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Existence results of solutions for Caputo Hadamard boundary value problems fractional differential inclusions

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Abstract

This study addresses the existence of solutions for a Caputo-Hadamard boundary value problem involving differential inclusions with nonlocal integral boundary conditions. The approach used in establishing the main results is based on fixed point theory within the framework of set-valued mappings. Several examples are presented to illustrate the applicability and validity of the theoretical findings.

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1. Introduction and Preliminaries

In recent decades, differential equations of fractional (non-integer) order have emerged as powerful tools for modeling various complex processes across multiple scientific disciplines, including chemistry, physics, thermodynamics, hydrology, control systems, and biomedical engineering-particularly in analyzing blood flow dynamics. Due to their ability to describe memory and hereditary properties of diverse materials and processes, fractional differential equations have become increasingly significant. Numerous studies have been conducted in this direction; for a detailed discussion, see [5, 9, 13, 14, 22] and the references therein.

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The question of existence and uniqueness of solutions to fractional differential equations has received growing attention, especially under various types of boundary conditions and using different definitions of fractional derivatives; relevant contributions can be found in [2, 3, 5, 12]. Notably, Jarad, Abdeljawad, and Baleanu [15] proposed in 2012 a novel operator known as the Caputo-Hadamard derivative which blends the Caputo and Hadamard approaches, offering additional flexibility in modeling real world systems. This operator has since inspired further analytical investigations and results, as reported in [6, 1, 23, 17, 19, 18].

Motivated by these developments, the present work aims to examine the existence of solutions for a class of differential inclusions involving the Caputo-Hadamard derivative, subject to nonlocal integral boundary conditions.

We focus on the following problem:

$$\begin{cases} \mathcal{D}^q w(s) \in \mathfrak{F}(s, w(s)), 1 < t < e, \\ w(1) = \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s)) ds, \\ w(e) = \lambda \int_{\beta_1}^{\beta_2} h(s, w(s)) ds, \end{cases} \tag{1}$$

where $\mathcal{D}^q$ is Caputo-Hadamard derivative, $1 < q \leq 2$, $0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2 < 1$, $\lambda, \mu \in \mathbb{R}$, $\mathfrak{F}: [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $g, h: \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Let $X = C([1, e])$ the space of continuous functions on $\mathcal{J} = [1, e]$ with real values, equipped with uniform norm

$$\|w\|_\infty = \sup_{1 \leq r \leq e} |w(r)|$$

This space is a Banach space.

We set

$$\mathcal{AC}_\delta^n(I, \mathbb{R}) = \{w: \mathcal{J} \to \mathbb{R}: \delta^{n-1} w(s) \in \mathcal{AC}(\mathcal{J}, \mathbb{R})\},$$

here $\mathcal{AC}(\mathcal{J}, \mathbb{R})$ denotes the space of absolutely continuous functions and $\delta = s \frac{d}{ds}$.

Denote by $\mathcal{CL}(X)$ the set of closed subsets of X, $\mathcal{CB}(X)$ the set of closed and bounded subsets of X and $K_c(\mathfrak{X})$ the set of compact and convex subsets of X.

Definition 1.1: [8] A set valued map $\mathfrak{F}: \mathcal{J} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a Carathéodory, if:

(1) $s \to \mathfrak{F}(s, \eta)$ is measurable for each $\eta \in \mathbb{R}$

(2) $\eta \to \mathfrak{F}(s, \eta)$ is upper semi -continuous for almost all $s \in \mathcal{J}$.

Lemma 1.2: [10] *Let X, Z be Banach spaces and $\mathfrak{F}: \mathcal{J} \times X \to \mathcal{CL}(Z)$ be a set valued map, if $\mathfrak{F}$ is L^1-Carathéodory and upper-semi continuous then it has a closed graph. Further, if $\mathfrak{F}$ is completely continuous, then it is upper-semi continuous.*

Lemma 1.3: [18] *Let* X *be a Banach space,* $\mathcal{I}$ *be a compact real interval,* $\mathfrak{F}$ *is a Carathéodory operator, and let* γ *be a linear continuous operator from* $L^1(\mathcal{I},X) \to \mathcal{C}(\mathcal{I},X)$ *such that*

$$\gamma \circ S_{\mathfrak{F},w}: L^1(\mathcal{I},X) \to K_c(X).$$

Then $T = \gamma \circ S_{\mathfrak{F},w}$ has a closed graph.

For each $w \in \mathcal{AC}(\mathcal{I},\mathbb{R})$, define the selection set of $\mathfrak{F}$ by

$$S_{\mathfrak{F},w} = \{v \in L^1(\mathcal{I},\mathbb{R}), \quad v \in \mathfrak{F}(s,w(s)) \qquad a.e.\, s \in \mathcal{I}\}.$$

Let $(X,\hat{d})$ be a metric space. The Hausdorff metric $\mathcal{H}: \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}_+$ is defined by

$$\mathcal{H}(\mathcal{A},\mathcal{B}) = \max\{\sup_{a\in\mathcal{A}} \hat{d}(a,\mathcal{B}), \sup_{b\in\mathcal{B}} \hat{d}(b,\mathcal{A}).\}$$

Definition 1.4: [16, 20] Hadamard's integral type of order $q > 0$ for $\xi \in L^1[1,\infty)$ is given by:

$${}^H I^q \xi(r) = \frac{1}{\Gamma_1(q)} \int_1^r \left(\log \frac{r}{s}\right)^{q-1} \frac{\xi(s)}{s} ds,$$

with the existence of the integral.

Definition 1.5: [15] Let $\xi \in \mathcal{AC}_\delta^n([\rho,\tau],\mathbb{R})$ with $0 < \rho < \tau$, then Caputo-Hadamard derivative of non integer order $q > 0$ is given by:

$$\mathcal{D}^q \xi(s) ={}^H D^q \left[\xi(s) - \sum_{k=0}^{n-1} \frac{\delta^i \xi(\rho)}{i!} \left(\log \frac{s}{\rho}\right)^q\right](s),$$

where $q > 0$ and $n = [q] + 1$.

Lemma 1.6: *Let* $\xi \in \mathcal{AC}_\delta^n([\rho,\tau],\mathbb{R})$ *or* $\mathcal{C}_\delta^n([\rho,\tau],\mathbb{R})$ *and* $q \in \mathbb{C}$. *Then*

$${}^H I^q(\mathcal{D}^q \xi)(r) = \xi(r) - \sum_{i=0}^{n-1} \frac{\delta^i \xi(\rho)}{i!} \left(\log \frac{r}{\rho}\right)^q.$$

Theorem 1.7: [21] *Let* $(X,\hat{d})$ *be a complete metric space and* $\mathcal{T}: X \to \mathcal{CB}(X)$ *be a set valued contraction, that's for all* $w,z \in X$ *there exists* $\rho \in [0,1)$ *such that:*

$$\mathcal{H}(\mathcal{T}w,\mathcal{T}z) \leq \rho \hat{d}(w,z).$$

Then $\mathcal{T}$ *admits a fixed point.*

Theorem 1.8: [11] *Let* $\mathcal{E}$ *be a Banach space and* $\mathcal{S}$ *be a nonempty and convex subset of* $\mathcal{E}$. *Let* Ω *be a nonempty convex subset of* $\mathcal{S}$ *with* $0 \in \Omega$. *If an operator* $\mathcal{T}: \bar{\Omega} \to K_c(\mathcal{E})$ *is upper semicontinuous and compact. So either*

(i) $\mathcal{T}$ *has a fixed point in* Ω, *or*

(ii) *there exist* $\zeta \in \partial\Omega$ *and* $\delta \in [0,1]$ *such that* $\zeta \in \delta\mathcal{T}(\zeta)$.

2. Main results

A function w is a solution of (1) if there exists $v \in \mathfrak{F}(r, w(r))$ satisfying the following problem:

$$\begin{cases} \mathcal{D}^q w(r) = v(r), 1 < r < e, \\ w(1) = \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds, \\ w(e) = \lambda \int_{\beta_1}^{\beta_2} h(s, w(s))ds, \end{cases}$$

Lemma 2.1: *If a function w is a solution of the problem:*

$$\mathcal{D}^q w(r) = v(r) \text{ for all } r \in [1, e] \tag{2}$$

with the boundary conditions:

$$w(1) = \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds \tag{3}$$

$$w(e) = \lambda \int_{\beta_1}^{\beta_2} h(s, w(s))ds, \tag{4}$$

then it is a solution of the following integral equation

$$w(r) = \int_1^e \mathrm{G}(r, s)v(s)ds + \lambda \log r \int_{\beta_1}^{\beta_2} h(s, w(s))ds + (\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds, \tag{5}$$

where

$$\mathrm{G}(r, s) = \frac{1}{s\Gamma(q)} \begin{cases} -\log r(1 - \log s)^{q-1}, 1 \le r \le s \le e \\ (\log \frac{r}{s})^{q-1} - \log r(1 - \log s)^{q-1}, 1 \le s \le r \le e, \\ \\ -\log r(1 - \log s)^{q-1}, 1 \le r \le s \le e \end{cases} \tag{6}$$

Proof: If w is a solution of (2), by Hadamard integral of order q to both sides of (2), and using Lemma (1.6), we get

$$w(r) = a_1 + a_2 \log(r) + {}^H I^q v(r)$$

$$w(r) = a_1 + a_2 \log(r) + \frac{1}{\Gamma(q)} \int_1^r \left(\log \frac{r}{s}\right)^{q-1} \frac{v(s)}{s} ds.$$

From (3)-(4), we get

$$w(1) = a_1 = \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds$$

and

$$w(e) = a_1 + a_2 + \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} \frac{v(s)}{s} ds = \lambda \int_{\beta_1}^{\beta_2} h(s, w(s))ds,$$

so

$$a_2 = \lambda \int_{\beta_1}^{\beta_2} h(s, w(s))ds - \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds - \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} \frac{v(s)}{s} ds.$$

By substitution, we find

$$w(r) = \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds + \frac{1}{\Gamma(q)} \int_1^r \left(\log \frac{r}{s}\right)^{q-1} \frac{v(s)}{s} ds$$
$$+\log(r)[\lambda \int_{\beta_1}^{\beta_2} h(s, w(s))ds - \mu \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds - \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} \frac{v(s)}{s} ds]$$

and

$$w(r) = \mu(1 - \log(t)) \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds + \lambda \log(r) \int_{\beta_1}^{\beta_2} h(s, w(s))ds$$
$$-\frac{1}{s\Gamma(q)} \int_r^e \log(t)(1 - \log(s))^{q-1} v(s)ds$$
$$+\frac{1}{s\Gamma(q)} \int_1^r \left[\left(\log \frac{r}{s}\right)^{q-1} - \log(r)(1 - \log(s))^{q-1}\right] v(s)ds,$$

thus

$$w(r) = \int_1^e \mathrm{G}(r, s)v(r)ds + \lambda \mathrm{log}t \int_{\beta_1}^{\beta_2} h(s, w(s))ds$$
$$+(\mu - \mu \mathrm{log}t) \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds,$$

where

$$\mathrm{G}(r, s) = \frac{1}{s\Gamma(q)} \begin{cases} \left(\log \frac{r}{s}\right)^{q-1} - \log(r)(1 - \log(s))^{q-1}, 1 \le s \le r \le e, \\ -\log(r)(1 - \log(s))^{q-1}, 1 \le r \le s \le e. \end{cases}$$

Convex case

Theorem 2.2: *Under the following hypotheses:*

- $(\mathfrak{H}_1)$: $\widetilde{\mathfrak{F}}: [1, e] \times \mathbb{R} \to K_c(\mathbb{R})$ *is a Carathéodory set valued map.*
- $(\mathfrak{H}_2)$ *There exists* $\varphi \in L^1([1, e])$ *such that*
  $$\left\|\widetilde{\mathfrak{F}}(r, w)\right\|_* = \sup\{|w|, w \in \mathfrak{F}\} \le \varphi(t), for\ all\ r \in \mathcal{J}\ and\ w \in \mathbb{R}.$$
- $(\mathfrak{H}_3)$ *There exist two non decreasing and continuous functions* $\psi, \phi: \mathbb{R}_+ \to \mathbb{R}_+$ and $\vartheta, \varsigma \in \mathcal{C}(\mathcal{J}, \mathbb{R}_+)$ *such that*
  $$|g(r, w_r)| \le \vartheta(r)\psi(\|w\|)\ for\ all\ r \in \mathcal{J}\ and\ w \in \mathbb{R}$$
  $$\left|h\big(r, w(r)\big)\right| \le \varsigma(r)\phi(\|w\|)\ for\ all\ r \in \mathcal{J}\ and\ w \in \mathbb{R}.$$
- $(\mathfrak{H}_4)$ *There exists* $\mathcal{L} > 0$ *such that*
  $$\frac{\mathcal{L}}{\lambda\vartheta_0\phi(\mathcal{L}) + \mu\varsigma_0\psi(\mathcal{L}) + \mathrm{G}^*\varphi_0} > 1, \tag{7}$$
  where
  $$\mathrm{G}^* = \sup_{r,s \in \mathcal{J}]} |\mathrm{G}(r, s)|$$
  $$\text{and } \varphi_0 = \|\varphi\|_{L^1}.$$
  the problem (1) has a solution on $\mathcal{J}$.

Proof: Firstly, we transform our problem (1) into a fixed point problem. Consider the set valued operator,

$$\mathcal{T}(w) = \{z \in \mathcal{C}(\mathcal{I}, \mathbb{R}), (r) = P_w(r) + \int_1^e \mathrm{G}(r,s)v(s)ds, \in S_{\mathfrak{F},w}\},$$

where

$$P_w(r) = \lambda \log r \int_{\beta_1}^{\beta_2} h(s, w(s))ds + (\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} g(s, w(s))ds$$

and the function G is given by (6). Clearly, from Lemma (2.1), any fixed point of $\mathcal{T}$ is a solution to (1). We show that $\mathcal{T}$ satisfies the coditions of the non-linear alternative of Leray-Schauder (1.8) and the proof will be given in five steps.

Step 1: $\mathcal{T}w$ is convex for each w.

If $z_1, z_2 \in \mathcal{T}w$, then there exist v_1, v_2 such that for each $r \in [1, e]$, we have

$$z_i = P_w(r) + \int_1^e \mathrm{G}(r,s)v_i(s)ds, i = 1,2.$$

Let $0 < \omega < 1$. Then, for each $r \in [1, e]$ we have

$$(\omega z_1 + (1-\omega)z_2)(r) = P_w(r) + \int_1^e \mathrm{G}(r,s)(\omega v_1 + (1-\omega)v_2)(s)ds.$$

Since $S_{\mathfrak{F},w}$ is convex ($\mathfrak{F}$ has convex values), we have

$$\omega z_1 + (1-\omega)z_2 \in \mathcal{T}w.$$

Consequently, $\mathcal{T}w$ is convex.

Step 2: $\mathcal{T}$ is bounded.

Let $B_R = \{w \in \mathcal{C}([1,e], \mathbb{R}), \|w\|_\infty \leq R\}$ be a set in $\mathcal{C}([1,e], \mathbb{R})$ and $w \in B_R$. For each $z \in \mathcal{T}w$, $r \in [1, e]$ and from $(\mathfrak{H}_2), (\mathfrak{H}_3)$, we have

$$|z(r)| \leq \int_1^e |\mathrm{G}(r,s)||v(r)|ds + \lambda \log r \int_{\beta_1}^{\beta_2} |h(s, w(s))|ds,$$
$$+|(\mu - \mu \log r)| \int_{\alpha_1}^{\alpha_2} |g(s, w(s))|ds,$$

so we have:

$$|z(r)| \leq \int_1^e |\mathrm{G}(r,s)||v(s)|ds + \lambda \log r \int_{\beta_1}^{\beta_2} (\phi(\|w(s)\|)ds$$
$$+|\mu(1 - \log r)| \int_{\alpha_1}^{\alpha_2} \psi(\mathrm{P}w\mathrm{P})ds$$
$$\leq \lambda \log r \phi(\|w\|) \int_1^e \varsigma(s)ds$$
$$+|(\mu - \mu \log r)\psi(\|w\| \int_1^e \vartheta(s)ds,$$

Then we get

$$\|z(r)\|_\infty \leq \mu \vartheta_0 \psi(R) + \lambda \varsigma_0 \phi(R) + \varphi_0 \mathrm{G}^* = C,$$

where $\vartheta_0 = \sup \int_0^e \vartheta(s)$ *and* $\varsigma_0 = \sup \int_1^e \varsigma(s)ds.$

So $\mathcal{T}w$ is bounded.

Step 3: The family $\mathcal{T}w$ is equicontinuous.

Let $r_1, r_2 \in \mathcal{I}$ with $r_1 < r_2$, $w \in B_R$ and let $z \in \mathcal{T}w$, we have

$$|z(r_2) - z(r_1)| \leq \lambda |\log r_2 - \log r_1| \int_{\beta_1}^{\beta_2} |h(s, w(s))| ds$$
$$+\mu |\log r_2 - \log r_1| \int_{\alpha_1}^{\alpha_2} |g(s, w(s))| ds + \int_1^e |\mathrm{G}(r_2, s) - \mathrm{G}(r_1, s)||v(s)| ds$$
$$\leq \lambda |\log r_2 - \log r_1| \int_{\beta_1}^{\beta_2} |\phi(\|w\|) + \mu |\log r_2 - \log r_1| \int_{\alpha_1}^{\alpha_2} |\psi(\|w\|$$
$$+\varphi_0 \int_1^e |\mathrm{G}(r_2, s) - \mathrm{G}(r_1, s)| ds$$
$$\leq |\log r_2 - \log r_1| [\lambda \int_{\beta_1}^{\beta_2} |\vartheta(s) \phi(\|w\|) + \mu \int_{\alpha_1}^{\alpha_2} |\varsigma(s) \psi(\mathrm{P}w\mathrm{P})$$
$$+ \int_1^e |\varphi(s)||\mathrm{G}(r_2, s) - \mathrm{G}(r_1, s)| ds].$$

Passing to the supermum, we get

$$\|z(r_2) - z(r_1)\|_\infty \leq \|\log r_2 - \log r_1\| [\lambda \vartheta_0 \phi(\|w\|) + +\mu \varsigma_0 \psi(\|w\|)]$$
$$+ \int_1^e |\varphi(s)| \mathrm{P}\mathrm{G}(r_2, s) - \mathrm{G}(r_1, s)\mathrm{P}.$$

For $r_1 \to r_2$, we get $\|z(r_2) - z(r_1)\|_\infty \to 0$, so from Ascoli theorem, we conclude that $\mathcal{T}: \mathcal{C}(\mathcal{J}, \mathbb{R}) \to \mathcal{P}(\mathcal{C}(I, \mathbb{R}))$ is completely continuous.

Step 4: The closness of the graph of $\mathcal{T}$.

Let $w_n \to w^*$ and $z_n \in \mathcal{T}w_n$ with $z_n \to z^*$. We show that $z^* \in \mathcal{T}w^*$.

$z_n \in \mathcal{T}w_n$ implies that there exists $v_n \in S_{\mathfrak{F}, w_n}$, such that, for each $r \in \mathcal{J}$, we have

$$z_n = P_{w_n}(r) + \int_1^e \mathrm{G}(r, s) v_n(s) ds.$$

Consider an operator $\gamma: L^1(\mathcal{J}, X) \to \mathcal{C}(\mathcal{J}, \mathbb{R})$ defined by

$$\gamma(v(r)) = z_n = P_w(r) + \int_1^e \mathrm{G}(r, s) v(s) ds.$$

We will prove the continuity of γ, in fact let (v_n) be a sequence in $L^1(\mathcal{J}, \mathbb{R})$ such that $v_n \to v^*$, we have

$$|z_n - z^*| \leq \int_1^e |\mathrm{G}(r, s)| v_n(s) - v^*(s)| ds$$
$$+\lambda \log t \int_{\beta_1}^{\beta_2} |h(s, w_n(s)) - h(s, w^*(s))| ds$$
$$+(\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} |g(s, w_n)(s)) - g(s, w^*(s))| ds.$$

Since

$$\int_1^e |\mathrm{G}(r, s)| \|v_n(r) - v_*(r)\|_\infty ds \leq \phi_0 \mathrm{G}^* \|w_n(r) - w^*(r)\|_\infty,$$

g and h are continuous, $\mathrm{P}z_n(r) - z^*(t)\mathrm{P}_\infty \to 0$ as $n \to \infty$. Then γ is continuous and so from Lemma 1.3 $T = \gamma \circ S_{\mathfrak{F}, w}$ has a closed graph. Consequently, and from Corollary 1 in [7] T is an upper semi continuous operator.

Step 5: A priori bounds on solutions.

Let w be such that $w \in \tau \mathcal{T} w$ with $\tau \in [0,1]$, so there exists $v \in S_{\mathfrak{F}, w}$ such that, for each $r \in \mathcal{J}$,

$$w(t) = \tau P_w(t) + \tau \int_1^e \mathrm{G}(r,s)v(s)ds$$

This implies by $(\mathfrak{H}_3)$ that, for each $r \in \mathcal{I}$, we have

$$|w(r)| \leq \int_1^e |\mathrm{G}(r,s)||v(t)|ds + \lambda \log r \int_{\beta_1}^{\beta_2} h(s,w(s))ds$$

$$+|(\mu - \mu \log r)| \int_{\alpha_1}^{\alpha_2} g(s,w(s)))ds$$

$$\|w\|_\infty \leq \mathrm{G}^*\varphi_0 + \lambda\vartheta_0\phi(\|w\|) + \mu\varsigma_0\psi(\|w\|)$$

Thus,

$$\frac{\|w\|_\infty}{\mathrm{G}^*\varphi_0 + \lambda\vartheta_0\phi(\|w\|) + \mu\varsigma_0\psi(\|w\|)} \leq 1.$$

Then by condition (7), there exists $\mathcal{L} > 0$ such that $\|W\|_\infty \neq \mathcal{L}$. Let $\Omega = \{w \in \mathcal{C}(\mathcal{I},\mathbb{R}) : \|W\|_\infty < \mathcal{M} + 1\}$. The map $\mathcal{T} : \overline{\Omega} \to \mathcal{P}(\mathcal{C}(\mathcal{I},\mathbb{R}))$ is upper semi continuous and compact. From the choice of Ω, there is no $w \in \partial\Omega$ such that $w \in \tau\mathcal{T}w)$, for some $\tau \in [0,1]$.

From the non-linear alternative of Leray-Schauder 1.8, we conclude that $\mathcal{T}$ has a fixed point $w \in \overline{\Omega}$, so the problem (1) admits a solution.

Non convex case

Theorem 2.3: *Assume the following hypotheses hold:*

- $(\mathfrak{H}_5)$: $\mathfrak{F} : [1,e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is compact and the* $w \mapsto \mathfrak{F}(r,w(r))$ *is measurable for each* $r \in [1,e]$.
- $(\mathfrak{H}_6)$: *There exist* $\eta, \chi \in L^1([1;e],\mathbb{R}_+)$ *such that*

$$|g(r,w_1) - g(r,w_2)| \leq \eta(r)|w_1 - w_2|,$$

$$|h(r,w_1) - h(r,w_2| \leq \chi(r)|w_1 - w_2|.$$

- $(\mathfrak{H}_7)$: *There exists* $j \in L^1([1,e],\mathbb{R}^+)$ *such that:*

$$H\left(\mathfrak{F}\big(r,w_1(r)\big),\mathfrak{F}\big(r,w_2(r)\big)\right) \leq j(r)|w_1 - w_2|, for\ each\ {}_1, w_2 \in \mathcal{C}([1,e],\mathbb{R})$$

$$and\ d(0,\mathfrak{F}(r,0)) \leq j(r)\ a.e.\, r \in [1,e].$$

Then the problem (1) admits at least a solution provided

$$\mathrm{G}^* \int_1^e d(s)ds + \lambda\|\chi\|_{L^1} + \mu\|\eta\|_{L^1} < 1$$

Proof: Define an operator $\mathcal{T}$ such that: $\mathcal{T} : \mathcal{C}([1,e],\mathbb{R}) \to \mathcal{C}(\mathcal{I},\mathbb{R})$

$\mathcal{T}w(r) \in \{z \in \mathcal{C}(\mathcal{I}],\mathbb{R}) : z(r) = \int_1^e \mathrm{G}(r,s)v(s)ds + \lambda \log r \int_{\beta_1}^{\beta_2} h(s,w(s))ds + (\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} g(s,w(s))ds\}$, where $v \in S_{\mathfrak{F},w}$.

From the condition (H_5) the set valued map $\mathfrak{F}(r,w)$ is measurable to closed and convex values, so by theorem selection measurable for each $w \in \mathcal{C}([1,e],\mathbb{R})$, there exists a measurable selection $v(r) \in \mathfrak{F}(r,w(r))$, then $\mathcal{T}w$ is non- empty and well defined.

Step 1: $\mathcal{T}w$ is closed for each $w \in \mathcal{C}(\mathcal{I},\mathbb{R})$:

Let $(z_n)_{n\geq 0} \in \mathcal{T}w$ such that $z_n \to z$. Then there exists $v_n \in S_{\mathfrak{F},w_n}$ converges to v, for all $r \in \mathcal{I}$ and verifies

$$z_n(r) = \int_1^e \mathrm{G}(r,s)v_n(s)ds + \lambda \log r \int_{\beta_1}^{\beta_2} h(s,w(s))ds$$
$$+(\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} g(s,w(s))ds.$$

We prove firstly that $v \in S_{\mathfrak{F},w}$.

Since $\mathfrak{F}(t,w)$ is compact we can extract a subsequence v_{n_k} converges in $S_{\mathfrak{F},w}$, since (v_{n_k}) has the same limit as (v_n), then $v \in S_{\mathfrak{F},w}$. Passing to the limit of the previous relation, we find:

$$z(t) = \int_1^e \mathrm{G}(r,s)v(s)ds + \lambda \log r \int_{\beta_1}^{\beta_2} h(s,w(s))ds$$
$$+(\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} g(s,w(s))ds.$$

Since $v \in S_{\mathfrak{F},w}$, then $z \in \mathcal{T}w$.

Step 2: $\mathcal{T}$ is a contraction.

We will prove there exists $0 < \rho < 1$ such that:

$$\mathcal{H}\big(\mathcal{T}w_1(r), \mathcal{T}w_2(r)\big) \leq \rho \|w_1(r) - w_2(r)\|_\infty, \text{ for all } w_1, w_2 \in \mathcal{C}(\mathcal{I}, \mathbb{R}).$$

Let $w_1, w_2 \in \mathcal{C}([1,e], \mathbb{R})$ and $z_1 \in \mathcal{T}w_1$ and there exists $v_1(r) \in \mathfrak{F}(r, w_1(r))$ such that for all $r \in \mathcal{I}$:

$$z(r) = \int_1^e \mathrm{G}(r,s)v_1(s)ds + \lambda \log r \int_{\beta_1}^{\beta_2} h(s,w_1(s))ds$$
$$+(\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} g(s,w_1(s))ds.$$

From $(\mathfrak{H}_7)$, there exists $w \in \mathfrak{F}(r, w_2(r))$ such that

$$|v_1(r) - w| \leq d(r)|w_1(r) - w_2(r)| \text{ for all } r \in \mathcal{I}.$$

Define an operator Q by:

$$Q(r) = \{w \in \mathbb{R}, |v_1(r) - w|\} \cap \mathcal{T}w_1$$

As the operator $Q \cap \mathcal{T}$ is measurable, there exists a measurable selection $v_2 \in \mathfrak{F}(r, w_2)$ such that for each $t \in \mathcal{I}$ we have:

$$|v_1(r) - v_2(r)| \leq d(r)|w_1(r) - w_2(r)| \text{ for all } r \in \mathcal{I}].$$

Let $z_2 \in \mathcal{T}w_2$ which given by

$$z_2 = P_{w_2} + \int_1^e \mathrm{G}(r,s)v_2(s)ds,$$

so we have:

$$|z_1(r) - z_2(r)| \leq \int_1^e |\mathrm{G}(r,s)||v_1(s) - v_2(s)|ds + \lambda \log r \int_{\beta_1}^{\beta_2} |h(s,w_1(s))$$
$$-h(s,w_2(s))|ds + (\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} |g(s,w_1(s)) - g(s,w_2(s))|ds.$$

From $(\mathfrak{H}_6)$ and $(\mathfrak{H}_7)$ we get

$|z_1(t) - z_2(r)| \leq \int_1^e |\mathrm{G}(r,s)|m(t)|w_1(s)$
$-w_2(s)|ds + \lambda \log r \int_{\beta_1}^{\beta_2} \xi(s)|w_1(s) - w_2(s)|ds$
$+(\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} \eta(s)|w_1(s) - w_2(s)|ds.$
$\leq \int_1^e |\mathrm{G}(r,s)|d(r)|w_1(s) - w_2(s)|ds + \lambda \log r \int_{\beta_1}^{\beta_2} \chi(s)|w_1(s) - w_2(s)|ds$
$+(\mu - \mu \log r) \int_{\alpha_1}^{\alpha_2} \eta(s)|w_1(s) - w_2(s)|ds.$
$\leq \int_1^e |\mathrm{G}(r,s)|m(t)|w_1(s) - w_2(s)|ds + \lambda \int_1^e \chi(s)|w_1(s) - w_2(s)|ds$
$+\mu \int_1^e \eta(s)|w_1(s) - w_2(s)|ds$

$$\|z_1(r) - z_2(r)\|_\infty \leq (\mathrm{G}^* \int_1^e d(s)ds + \lambda\|\chi\|_{L^1} + \mu\|\eta\|_{L^1})\|w_1 - w_2\|_\infty$$

Since

$$\rho = \mathrm{G}^* \int_1^e dm(s)ds + \lambda\|\chi\|_{L^1} + \mu\|\eta\|_{L^1} < 1,$$

so by interchanging the role of z_1 and z_2 we get

$$\mathcal{H}\big(\mathcal{T}w_1(r), \mathcal{T}w_2(r)\big) \leq \rho\|w_1 - w_2\|_\infty, \text{ for all } {}_1, w_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}).$$

So, $\mathcal{T}$ is a contraction. Then from Nadler Theorem (1.7) $\mathcal{T}$ has a point fixed which .

Example 2.4:

$$\begin{cases} \mathcal{D}^{\frac{3}{2}}w(r) \in \mathfrak{F}(r, w(r)), 1 < r < e, \\ \\ w(1) = \frac{1}{10}\int_{\frac{1}{3}}^{\frac{1}{2}} (\ln s + \frac{w}{w+3})ds, \\ \\ w(e) = \frac{1}{100}\int_{\frac{2}{3}}^{\frac{3}{4}} (4 + s^2 \sin w)ds, \end{cases} \tag{8}$$

$\mathfrak{F}: [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set valued map given by

$$\mathfrak{F}(r, w(r)) = \left[\frac{r}{|w(r)|^3+2}, \frac{r}{|w(r)|+1}\right]$$

1. $\mathfrak{F}$ is a Carathéodory mapping:
   a) $r \to \mathfrak{F}(r, w)$ is measurable for each $w \in \mathbb{R}$,
   let $r_n \to r$ so $w(r_n) \to w(r)$ and

$$\begin{aligned} \lim_{n\to\infty} \mathfrak{F}(r_n, w(r_n)) &= \left[\frac{t_n}{|w(t_n)|^3+2}, \frac{r_n}{|w(r_n)|+1}\right] \\ &= \left[\frac{r}{|w(r)|^3+2}, \frac{r}{|w(r)|+1}\right] \\ &= \mathfrak{F}(r, w(r)), \end{aligned}$$

so

$$\lim_{n\to\infty} \mathcal{H}(\mathfrak{F}(r_n, w(r_n)), \mathfrak{F}(r, w(r))) = \mathcal{H}(\mathfrak{F}(r, w(r)), \mathfrak{F}(r, w(r)))$$

as $\mathcal{H}$ is a metric then

$$\lim_{n\to\infty} H(\mathfrak{F}(r_n, w(r_n)), \mathfrak{F}(r, w(r))) = 0.$$

So, $r \to \mathfrak{F}(r, w)$ is a continuous map. then $\mathfrak{F}$ is measurable for each $w \in \mathbb{R}$.

b) $w \to \mathfrak{F}(r, w)$ is upper semi-continuous for almost all $r \in [1, e]$.

Let $w_n \to w$, for each $t \in I$ we have

$$\begin{aligned}\lim_{n\to\infty} \mathfrak{F}(r, w_n(r)) &= \left[\frac{r}{|w_n(r)|^3+2}, \frac{t}{|w_n(r)|+1}\right] \\ &= \left[\frac{r}{|w(r)|^3+2}, \frac{r}{|w(r)|+1}\right] \\ &= \mathfrak{F}(r, w(r)),\end{aligned}$$

so

$$\lim_{n\to\infty} \mathcal{H}(\mathfrak{F}(t, w_n(t)), \mathfrak{F}(r, w(r))) = \mathcal{H}(\mathfrak{F}(r, w(r)), \mathfrak{F}(r, w(r))) = 0.$$

So, $w \to \mathfrak{F}(r, w)$ is continuous for each $r \in \mathcal{I}$.

2. $\mathcal{T}$ is compact:

To prove that $\mathcal{T}$ is compact, we use the Arzelá-Ascoli theorem.

We go first show that FB_R is bounded, then we will prove that it is equicontinuous, where $B_R := \{w \in \mathcal{C}(\mathcal{I}, \mathbb{R}), \|w\|_\infty \leq R\}$.

a) $\mathfrak{F}$ is bounded.

For $v \in \mathfrak{F}$, we have:

$$|v(r)| \leq \max\left\{\frac{t}{|w(r)|^3+2}, \frac{r}{|w(r)|+1}\right\} \leq r,$$

so

$$\|\mathfrak{F}(r, w(r))\|_\infty = \sup\{|v(r)|, v \in \mathfrak{F}(r, w(r))\} \leq \varphi(r),$$

where $\varphi(r) = r$ and $\varphi_0 = e^2 - 1$.

b) $\mathfrak{F}B_R$ is equicontinuous.

Let $r_1, r_2 \in I$ with $r_1 < r_2$ and $v \in \mathfrak{F}$ we have:

$$\frac{r_1}{|w(r_1)|^3+2} \leq v(r_1) \leq \frac{r_1}{|w(r_1)|+1},$$
$$\frac{r_2}{|w(r_2)|^3+2} \leq v(r_2) \leq \frac{r_2}{|w(r_2)|+1}.$$

So

$$\left|\frac{r_1}{|w(r_1)|^3+2} - \frac{r_2}{|w(r_2)|^3+2}\right| \leq |v(r_1) - v(r_2)| \leq \left|\frac{r_1}{|w(r_1)|+1} - \frac{r_2}{|w(r_2)|+1}\right|.$$

Then

$$\begin{aligned}&\lim_{r_1\to r_2} \left\|\frac{r_1}{|w(r_1)|^3+2} - \frac{r_2}{|w(r_2)|^3+2}\right\|_\infty \leq \lim_{r_1\to r_2} \|v(r_1) - v(r_2)\|_\infty \\ &\leq \lim_{r_1\to r_2} \left\|\frac{r_1}{|w(r_1)|+1} - \frac{t_2}{|w(r_2)|+1}\right\|_\infty\end{aligned}$$

we have

$$\lim_{r_1\to r_2}\left\|\frac{r_1}{|w(r_1)|^3+2}-\frac{r_2}{|w(r_2)|^3+2}\right\|_\infty \leq \lim_{r_1\to r_2}\left\|\frac{r_1}{|w(r_1)|+1}-\frac{r_2}{|w(r_2)|+1}\right\|_\infty = 0$$

then

$$\lim_{r_1\to r_2}\|v(r_1)-v(r_2)\|_\infty = 0.$$

So $\mathfrak{F}B_R$ is equicontinuous and bounded, then $\mathfrak{F}$ is compact.

3. $\mathfrak{F}$ is convex:
let $v_1, v_2 \in \mathfrak{F}(r, w(r))$ and $\omega \in [0,1]$, and for each $r \in \mathcal{I}$ we have:

$$\frac{\omega r}{|w(r)|^3+2} \leq \omega v_1(r) \leq \frac{\omega r}{|w(r)|+1},$$

$$\frac{(1-\omega)r}{|w(r)|^3+2} \leq (1-\omega)v_2(r) \leq \frac{(1-\omega)t}{|w(r)|+1}.$$

Then

$$\frac{t}{|w(r)|^3+2} \leq \omega v_1(t) + (1-\omega)v_2(r) \leq \frac{r}{|w(r)|+1}.$$

$$\omega v_1(r) + (1-\omega)v_2(r) \in \left[\frac{r}{2+|w(r)|^3}, \frac{r}{1+|w(r)|}\right] = \mathfrak{F}(r, w(r)).$$

4. Taking $\vartheta(r) = \ln r + 1$ and $\phi(\|w\|) = 1$, we get

$$|g(r, w(r)))| \leq r.$$

For h we take $\varsigma(r) = 5 + r^2$ and $\phi(\|W\|) = 1$, we get

$$|h(r, w(r))| \leq 5 + r^2$$

5. There is $\mathcal{L} > 0$ such that

$$\mathcal{L} > G^*\varphi_0 + \mu\vartheta_0\phi(\|w\|) + \lambda\varsigma_0\psi(\|w\|)$$

$$= \frac{1}{e\sqrt{\pi}}(e^2-1) + (10^{-1}(e-1) + 10^{-2}[5(e-1) + \frac{1}{3}(e^3-1)] = 0.67799.$$

Then according to Theorem (2.2) the problem (8) has a solution on $\mathcal{I}$.

Example 2.5:

$$\begin{cases} \mathcal{D}^{\frac{3}{2}}w(r) \in \mathfrak{F}(r, w(r)), 1 < t < e, \\ w(1) = \frac{1}{10}\int_{\frac{1}{3}}^{\frac{1}{2}} (\ln s + \frac{w}{3+w})ds, \\ w(e) = \frac{1}{100}\int_{\frac{2}{3}}^{\frac{3}{4}} (4 + s^2 \sin w)ds, \end{cases} \tag{9}$$

$F: [1, e] \times \mathbb{R} \to P(\mathbb{R})$ is a set valued map given by

$$w \to \mathfrak{F}(r, w(r)) \in \left\{\frac{|w(r)|}{2} + nr\right\}, n = \{0,1,2,3\}.$$

Clearly that $\mathfrak{F}$ is compact, so it is closed and bounded.
For $v_1 \in \mathfrak{F}(r, w_1)$ and $v_2 \in \mathfrak{F}(r, w_2)$, we have:

$$\begin{aligned}|v_1(r) - v_2(r)| &= \left|\left(\frac{|w_1|}{2} + 3r^3 + 5n\right) - \left(\frac{|w_2|}{2} + 3r^3 + 5n\right)\right| \\ &= \left|\frac{|w_1|}{2} - \frac{|w_2|}{2}\right| \\ &\leq \frac{1}{2}|w_1 - w_2|.\end{aligned}$$

Then for $j(r) = \frac{1}{2}$, we get

$$\|v_1(r) - v_2(t)\|_\infty \leq \frac{1}{2}\|w_1(r) - w_2(r)\|_\infty.$$

So

$$\mathcal{H}(\mathfrak{F}(r, w_1), \mathfrak{F}(r, w_2)) \leq \frac{1}{2}\|w_1 - w_2\|_\infty.$$

Remark that for $\eta(r) = \frac{1}{3}$, we get:

$$\begin{aligned}|g(r, w_1(r)) - g(r, w_2(r))| &= \left|\ln r + \frac{w_1(r)}{3+w_1} - \ln r - \frac{w_2}{3+w_2}\right|, \\ &\leq \frac{1}{3}|w_1 - w_2|,\end{aligned}$$

also for $\xi(r) = 1$ we get:

$$\begin{aligned}|h(r, w_1(r)) - h(r, w_2(r))| &= |\sin w_1 + 2r^2 + 4 - \sin w_2 - (2r^2 + 4)| \\ &= |\sin w_1(r) - \sin w_2(r)|.\end{aligned}$$

The mean value theorem gives:

$$\begin{aligned}|\sin w_2 - \sin w_1| &\leq |w_2 - w_1| \\ &\leq |w_1 - w_2|.\end{aligned}$$

On other hand, we have:

$$\begin{aligned}k = \mathrm{G}^* \int_1^e j(s)ds + \lambda\|\eta\|_{L^1} + \mu\|\xi\|_{L^1} &= \left(\frac{1}{e\sqrt{\pi}} + \frac{1}{30} + 10^{-2}\right)(e - 1) \\ &= 0.4310941407 < 1.\end{aligned}$$

Then from Theorem 1.7 the problem (9) has at least a solution on $\mathcal{J}$.

Remark 2.6: The obtained results considering as a generalization of those due to Ahmad and Ntouyas [4], where we tried to change the derivative type to Caputo-Hadamard one, and we modified in the boundary conditions.

Conclusion

In this work, we used two different fixed point theorems to prove the existence of solution for a Caputo-Hadamard boundary valued problem of fractional differential inclusions, the first one is topologic which needs more conditions, but in the second of Nadler [21], where we proved only the closeness of the rang, boundless and contractive condition of the operator. So consequently, we think that the metric fixed point theory rests one of the efficacy tools in the study of non linear problems.

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