1\tanh(u_2(s-5)) - \frac{e^{-s}}{1+s^2}. \end{cases} \tag{4.3}$$

System (4.3) corresponds to the general model (1.1) with:

$$a_i(u_i) = 1, \quad b_i(u_i) = u_i, \quad W_j(u_j) = \tanh(u_j), \quad \tau = 5,$$

$$= \begin{bmatrix} -0.5 & -0.3 \\ -0.4 & -0.6 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} -0.1 & -0.1 \\ -0.1 & -0.1 \end{bmatrix},$$

$$I_1(s) = -e^{-s}, \quad I_2(s) = -\frac{e^{-s}}{1+s^2}.$$

The system parameters satisfy assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ with:

$$L_j = 1, \quad M_j = 1, \quad \alpha_i = 1, \quad \overline{a}_i = 1, \quad \overline{I}_i = 1 \quad (i,j = 1,2).$$

Since all conditions of Theorem 3.4 are met, the network (4.3) is globally practically exponentially stable.

Numerical simulations confirm this result. Figure 3 shows the state trajectories and the Euclidean norm $\|u(s)\|$ for the initial condition:

$$\phi_1(\varepsilon) = \sin(\varepsilon), \quad \phi_2(\varepsilon) = \cos(\varepsilon), \quad \varepsilon \in [-5,0].$$

The norm $\|u(s)\|$ converges exponentially to a bounded region near zero, demonstrating the practical exponential stability of the system.

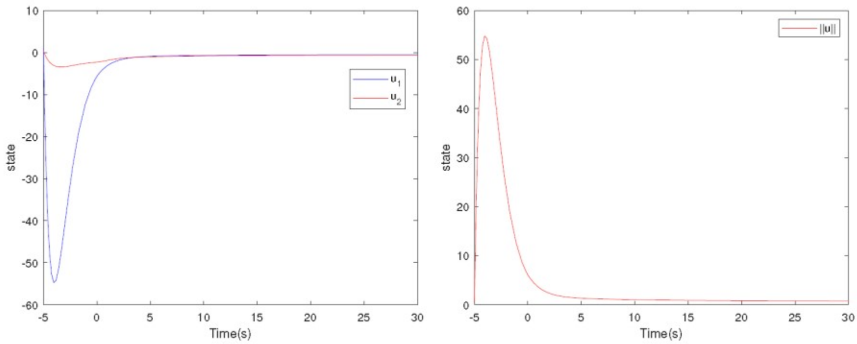

Figure 3
State trajectories (left) and norm evolution (right) for system (4.3). The convergence of $\|u(s)\|$ to a neighborhood of the origin confirms GPES.

5. Conclusion

This Letter establishes a new sufficient condition that guarantees GES and GPES for a broad class of generalized delayed Cohen–Grossberg neural networks. This unifying framework encompasses numerous prominent neural models, including delayed and non-delayed variants of Hopfield and cellular neural networks. In contrast to traditional stability techniques reliant on Lyapunov functionals, our approach is fundamentally based on generalized Halanay's inequality. The derived condition is shown to be delay-independent and less restrictive than several existing results. The practical validity and effectiveness of the theoretical findings are demonstrated through supporting numerical examples and simulations.

Acknowledgment

The authors are deeply grateful to the editor and the reviewers for their meticulous review and constructive criticism. Their insightful feedback was instrumental in enhancing the rigor and clarity of this work.

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Received April, 2025