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Fixed point theorems in fuzzy partial metric spaces

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Abstract

Main focus of this paper is to introduce fixed point (*FP*) results in the context of fuzzy partial metric spaces (*FPMS*). Using control operations, we initiate recent contractive conditions and demonstrate the permanence and distinctiveness of fixed points (*FP*) of self-functions. Our results generalize the theorems existing in the spaces of a partially metric and a fuzzy metric (*FMS*). By examples, we demonstrate the usefulness of our findings. The presented work is a combination and generalization of several known facts in the earlier works.

Subject Classification: *47H10, 54E70.*

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1. Introduction

The foundation of fuzzy mathematics is laid by Zadeh [1] in 1965. The fixed-point theory is still a staple of nonlinear functional analysis, providing invaluable insight into the solution of equations in mathematics, physics, economics and engineering. Since the seminal contraction principle of Banach, the discipline has developed further in abstract spaces, to cover more complicated systems. Among them, the *FMS*, introduced by Kramosil and Michalek [4] and developed by George and Veeramani [5] represent uncertainty that is present in the real-world phenomena by approximating the concept of nearness through fuzzy sets. At the same time, partial metric spaces are

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generalization of classical metrics, which is that they can have non-zero self-distances, which are suited to fields such as computer science where self-similarity is important. Their combination into fuzzy partial metric spaces forms a powerful model to deal concurrently with imprecision and non-zero self-distances, but their theory of fixed-point is not properly studied.

There have been considerable developments of fuzzy metric spaces and Gregori and Sapena [6] introduced fuzzy contractive mappings and Wardowski [7] introduced F-contractions to obtain more refined fixed-point results. Simultaneously, Matthews [3] and then others developed parallel metrics space, with the purpose of developing fixed-point theorems of function and multifunction mappings. These streams however grew very independently. Whereas fuzzy metrics are best able to model vagueness, partial metrics are able to reflect self-distance phenomena- however their combination is immature. More recent sources by Banach [2], Tirado [8], Huang et al. [9] and Moussaoui, Amir, and Omari [10] have been filling this gap, but there are significant limitations to overcome: the existing theorems frequently have restrictive conditions, have no generalizations to integral/differential equations, and do not do enough to deal with compatibility conditions among multiple mappings. This discontinuity highlights the necessity of a single theory in the fuzzy partial metric space.

Although this has been improved, three key gaps still remain in fuzzy partial metric fixed-point theory. First of all, contraction conditions are developed using single valued mappings, and the multiplexing cases (e.g., pairs or quadruples) are not developed. Second, Aamri and Moutawakil [11], Jungck [12], Abbas, Sen, and Nazir [13] discussed about the compatible requirements, including the E.A. property and weak compatibility, have not been technically incorporated, restricting findings to idealized situations. Third, it is limited to a small number of applications to real-world problems (e.g. integral equations), which limits the usefulness of the framework. An example is that the work of Wardowksi [7] and Triado [8] deals with contractions in fuzzy quasi-metric spaces, whereas they do not consider partiality Huang et al. [9] considers fuzzy F-contractions, but does not consider multi-mapping compatibility. The paper has addressed these gaps by proposing new contractive conditions that integrate both single and multi-mapping models and are compatible with one another and show practical usage in the real-world.

Four contributions are made in this paper. Our conceptual definition of generalized contractive conditions is initially defined in terms of control functions (φ) which generalizes the Banach-type contractions to the $FPMS$ to enable the generalized mappings of a single mapping and multiple mappings. Second, we prove special fixed-point theorems of endomorphism (Theorem 3.1), pairs of mappings (Theorem 3.2) as well as quadruples of mappings under E.A. property and weak compatibility (Theorem 3.3). Third, we define shared fixed-point outcomes that integrate and broaden theorems developed by Wardowski [7], Triado [8], and Huang et al. [9], weakening their hypotheses. Fourth, we give practical utility through the application of our framework to solving of integral equations and this accentuates the wider applicability of our framework in nonlinear analysis. These inventions do not only address gaps in theory; they give available tools to model complex systems in applied mathematics.

Jasim, Harbi, and Mohammed [14], and Jasim et al. [15] presented coupled fixed points theorems, and common FP theorems in fuzzy Fréchet spaces. The results established several FP theorems in FMS. Ali and Al-Yaseen [16] introduced the characterization of expansive mappings within the FMS frame work and established the shadowing property. Mohanraj and Rajesh [17] introduced new FP results in multiplicative metric spaces. Siddamsetti et al. [18] demonstrated FP theorems for non-linear mappings in the frame work of modular metric spaces.

This paper follows the following structure. Section 2 examines groundworks, along with the definition of $FPMS$, control functions, and compatibility conditions. Section 3 provides our theorems, and proofs are rather rigorous and illustrative. Section 4 applies to the case of integral equations, and it confirms the helpfulness of our method. Section 5 ends with a inference and recommendations of future research, including applying results to fuzzy differential equations or relation-theoretic fixed points. With this framework, we are trying to provide a complete base in fixed-point theory in $FPMS$.

2.Preliminaries

Definition 2.1: A PMS on Ӽ is denoted by (Ӽ, δ), where Ӽ is non - empty and δ: Ӽ × Ӽ → R^+ is a mapping subject to the said requirements.

For all ꝳ, ꝴ, ꝵ ∈ Ӽ,
(PM1) δ (ꝳ ꝳ) ≤ δ (ꝳ, ꝴ),
(PM2) δ (ꝳ ꝳ) = δ (ꝳ ꝴ) = δ (ꝴ, ꝴ) if and only if ꝳ = ꝴ,
(PM3) δ (ꝳ ꝴ) = δ (ꝳ, ꝴ),
(PM4) δ (ꝳ ꝵ) ≤ δ (ꝳ, ꝴ) + δ (ꝴ, ꝵ) – δ (ꝴ, ꝴ).

Definition 2.2: A dyadic operator ∗: [0, 1] x [0, 1] ⟶ [0, 1] is a c°ntinuous t-norm provided that it fulfils the subsequent requirements, ∀ α1, β1, γ_1 ∈ [0, 1],

(T1) α1 ∗ β1 = β1 ∗ α1, α1 ∗ (β1 ∗ γ_1) = (α1 ∗ β1) ∗ γ_1,
(T2) ∗ is continuous on [0, 1],
(T3) α1 ∗ 1 = α1 for all α1 ∈ [0, 1],
(T4) α1 ∗ β1 ≤ γ_1 ∗ δ_1 whenever α1 ≤ γ_1 and β1 ≤ δ_1.

Illustrations of continuous t – norm:

(1) α1 ∗ β1 = α1β1,
(2) α1 ∗ β1 = min {α1, β1}.

Definition 2.3: A triad (Ӽ, Ɱ, ∗) is termed a FMS provided that, Ӽ is a non-empty set, ∗ is a c°ntinuous t-norm and Ɱ denotes FS on $Ӽ^2$ × (0, ∞), fulfilling the subsequent requirements for any l, m, n ∈ Ӽ and each s, ƭ > 0.

(FM1) Ɱ $(l, m,$ ƭ$) > 0$,
(FM2) Ɱ $(l, m,$ ƭ$) = 1 \Leftrightarrow l = m$,
(FM3) Ɱ $(l, m,$ ƭ$)$ = Ɱ $(m, l,$ ƭ$)$,
(FM4) Ɱ $(l, m,$ ƭ$)$ ∗ Ɱ (m, n, s) ≤ Ɱ $(l, n,$ ƭ $+ s)$,

(FM5) Ɱ $(l, m, .)$: $(0, \infty) \rightarrow [0,1]$ is continuous.

Example 1: Let Ҳ - set of all rational and irrationals, d(distance function) stands for Euclidean metric. Let $l * m = min\{l, m\} \forall\, l, m \in [0,1]$. For each ŧ > 0 and $l, m \in$ Ҳ,

Let Ɱ $(l, m,$ ŧ$) =$ ŧ$/$ (ŧ $+ d\,(l, m))$, then (Ҳ, Ɱ, $*$) is a FMS.

Proposition 1: If (Ҳ, Ɱ, $*$) is a FMS, then Ɱ $(l, m,$ ŧ$)$: $(0, \infty) \rightarrow [0,1]$ is a non – decreasing function $\forall\, l, m \in$ Ҳ.

Definition 2.5: Let (Ҳ, Ɱ, $*$) be a FMS, and $\{l_n\}$ a sequence in it. One says that the sequence converges to a point l of Ҳ (or that the limit of the sequence is l), when it is satisfied that $\lim_{n\rightarrow\infty}$ Ɱ$(l_n, l,$ ŧ$)$ = 1 for any ŧ > 0.

Definition 2.6: A Cauchy sequence *(CS)* $\{l_n\}$ is one that satisfies $\lim_{n\rightarrow\infty}$ Ɱ $(x_n, x_m,$ ŧ$)$ = 1, ∀ ŧ > 0.

Definition 2.7: A FMS (Ҳ, Ɱ, $*$) is termed complete when every *CS* in Ҳ is c°nvergent in Ҳ.

Definition 2.8: A triad (Ҳ, Ɱ, $*$) is said to be a $PFMS$, if Ҳ is a non-empty unite, $*$ is a c°ntinuous ŧ–norm and Ɱ is a *FSS* of Ҳ ^{2} × (0, ∞) fulfil the conditions ensuing on, for all $l, m, n \in$ Ҳ and each $s,$ ŧ > 0,

(PFM1) Ɱ $(l, m,$ ŧ$) =$ Ɱ $(l, l,$ ŧ$) =$ Ɱ $(m, m,$ ŧ$) \Leftrightarrow l = m$,
(PFM2) Ɱ $(l, l,$ ŧ$) \geq$ Ɱ $(l, m,$ ŧ$) > 0$,
(PFM3) Ɱ $(l, m,$ ŧ$) =$ Ɱ $(l, m,$ ŧ$)$,
(PFM4) Ɱ $(l, m,$ ŧ$) *$ Ɱ $(m, n, s) \leq$ Ɱ $(l, n,$ ŧ $+ s)$,
(PFM5) Ɱ $(l, m, .)$: $(0, \infty) \rightarrow [0,1]$ is continuous.

Result: Let (Ҳ, Ɱ, $*$) denotes $PFMS$,

1. If Ɱ$(l, m,$ ŧ$) = 1$, then $l = m$ from settings (PFⱮ1) and (PFⱮ2). But the inverse of this implication is no need of essentially true, that is Ɱ (l, m, ŧ) may not be equal to 1 whenever $l = m$.
2. It is clear that Ɱ $(l, n,$ ŧ$) *$ Ɱ $(n, m,$ ŧ$) \leq$ Ɱ $(l, m,$ ŧ$) *$ Ɱ $(n, n,$ ŧ$), \forall\, l, m,$ $n \in$ Ҳ and each ŧ > 0.

Example 2: Let Ҳ - the set of all rational and irrationals d(distance function) be the metric. Let $l * m = lm$ for all $l, m \in [0,1]$. Consider the mapping Ɱ: Ҳ ^{2} × (0, ∞) → [0, 1] expressed by Ɱ $(l, m,$ ŧ$) =$ ŧ $/$ (ŧ $+ d\,(l, m))$, then (Ҳ, Ɱ, $*$) is a $PFMS$.

Proposition 2: Consider a $PFMS$ (Ҳ, Ɱ, $*$). If $m \geq n$ whenever $l_1 * m_1 \geq l_1 * n_1$, then Ɱ $(l_1, l_1, .)$: (0, ∞) → [0, 1], is the non-decreasing function of all l_1, m_1 ∈ Ҳ.

Definition 2.9: Let (Ꭓ, Ɱ, $*$) be a $PFMS$ and $\{l_n\} \in$ Ꭓ. $\{l_n\}$ is considered to be convergence to a limit $l \in$ Ꭓ if $\lim_{n\to\infty}$ Ɱ $(l_n, l, \mathfrak{t})$ = Ɱ $(l, l, \mathfrak{t})$ $\forall\, \mathfrak{t} > 0$.

Definition 2.10: A sequence $\{l_n\}$ is called a *CS* in Ꭓ if Ɱ $(l_n, l_m, \mathfrak{t}) = 1$.

Definition 2.11: A $PFMS$ (Ꭓ, Ɱ, $*$) is named complete if each *CS* $\{l_n\}$ converges to a limit l in Ꭓ.

Proposition 3: Let (Ꭓ, Ɱ, $*$) be a $PFMS$ and $\{l_n\}$ be a sequence in Ꭓ such that $\lim_{n\to\infty}$ Ɱ$(l_n, l_n, \mathfrak{t})$ = $\lim_{n\to\infty}$ Ɱ $(l_n, l, \mathfrak{t})$ = Ɱ $(l, l, \mathfrak{t})$. Then $\lim_{n\to\infty}$ Ɱ $(l_n, m, \mathfrak{t})$ = $M(l, m, \mathfrak{t})$ $\forall\, l, m \in$ Ꭓ and $\mathfrak{t} > 0$.

Result: Consider a continuous mapping Φ: [0, 1] → [0, 1] which satisfies the ensuing requirements listed below,

1. Φ is a non – decreasing function on [0, 1].
2. Φ (ƭ) ≥ ƭ for each ƭ Є (0, 1). We note that Φ (1) = 1 and Φ (ƭ) ≥ ƭ for all ƭ ∈ (0, 1).

3. Main result:

Theorem 3.1: *Let (Ꭓ, Ɱ, $*$) be a c°mplete PFMS such that* Ɱ $(l, l, \mathfrak{t}) = 1\ \forall\, l \in$ Ꭓ$, \mathfrak{t} > 0$ *and* f: Ꭓ → Ꭓ *be a self – function such that for all* $l, p \in$ Ꭓ *and* $k \in (0,1)$,

Ɱ$(f(l), f(p), k\,\mathfrak{t}) \geq \varphi$ {Ɱ $(l, f(l), \mathfrak{t})$} -----------(1) hold, then there exists a distinctive FP (invariant point) for f in Ꭓ.

Proof: Let $\{l_n\}$ be a sequence in Ꭓ such that $\lim\limits_{n\to\infty} l_n = l$, and let $f(l_n) = l_{n+1}$.

From (1) we have Ɱ $(l_n, l_{n+1}, k\mathfrak{t})$ = Ɱ $(f(l_{n-1}), f(l_n), k\mathfrak{t})$
$\geq \varphi$ {Ɱ $(l_{n-1}, f(l_{n-1}), \mathfrak{t})$
$= \varphi$ {Ɱ $(l_{n-1}, l_n, \mathfrak{t})$}
$\geq$ Ɱ $(l_{n-1}, l_n, \mathfrak{t})$.

That is Ɱ $(l_n, l_{n+1}, k\mathfrak{t}) \geq$ Ɱ $(l_{n-1}, l_n, \mathfrak{t})$.

Then Ɱ $(l_n, l_{n+1}, \mathfrak{t}) \geq$ Ɱ $(l_{n-1}, l_n, \frac{\mathfrak{t}}{k}) \geq$ Ɱ $(l_{n-2}, l_{n-1}, \frac{\mathfrak{t}}{k^2}) \geq \ldots\ldots \geq$ Ɱ $(l_0, l_1, \frac{\mathfrak{t}}{k^n})$

Taking limit as n → ∞ we get, $\lim\limits_{n\to\infty}$ Ɱ $(l_n, l_{n+1}, \mathfrak{t}) = 1$, for all $\mathfrak{t} > 0$.

We have to show that $\{l_n\}$ is a *CS*,

Let $n, m \in N\ \&\ n < m$,

Ɱ $(l_n, l_m, \mathfrak{t})$ = Ɱ $(l_n, l_m, \mathfrak{t})$ $*$ Ɱ $(l_{n+1}, l_{n+1}, \mathfrak{t})$
$\geq$ Ɱ $(l_n, l_{n+1}, \mathfrak{t})$ $*$ Ɱ $(l_m, l_{n+1}, \mathfrak{t})$
= Ɱ $(l_n, l_{n+1}, \mathfrak{t})$ $*$ Ɱ $(l_m, l_{n+1}, \mathfrak{t})$ $*$ Ɱ $(l_{n+2}, l_{n+2}, \mathfrak{t})$
$\geq$ Ɱ $(l_n, l_{n+1}, \mathfrak{t})$ $*$ Ɱ $(l_m, l_{n+2}, \mathfrak{t})$ $*$ Ɱ $(l_{n+1}, l_{n+2}\, \mathfrak{t})$
$\geq \ldots\ldots\ldots\ldots\ldots\ldots \geq$ Ɱ $(l_n, l_{n+1}, \mathfrak{t})$ $*$ Ɱ $(l_m, l_{n+2}, \mathfrak{t})$ $*$
Ɱ $(l_{n+1}, l_{n+2}\, \mathfrak{t})$ $*\ldots\ldots\ldots*$ Ɱ $(l_{m-1}, l_m, \mathfrak{t})$

Now taking $n \to \infty$ and $m \to \infty$ we get

$\lim_{n\to\infty, m\to\infty}$ Ɱ $(l_n, l_m, \mathfrak{t}) \geq$ Ɱ $(l, l, \mathfrak{t}) *$ Ɱ $(l, l, \mathfrak{t}) * \ldots\ldots\ldots\ldots\ldots\ldots\ldots .*$ Ɱ $(l, l, \mathfrak{t})$.

$= 1 * 1 * 1 * \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .* 1$

$\lim_{n\to\infty, m\to\infty}$ Ɱ $(l_n, l_m, \mathfrak{t}) = 1$.

Hence $\{l_n\}$ is a *CS* in (Ӽ, Ɱ, $*$).

Since (Ӽ, M, $*$) is c^{o}mplete, every *CS* converges to l.

Now, Ɱ $(f(l), l, \mathfrak{t}) =$ Ɱ $(f(l), l, \mathfrak{t}) *$ Ɱ $(l_n, l_n, \mathfrak{t})$

$\geq$ Ɱ $(f(l), l_n, \mathfrak{t}) *$ Ɱ $(l_n, l, \mathfrak{t})$

$=$ Ɱ $(f(l), f(l_{n-1}), \mathfrak{t}) *$ Ɱ $(l_n, l, \mathfrak{t})$

$\geq$ Ɱ $(l, l_n, \mathfrak{t}) *$ Ɱ $(l_n, l, \mathfrak{t})$

Taking $n \to \infty$ we get

Ɱ $(f(l), l, \mathfrak{t}) \geq$ Ɱ $(l, l, \mathfrak{t}) *$ Ɱ $(l, l, \mathfrak{t})$

$= 1 * 1$

Therefore Ɱ $(f(l), l, \mathfrak{t}) = 1$.

Hence l is FP of f.

Uniqueness:

Assume that l and p are two different FP of f.

That is $l \neq p$. Ɱ $(l, p, \mathfrak{t}) =$ Ɱ $(f(l), f(p), \mathfrak{t})$

$\geq \varphi\{$Ɱ $(f(l), f(p), \mathfrak{t})\}$

$= \varphi\{$Ɱ $(l, p, \mathfrak{t}))$

$>$ Ɱ $(l, p, \mathfrak{t})$.

This is a contradiction. Hence $l = p$.

Therefore, f has a distinctive FP

Theorem 3.2: L*et (Ӽ, Ɱ,* $*$) *be a c^{o}mplete PFMS such that* Ɱ $(l, l, \mathfrak{t}) = 1\ \forall\, l \in$ Ӽ*, and* f, g: Ӽ $\to$ Ӽ *be two self - functions such that* $\forall\, l, p \in$ Ӽ*,*

Ɱ $(f(l), f(p), k\mathfrak{t}) \geq \varphi\{$Ɱ $(\mathrm{g}(l), \mathrm{g}(p), \mathfrak{t})\}$ ---------- (1) hold then there exist a distinctive FP for f and g in Ӽ where $k \in (0, 1)$.

Proof: Let $\{l_n\}$ be a sequence in Ӽ such that $\lim_{n\to\infty} l_n = l$, and let $f(l_n) = l_{n+1}$, and $\mathrm{g}(l_{n+1}) = l_{n+2}$.

Then Ɱ $(l_{(n+1)}, l_{(n+2)}, k\mathfrak{t}) =$ Ɱ $(f(l_n), f(l_{n+1}), \mathfrak{t})$

$\geq \varphi\{$Ɱ $(\mathrm{g}(l_n), \mathrm{g}(l_{n+1}), \mathfrak{t})\}$ from (1)

$\geq$ Ɱ $(\mathrm{g}(l_n), \mathrm{g}(l_{n+1}), \mathfrak{t})$

$=$ Ɱ $(l_{n+1}, l_{n+2}, \mathfrak{t})$

That is Ɱ $(l_{n+1}, l_{n+2}, k\mathfrak{t}) \geq$ Ɱ $(l_{n+1}, l_{n+2}, \mathfrak{t})$.

Hence Ɱ $(l_{n+1}, l_{n+2}, \mathfrak{t}) \geq$ Ɱ $(l_{n+1}, l_{n+2}, \frac{\mathfrak{t}}{k}) \geq$ Ɱ $(l_{n+1}, l_{n+2}, \frac{\mathfrak{t}}{k^2}) \geq \ldots\ldots\ldots\ldots\ldots \geq$ Ɱ $(l_{n+1}, l_{n+2}, \frac{\mathfrak{t}}{k^n})$.

Taking $n \to \infty$ we get $\lim_{n\to\infty}$ Ɱ $(l_{n+1}, l_{n+2}, \mathfrak{t}) = 1, \forall\, \mathfrak{t} > 0$.

Next, we have to prove $\{l_n\}$ is a *CS*.

Let $n, m \in N\ \& n < m$

$\mathcal{M}(l_n, l_m, \mathfrak{t}) = \mathcal{M}(l_n, l_m, \mathfrak{t}) * \mathcal{M}(l_{n+1}, l_{n+1}, \mathfrak{t})$
$\geq \mathcal{M}(l_n, l_{n+1}, \mathfrak{t}) * \mathcal{M}(l_m, l_{n+1}, \mathfrak{t})$
$= \mathcal{M}(l_n, l_{n+1}, \mathfrak{t}) * \mathcal{M}(l_m, l_{n+1}, \mathfrak{t}) * \mathcal{M}(l_{n+2}, l_{n+2}, \mathfrak{t})$
$\geq \mathcal{M}(l_n, l_{n+1}, \mathfrak{t}) * \mathcal{M}(l_m, l_{n+2}, \mathfrak{t}) * \mathcal{M}(l_{n+1}, l_{n+2}, \mathfrak{t})$
$\geq \ldots\ldots\ldots\ldots \geq \mathcal{M}(l_n, l_{n+1}, \mathfrak{t}) * \mathcal{M}(l_m, l_{n+2}, \mathfrak{t}) * \mathcal{M}(l_{n+1}, l_{n+2}, \mathfrak{t}) * \ldots\ldots * \mathcal{M}(l_{m-1}, l_m, \mathfrak{t})$

Now taking $n \to \infty$ *and* $m \to \infty$ we get
$\lim_{n\to\infty, m\to\infty} \mathcal{M}(l_n, l_m, \mathfrak{t}) \geq \mathcal{M}(l, l, \mathfrak{t}) * \mathcal{M}(l, l, \mathfrak{t}) * \ldots\ldots\ldots\ldots * \mathcal{M}(l, l, \mathfrak{t})$.
$= 1 * 1 * 1 * \ldots\ldots\ldots\ldots * 1$.
That is $\lim_{n\to\infty, m\to\infty} \mathcal{M}(l_n, l_m, \mathfrak{t}) = 1$.
Hence $\{l_n\}$ is a *CS* in $(\mathfrak{X}, \mathcal{M}, *)$.
Since $(\mathfrak{X}, \mathcal{M}, *)$ is complete, every *CS* converges to l.
$\lim_{n\to\infty} \mathcal{M}(l_n, l, \mathfrak{t}) = \mathcal{M}(l, l, \mathfrak{t}) = \lim_{n\to\infty, m\to\infty} \mathcal{M}(l_n, l_m, \mathfrak{t}) = 1, \forall\, \mathfrak{t} > 0$.
Now, $\mathcal{M}(f(l), l, \mathfrak{t}) = \mathcal{M}(f(l), l, \mathfrak{t}) * \mathcal{M}(l_n, l_n, \mathfrak{t})$
$\geq \mathcal{M}(f(l), l_n, \mathfrak{t}) * \mathcal{M}(l_n, l, \mathfrak{t})$
$= \mathcal{M}(f(l), f(l_{n-1}), \mathfrak{t}) * \mathcal{M}(l_n, l, \mathfrak{t})$
$\geq \mathcal{M}(l_n, l, \mathfrak{t}) * \mathcal{M}(l_n, l, \mathfrak{t})$
Taking limit as $n \to \infty$ we get
$$\mathcal{M}(f(l), l, \mathfrak{t}) \geq \mathcal{M}(l, l, \mathfrak{t}) * \mathcal{M}(l, l, \mathfrak{t})$$
$= 1 * 1$
Therefore $\mathcal{M}(f(l), l, \mathfrak{t}) = 1$. This means $f(l) = l$.
Hence l is the FP of f.
Similarly, we can prove $g(l) = l$.
Therefore, l is the FP of both f and g.

Uniqueness:

To prove l is the unique FP of both f *and* g.
Suppose that l *and* p are two different FP of f *and* g. That is $l \neq p$.
$\mathcal{M}(l, p, \mathfrak{t}) = \mathcal{M}(f(l), f(p), \mathfrak{t})$
$\geq \varphi\{\mathcal{M}(g(l), g(p), \mathfrak{t})\}$
$= \varphi\{\mathcal{M}(l, p, \mathfrak{t})\}$
$$> \mathcal{M}(l, p, \mathfrak{t}).$$
Hence $\mathcal{M}(l, p, \mathfrak{t}) > \mathcal{M}(l, p, \mathfrak{t})$.
This is a contradiction. We have $l = p$.
l is the unique FP for both f and g

Definition 2.12: Two self – functions f and g of a *PFMS* $(\mathfrak{X}, \mathcal{M}, *)$ are compatible if $\lim_{n\to\infty} \mathcal{M}(f(g(l_n)), g(f(l_n)), \mathfrak{t}) = 1 \forall\, t > 0$ and $\{l_n\}$ is a sequence in $\mathfrak{X}$ such that $\operatorname{limit}_{n\to\infty} f(l_n) = \operatorname{limit}_{n\to\infty} g(l_n) = l$ for some $l \in \mathfrak{X}$.

Definition 2.13: Two self – functions f *and* g of a *PFMS* $(\mathfrak{X}, \mathcal{M}, *)$ are weakly compatible if $f(l) = g(l)$ for all $l \in \mathfrak{X}$ implies that $f(g(l)) = g(f(l))$.

Definition 2.14: Let f, g, h and l be four self – functions of a $PFMS$ (Ꭓ, Ɱ, $*$). Then the pairs (f, g) and (h, l) are meet the requirement common *E.A* property if, ∃ two sequences $\{l_n\}$ and $\{p_n\}$ in Ꭓ such that $\lim_{n\to\infty} f(l_n) = \lim_{n\to\infty} h(l_n) = \lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} l(p_n) = l$ where $l \in f$(Ꭓ) ∩ g(Ꭓ) or $l \in h$(Ꭓ) ∩ l(Ꭓ)).

***E. A* Property (Existing & Approaching):** The *E.A* property is used to find whether a sequence $\{l_n\}$ in a $PFMS$ converges to a point l with in the space.

Lemma 1: *Let (Ꭓ, Ɱ, $*$) be a PFMS. If,* $\exists\ k \in (0,1)$ *such that* Ɱ$(l, p, k$ƭ$) \geq$ Ɱ$(l, p,$ ƭ$)$ *then* $l = p\ \forall\ t > 0,\ k \in (0,1),$ *and* $l, p \in$ Ꭓ.

Theorem 3.3: *Let f, g, h and l be four self – functions of $PFMS$ (Ꭓ, Ɱ, $*$) with $a * b = \min\{a, b\}$ satisfies subsequent conditions,*

(i) The pairs (f, h) and (g, l) satisfies the E.A property,

(ii) The pairs (f, h) and (g, l) are weakly compatible,

(iii) There exists k ∈ (0, 1), ∀ l, p ∈ Ꭓ and t > 0

Ɱ(f(l), g(p), kƭ) ≥ min{Ɱ (f(l), l(p), ƭ), Ɱ(g(p), l(p), ƭ), Ɱ(h(l), l(p), ƭ), Ɱ(f(l), h(l), ƭ)}….(1). Then f, g, h and l have a distinctive CFP.

Proof: Since the pairs (f, h) and (g, l) fulfills the requirement *E.A* property, then ∃ two sequences $\{l_n\}$ & $\{p_n\}$ in Ꭓ such that $\lim_{n\to\infty} f(l_n) = \lim_{n\to\infty} h(l_n) = \lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} l(p_n)$ = z where z ∈ f(Ꭓ) ∩ g(Ꭓ) or z ∈ h(Ꭓ) ∩ l(Ꭓ). Then h(u) = z for some u ∈ Ꭓ.

Now we have to prove f(u) = z. Put l = u and p = p_n in (1), we get

Ɱ(f(u), $g(p_n)$, kƭ) ≥ min {Ɱ (f(u), $l(p_n)$, ƭ), Ɱ ($g(p_n)$, $l(p_n)$, ƭ), Ɱ (h(u), $l(p_n)$, ƭ), Ɱ (f(u), h(u), ƭ)}

Taking $\lim_{n\to\infty}$ both sides we get,

Ɱ (f(u), z, kƭ) ≥ min {Ɱ (f(u), z, ƭ), Ɱ (z, z, ƭ), Ɱ (z, z, ƭ), Ɱ (f(u), z, ƭ)}

= min {Ɱ (f(u), z, ƭ), Ɱ (z, z, ƭ)}

= min {Ɱ (f(u), z, ƭ), 1}

Ɱ (f(u), z, kƭ) ≥ Ɱ (f(u), z, ƭ)

By Lemma f(u) = z.

Therefore f and h have a coincident point z.

Next, we take l(w) = z for some w ∈ Ꭓ.

We have to prove $g(w)$ = z.

Put l = u and p = w in (1) we get,

Ɱ (f(u), $g(w)$, kƭ) ≥ min {Ɱ (f(u), l(w), ƭ), Ɱ ($g(w)$, l(w), ƭ), Ɱ (h(u), l(w), ƭ), Ɱ (f(u), h(u), ƭ)}

Ɱ(z, $g(w)$, kƭ) ≥ min {Ɱ (z, z, ƭ), Ɱ ($g(w)$, z, ƭ), Ɱ (z, z, ƭ), Ɱ (z, z, ƭ)}

= min {1, Ɱ ($g(w)$, z, ƭ)}

= Ɱ ($g(w)$, z, ƭ)

Ɱ ($g(w)$, z, kƭ) ≥ Ɱ ($g(w)$, z, ƭ)

By lemma $g(w) = z$.

There g and l have a coincident point z.

Since the pair (f, h) is weakly compatible, we get

$f(z) = f(h(z)) = h(f(z)) = h(z)$.

Next, we to show z is the CFP of f and h.

Put x = z and p = w in (1), we get

$\mathcal{M}(f(z), g(w), kt) \geq \min\{\mathcal{M}(f(z), l(w), t), \mathcal{M}(g(w), l(w), t), \mathcal{M}(h(z), l(w), t), \mathcal{M}(f(z), h(z), t)\}$

$\mathcal{M}(f(z), z, kt) \geq \min\{\mathcal{M}(f(z), z, t), \mathcal{M}(z, z, t), \mathcal{M}(f(z), z, t), \mathcal{M}(f(z), f(z), t)\}$

$= \min\{\mathcal{M}(f(z), z, t), 1, \mathcal{M}(f(z), z, t), 1\}$

$= \mathcal{M}(f(z), z, t)$

$\mathcal{M}(f(z), z, kt) \geq \mathcal{M}(f(z), z,)$

By Lemma $f(z) = z$.

Therefore, z is the CFP of f and h.

From the Similar argument we can prove z is the CFP of g and l.

Therefore, z is the CFP of f, g, h and l.

Uniqueness:

Assume that r and s be two distinct CFP of f, g , h and l.

That is $r \neq s$.

Put l = r and p = s in equation (1).

Now, $\mathcal{M}(f(r), g(s), kt) \geq \min\{\mathcal{M}(f(r), l(s), t), \mathcal{M}(g(s), l(s), t), \mathcal{M}(h(r), l(s), t), \mathcal{M}(f(r), h(r), t)\}$

$\mathcal{M}(r, s, kt) \geq min\{\mathcal{M}(r, s, t), \mathcal{M}(s, s, t), \mathcal{M}(r, s, t), \mathcal{M}(r, r, t)\}$

$\mathcal{M}(r, s, kt) \geq min\{\mathcal{M}(r, s, t), 1, \mathcal{M}(r, s, t), 1\}$

$\mathcal{M}(r, s, kt) \geq \mathcal{M}(r, s, t)$.

This is a contradiction to our assumption $r \neq s$.

By lemma $r = s$.

Therefore f, g , h, and l have a distinctive CFP

4. Examples

Example 4.1: We apply the theorem 3.1 to solve the integral equation

$x(t) = f(t) + \lambda\int_0^t k(t, s, x(s))ds, t \in [0,1]$

where $\lambda \in R$, $f : [0,1] \to R$ is continuous, and $k : [0,1]\, x\, [0.1]\, x\, R \to R$ is continuous.

Let $\mathcal{X} = [0, 1]$ with fuzzy partial metric $\mathcal{M}(x, y, t) = exp\,(-d\,(x, y)/t + 1)$

Where $d(x, y) = sup\,|x(t) - y(t)|$ where $t \in [0, 1]$

Let $(\mathcal{X}, \mathcal{M}, *)$, $a * b = \min\{a, b\}$ is a complete $PFMS$.

Define $T : \mathcal{X} \to \mathcal{X}$ by $T_x(t) = f(t) + \lambda\int_0^t k(t, s, x(s))ds, t \in [0,1]$.

Assume that $|k(t, s, u) - k(t, s, v)| \leq L|u - v|\ \forall\ t, s \in [0, 1], u, v \in R$, with $|\lambda|\, t < 1$,

Then $d(T(x), T(y)) \leq |\lambda| Ld(x, y)$,

This implies $\mathcal{M}(T(x), T(y), k\,t) \geq exp\,(-|\lambda| L\, d(x, y)/k\,t + 1)$.

Let $\varphi(s) = s^k$, for, $k > 1$ such that $(|\lambda| L)^k < 1$, then
$\mathcal{M}(T(x), T(y), k\,\mathfrak{t}) \geq \varphi(\mathcal{M}(x, y, \mathfrak{t}))$
By lemma T has a unique FP.

Example 4.2: Let $(\mathfrak{X}, \mathcal{M}, *)$ be a $PFMS$, where $\mathfrak{X} = [0, 1]$, and $a * b = ab$ with,
$\mathcal{M}(r, s, \mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t}+|r-s|}$ for all $\mathfrak{t} > 0$.

Define the self - mappings f, g, h and l by, $f(r) = \begin{cases} 1, if\ r \in [0,1] \cap Q^+ \\ \frac{1}{2}, if\ r \notin [0,1] \cap Q^+ \end{cases}$

$$g(r) = \begin{cases} 1, if\ r \in [0,1] \cap Q^+ \\ \frac{1}{4}, if\ r \notin [0,1] \cap Q^+ \end{cases}$$

$$h(r) = \begin{cases} 1, if\ r \in [0,1] \cap Q^+ \\ 0, if\ r \notin [0,1] \cap Q^+ \end{cases}$$

$$l(r) = \begin{cases} 1, if\ r \in [0,1] \cap Q^+ \\ 0, if\ r \notin [0,1] \cap Q^+ \end{cases}$$

Clearly $f(\mathfrak{X}) = \{1, \frac{1}{2}\}$, $g(\mathfrak{X}) = \{1, \frac{1}{4}\}$, $h(\mathfrak{X}) = \{0, 1\}$ and $l(\mathfrak{X}) = \{0, 1\}$.
Here $f(\mathfrak{X})$ is not a subset of $h(\mathfrak{X})$ and $g(\mathfrak{X})$ is not a subset of $l(\mathfrak{X})$.
Let $l \in [0,1] \cap Q^+$ and $p \notin [0,1] \cap Q^+$ and $\mathfrak{t} > 0$, we get
$\mathcal{M}(f(l), g(p), k\mathfrak{t}) \geq \min\{\mathcal{M}(f(l), l(p), \mathfrak{t}), \mathcal{M}(g(p), l(p), \mathfrak{t}), \mathcal{M}(h(l), l(p), \mathfrak{t}), \mathcal{M}(f(l), h(l), \mathfrak{t})\}$
$\mathcal{M}(1, \frac{1}{4}, k\mathfrak{t}) \geq \min\{\mathcal{M}(1, \frac{1}{4}, \mathfrak{t}), \mathcal{M}(\frac{1}{4}, 0, \mathfrak{t}), \mathcal{M}(1, 0, \mathfrak{t}), \mathcal{M}(1, 1, \mathfrak{t})\}$
$\mathcal{M}(1, \frac{1}{4}, k\mathfrak{t}) > \mathcal{M}(1, 0, \mathfrak{t})$.
Which is true for all $\mathfrak{t} > 0$.
Thus, all requirements of theorem 3.3 are fulfilled.
Therefore 1 is the only one CFP of f, g, h and l.

5. Conclusion

This paper presents fixed-point theorems in $PFMS$ by way of novel contractive conditions with control functions. Our results generalize existing theorems in FMS and $PFMS$, proving unique fixed points for single mappings, pairs of mappings, and quadruples under E.A. property and weak compatibility. The utility of our findings was demonstrated through examples and an application to integral equations, highlighting broader applicability in nonlinear analysis. This work unifies and extends key results in the literature, providing a foundation for future research in fuzzy FP theory and its applications.

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