

Effect of induced topological folding on the homology group of knots

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Abstract

In this article, we introduce the topological folding of some operations in a special family of knots and their homology groups. We obtain the relationship between the topological folding in the homology group of knots and their subgroups. We deduce the homology group of the end limits resulting from iterative topological foldings on the n -dimensional knot. We discuss the limit unfolding that involves varying curvature and torsion, which are related to the homology group of the unknot. We present some applications in biochemistry.

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Keywords: Knot, Topological folding, Homology group.

1. Introduction

Topology is an intriguing branch of geometry and mathematics that looks at shapes and spaces that can be reshaped without tearing, in geometry, two figures are the same if we can shift, rotate, or flip one to line up with the other. In topology, two figures are considered the same if a continuous function maps one shape perfectly onto the other; we call these shapes "homomorphic" or "topologically equivalent". Knot theory is an area of topology focused on studying figures [15]. Chemists work on knotting molecules to build material with special features. Biochemists focus on the role of enzymes in knotting and unknotting DNA [9]. A mathematical knot is a tamely embedded closed curve embedded in R . A knot is a continuous, closed loop in space, with no beginning and no end, the knot with no crossings is called the unknot. An n -dimensional knot (M, N, k) is an embedding of an n -dimensional manifold N^n in an $(n + 2)$ -dimensional manifold M^{n+2} , $k: N^n \subset M^{n+2}$ [7, 16]. A singular

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n -simplex is defined as a map $\sigma: \Delta^n \rightarrow X$; meanwhile, $C_n(X)$ is the free abelian group generated by the collection of n -simplex of X . The n -th homology group of X is defined as the quotient group $\hat{H}_n(X) = \ker\{\partial_n: \hat{C}_n(X) \rightarrow \hat{C}_{n-1}(X)\} / \text{Im}\{\partial_{n+1}: \hat{C}_{n+1}(X) \rightarrow \hat{C}_n(X)\}$ [6, 8, 10, 11, 14]. Given Riemannian manifolds M_1 and M_2 , in which $\dim M_1 = m_1$ and $\dim M_2 = m_2$, respectively. We say $\xi: M_1 \rightarrow M_2$ represents an isometric folding of M_1 into M_2 whenever $\xi: I \rightarrow M_1$ is a piecewise geodesic path, the composed path $\xi \circ \xi: I \rightarrow M_2$ is a piecewise geodesic path, and it has the same length as ξ [15]. When ξ does not stay at the same length, it is called a topological folding [13]. Also, if M_1 and M_2 are identical dimensions. We say $\xi: M_1 \rightarrow M_2$ is an unfolding of M_1 into M_2 whenever $\xi: I \rightarrow M_1$ is a piecewise geodesic path, the composed path $\xi \circ \xi: I \rightarrow M_2$ is a piecewise geodesic path with length greater than ξ [12]. Let us have spaces X_1 and X_2 with $x_1 \in X_1$, $x_2 \in X_2$, and $X_1 \cap X_2 = \emptyset$, then the wedge sum $X_1 \dot{\cup} X_2$ is the union of X_1 and X_2 after identifying x_1 and x_2 [11]. Different types of folding on manifolds and graphs associated with certain classes of groups were presented in [1-5]. In this paper, we write F_m^G for the free group of rank m .

2. The main results

Theorem 2.1: Let $L_1^m, L_2^m, \dots, L_r^m$ be an m -dimensional knots embedded in R^{m+2} . Then, the topological folding $\gamma: \bigcup_{i=1}^r L_i^m \rightarrow \bigcup_{i=1}^r L_i^m$, induces $\gamma: \bigoplus_{i=1}^r \hat{H}_m(L_i^m) \rightarrow \bigoplus_{i=1}^r \hat{H}_m(L_i^m)$ for which $(\bigoplus_{i=1}^r \hat{H}_m(L_i^m))$ reduces the rank of $\bigoplus_{i=1}^r \hat{H}_m(L_i^m)$.

Proof: Suppose $\gamma: \bigcup_{i=1}^r L_i^m \rightarrow \bigcup_{i=1}^r L_i^m$ is a topological folding that maps $\bigcup_{i=1}^r L_i^m$ into itself. Then

$F: \bigcup_{i=1}^r L_i^m \rightarrow \bigcup_{i=1}^r L_i^m$ has the following forms:

If $F(\bigcup_{i=1}^r L_i^m) = L_1^m \gamma L_2^m \gamma \dots \gamma F(L_s^m) \gamma \dots \gamma L_r^m$, for $s = 1, 2, \dots, r$, then $(\bigoplus_{i=1}^r \hat{H}_m(L_i^m)) = \hat{H}_m((\bigcup_{i=1}^r L_i^m)) \approx \hat{H}_m(L_1^m) \oplus \hat{H}_m(L_2^m) \oplus \dots \oplus \hat{H}_m(L_s^m) \oplus \dots \oplus \hat{H}_m(L_r^m)$. Because $\text{rank}(\hat{H}_m((L_s^m))) \leq \text{rank}(\hat{H}_m(L_s^m))$, leads to a lower rank of $\bigoplus_{i=1}^r \hat{H}_m(L_i^m)$.

Also, if $F(\bigcup_{i=1}^r L_i^m) = L_1^m \gamma L_2^m \gamma \dots \gamma (L_s^m) \gamma \dots \gamma (L_k^m) \gamma \dots \gamma L_r^m$, for $k = 1, 2, \dots, r$, $s < k$, then $\bar{F}(\bigoplus_{i=1}^r \hat{H}_m(L_i^m)) = \hat{H}_m((\bigcup_{i=1}^r L_i^m)) \approx \hat{H}_m(L_1^m) \oplus \hat{H}_m(L_2^m) \oplus \dots \oplus \hat{H}_m((L_s^m)) \oplus \dots \oplus \hat{H}_m((L_k^m)) \oplus \dots \oplus \hat{H}_m(L_r^m)$ thus $\bar{F}$ leads to a lower rank of $\bigoplus_{i=1}^r \hat{H}_m(L_i^m)$. Continuing in this way, if $F(\bigcup_{i=1}^r L_i^m) = \bigcup_{i=1}^r F(L_i^m)$. Then,

$(\bigoplus_{i=1}^r \hat{H}_m(L_i^m)) = \hat{H}_m(F(\bigvee_{i=1}^r L_i^m)) = \hat{H}_m(\bigvee_{i=1}^r F(L_i^m)) \approx \bigoplus_{i=1}^r \hat{H}_m(F(L_i^m)).$ Hence, reduces the rank of $\bigoplus_{i=1}^r \hat{H}_m(L_i^m).$

Lemma 2.2: Consider two disjoint n -dimensional unknots, L_1^n and L_2^n . Then there is a type of unfolding $\hat{o}: L_1^n \cup L_2^n \rightarrow \hat{L}_1^n \cup \hat{L}_2^n$, which induces $\bar{F}: \hat{H}_m(L_1^n \cup L_2^n) \rightarrow \hat{H}_m(\hat{L}_1^n \cup \hat{L}_2^n)$ such that $\bar{F}(\hat{H}_m(L_1^n \cup L_2^n)) \approx \begin{cases} Z & \text{if } m = 0, \\ Z \oplus Z & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$

Proof: Let $F: L_1^n \cup L_2^n \rightarrow \hat{L}_1^n \cup \hat{L}_2^n$ be an unfolding map such that $F(L_1^n \cup L_2^n) = F(L_1^n) \vee F(L_2^n)$, as in Fig. 1 (for $n = 1$), which induces $\bar{F}: \hat{H}_m(L_1^n \cup L_2^n) \rightarrow \hat{H}_m(\hat{L}_1^n \cup \hat{L}_2^n)$ such that $\bar{F}(\hat{H}_m(L_1^n \cup L_2^n)) = \hat{H}_m(F(L_1^n \cup L_2^n))$. Clearly for $m = 0$, $\bar{F}(\hat{H}_0(L_1^n \cup L_2^n)) = \hat{H}_0(F(L_1^n \cup L_2^n)) \approx Z$. Also, if $m = n$, $\bar{F}(\hat{H}_m(L_1^n \cup L_2^n)) = \hat{H}_m(F(L_1^n \cup L_2^n)) \approx \hat{H}_m(F(L_1^n)) \oplus \hat{H}_m(F(L_2^n))$. Since $\hat{H}_m(F(L_i^n)) \approx Z$, $i = 1, 2$, it follows that $\bar{F}(\hat{H}_m(L_1^n \cup L_2^n)) \approx Z \oplus Z$. And, if $m \neq n, 0$, we get from $\hat{H}_m(F(L_i^n)) \approx 0$, $i = 1, 2$, and from $\hat{H}_m(F(L_1 \cup L_2)) = \hat{H}_m(F(L_1^n) \oplus \hat{H}_m(F(L_2^n))$ that $\bar{F}(\hat{H}_m(L_1^n \cup L_2^n)) \approx 0$.

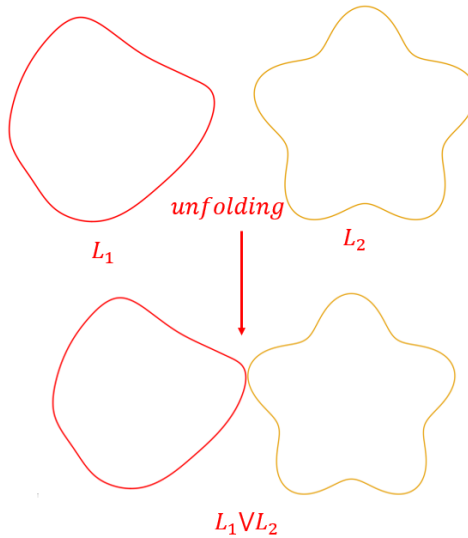


Figure 1

Theorem 2.3 Let L_i , $i = 1, 2, \dots, r$, be disjoint unknots. Then, whenever $k \leq r$, there exists a sequence of topological folding F_k of $\bigvee_{i=1}^r L_i$ into itself, which induces a sequence of topological folding F_k of $\bigoplus_{i=1}^r \hat{H}_m(L_i)$ into itself such that

- (a) $\bar{F}_k(\bigoplus_{i=1}^r \hat{H}_1(L_i)) = F_{r-k}^G.$
- (b) $\bar{F}_k(\bigoplus_{i=1}^r \hat{H}_0(L_i)) \approx Z.$
- (c) $\bar{F}_k(\bigoplus_{i=1}^r \hat{H}_m(L_i)) \approx 0, m \geq 2.$

Proof: We give the proof for point (a), points (b) and (c) are clear. Let $F_1: \bigcup_{i=1}^r L_i \rightarrow \bigcup_{i=1}^r L_i$ be a topological folding such that $F_1(\bigcup_{i=1}^r L_i) = L_1 Y L_2 Y \dots Y_1(L_t) Y \dots Y L_r$ for $t = 1, 2, \dots, r$ and $F_1(L_t) \neq L_t$ is a topological folding by cut as seen in Fig.2, then we obtain the induced $\bar{F}_1: \bigoplus_{i=1}^r \hat{H}_1(L_i) \rightarrow \bigoplus_{i=1}^r \hat{H}_1(L_i)$ such that $\bar{F}_1(\bigoplus_{i=1}^r \hat{H}_1(L_i)) = \hat{H}_1(\bigcup_{i=1}^r L_i)$ and so $\bar{F}_1(\bigoplus_{i=1}^r \hat{H}_1(L_i)) \approx \hat{H}_1(L_1) \oplus \hat{H}_1(L_2) \oplus \dots \oplus \hat{H}_1(L_t) \oplus \dots \oplus \hat{H}_1(L_r)$. Since $\hat{H}_1(\bigcup_{i=1}^r L_i) = 0$ and $\hat{H}_1(L_i) \approx Z, i = 1, 2, r, i \neq t$, we get $\bar{F}_1(\bigoplus_{i=1}^r \hat{H}_1(L_i)) = F_{r-1}^G$. Additionally, let $F_2: \bigcup_{i=1}^r L_i \rightarrow \bigcup_{i=1}^r L_i$ be a topological folding by cut such that $F_2(\bigcup_{i=1}^r L_i) = L_1 Y L_2 Y \dots Y_2(L_s) Y \dots Y_2(L_t) \dots Y L_r$ for $s = 1, 2, \dots, r, s < t$ and $F_2(L_s) \neq L_s, F_2(L_t) \neq L_t$, then we see the induced $\bar{F}_2: \bigoplus_{i=1}^r \hat{H}_1(L_i) \rightarrow \bigoplus_{i=1}^r \hat{H}_1(L_i)$ in which $\bar{F}_2(\bigoplus_{i=1}^r \hat{H}_1(L_i)) = F_{r-2}^G$. Using this method we have $F_r: \bigcup_{i=1}^r L_i \rightarrow \bigcup_{i=1}^r L_i$ in which $F_r(\bigcup_{i=1}^r L_i) = \bigcup_{i=1}^r L_i$ and $F_r(L_i) \neq L_i$ which induces $\bar{F}_r: \bigoplus_{i=1}^r \hat{H}_1(L_i) \rightarrow \bigoplus_{i=1}^r \hat{H}_1(L_i)$ for which $\bar{F}_r(\bigoplus_{i=1}^r \hat{H}_1(L_i)) = F_0^G$.



Figure 2

Theorem 2.4: Let L^n be any n -dimensional knot embedded in R^{n+2} , and $F: L^n \rightarrow M$ is an isometric folding of L^n into M , where $M \subset L^n$. Then, there is an induced folding $\bar{F}: \hat{H}_m(L^n) \rightarrow \hat{H}_m(M)$ such that $\bar{F}(\hat{H}_m(L^n))$ is isomorphic to a subgroup of $\hat{H}_m(L^n)$.

Consider the inclusion map $I: M \rightarrow L^n$, so $I \circ I = 1_M$. Hence, $\bar{F} \circ \bar{I} = 1_{\hat{H}_m(M)}$. Thus, $\bar{I}: \hat{H}_m(M) \rightarrow \hat{H}_m(L^n)$ is injective, so that $\hat{H}_m(L^n)$ is isomorphic to its image, which is a subgroup of $\hat{H}_m(M) = \bar{F}(\hat{H}_m(M))$.

Theorem 2.5: *The homology group of the end limits of topological foldings on the n -dimensional knot L^n into itself is either zero or isomorphic to Z .*

Proof: Suppose that L^n be an n -dimensional knot, which is an embedding in an $(n+2)$ -dimensional manifold M^{n+2} . For $i = 1, 2, \dots, r$, $j = 1, 2, \dots, n$, let F_i^j be a sequence of topological folding on the n -dimensional knot L^n into itself. Then,

we get the following chains: $L^n \xrightarrow{F_1^1} L_1^n \xrightarrow{F_2^1} L_2^n \rightarrow \dots \rightarrow L_{r-1}^n \xrightarrow{\lim_{r \rightarrow \infty} F_r^1} L^{n-1}$,

$$L^{n-1} \xrightarrow{F_1^2} L_1^{n-1} \xrightarrow{F_2^2} L_2^{n-1} \rightarrow \dots \rightarrow L_{r-1}^{n-1} \xrightarrow{\lim_{r \rightarrow \infty} F_r^2} L^{n-2}, \dots,$$

$$L^1 \xrightarrow{F_1^n} L_1^1 \xrightarrow{F_2^n} L_2^1 \rightarrow \dots \rightarrow L_{r-1}^1 \xrightarrow{\lim_{r \rightarrow \infty} F_r^n} L^0.$$

Thus, we conclude that the end limit of an n -dimensional knot is a 0-dimensional knot, so it is a point, which is an embedding in a 3-dimensional manifold, and the homology group of a point is a free abelian group of rank less than or equal to 1.

Theorem 2.6: *For any path J_s , $s = 1, 2, \dots, r$ there is a sequence of unfoldings $F_s: J_s \rightarrow J_{s+1}$, with variation in curvature and torsion, which induces $\bar{F}_s: \hat{H}_m(J_s) \rightarrow \hat{H}_m(J_{s+1})$ in which*

- (a) $\lim_{s \rightarrow \infty} \bar{F}_s(\hat{H}_m(J_s)) \approx Z$, $m = 0, 1$,
- (b) $\lim_{s \rightarrow \infty} (\bar{F}_s(\hat{H}_m(J_s))) \approx 0$, for $m \geq 2$.

Proof: Let us consider the unfolding sequence that involves changes in curvature and torsion for which $F_2: J_1 \rightarrow J_2, F_3: J_2 \rightarrow J_3, \dots, F_r: J_r \rightarrow J_{r+1}$ such that

$$\lim_{s \rightarrow \infty} (F_s(J_s)) = J, \text{ where } J \text{ is unknot as in Fig.3, thus we obtain the}$$

induced unfoldings $\bar{F}_s: \hat{H}_m(J_s) \rightarrow \hat{H}_m(J_{s+1}), s = 1, 2, \dots, r$ such that

$$\lim_{s \rightarrow \infty} (\bar{F}_s(\hat{H}_m(J_s))) = \hat{H}_m(\lim_{r \rightarrow \infty} (F_s(J_s))).$$

Now, for $n = 0$, $\lim_{r \rightarrow \infty} (\bar{F}_s(\hat{H}_0(J_s))) = \hat{H}_0(\lim_{s \rightarrow \infty} (F_r(J_r))) = \hat{H}_0(J) \approx Z$. Also,

if $n = 1$, it follows that $\lim_{s \rightarrow \infty} (\bar{F}_s(\hat{H}_1(J_s))) = \hat{H}_1(\lim_{s \rightarrow \infty} (F_s(J_s))) = \hat{H}_1(J) \approx Z$.

Moreover, if $n \geq 2$, then $\lim_{s \rightarrow \infty} (\bar{F}_s(\hat{H}_m(J))) = \hat{H}_m(\lim_{s \rightarrow \infty} (F_r(J))) \approx 0$. Hence, we have the induced unfoldings

$$\bar{F}_s: \hat{H}_m(J) \rightarrow \hat{H}_m(J), r = 1, 2, \dots \text{ such that}$$

- (a) $\lim_{s \rightarrow \infty} (\bar{F}_s(\hat{H}_m(J_s))) \approx Z$, $m = 0, 1$.
- (b) $\lim_{s \rightarrow \infty} (\bar{F}_s(\hat{H}_m(J_s))) \approx 0$, for $m \geq 2$.

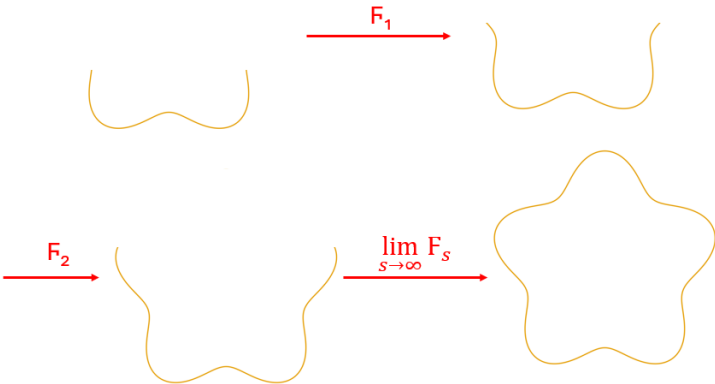


Figure 3

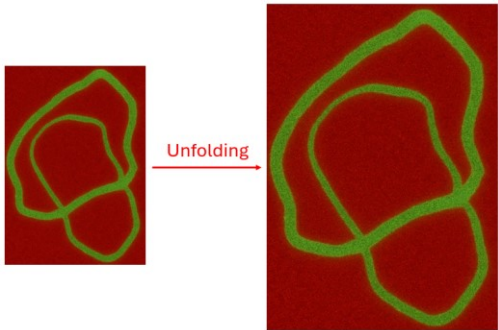


Figure 4

3. Applications

- (a) In biological activities of a *DNA* molecule, a knot, which is the case for circular *DNA*, is determined by an enzymatic mechanism. There is an unfolding of knots and enzymatic mechanisms as in Fig. 4.

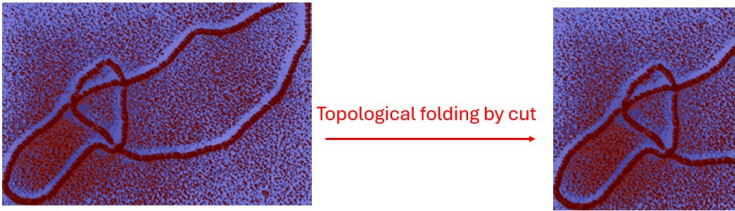


Figure 5

- (b) The topological folding by cut on knots is the same as the surgical operations as in Fig. 5. This allows a knot to be folded into another by

changing it where the strands cross. However, these mathematical operations have biochemical equivalents performed by topoisomerases. These enzymes, important for every cell, can first cut one or both strands of the circular *DNA*. They can pass a part of the ring through the cut and then close the ends to form a knot.

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