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Certain properties of the Wright function

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Abstract

In this paper, we have demonstrated several results for the Wright functions. We studied some basic properties, successive differentiation recurrence relations, various types of integral representations, integral duplication formulae, multiplication theorem, etc. Several particular cases give important results in the forms of well known Special Functions.

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1. Introduction

In 1933, Wright [1] introduced the function $W_{\xi,\tau}(z)$ for real and positive ξ and τ is any number, real or complex. It was modified in accordance to his theory of asymptotic partitions in [2]. Later, in 1940, he consider $-1 < \xi < 0$, [3]. It is given by the series

$$W_{\xi,\tau}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(\xi j+\tau)}, \tag{1.0.1}$$

where z, $\tau \in \mathbb{C}$: $\xi > -1$ and $\Gamma(\cdot)$ is the Gamma function.

For $\xi = 0$ and $\tau = 1$, the Wright function reduces to the exponential function.

The generalization of the Wright function [4] defined for $z, d_i, h_r \in \mathbb{C}$ and $D_i, H_r \in \mathbb{R}$; $d_i, h_r \neq 0$; $i = 1,2,\dots,p$; $r = 1,2,\dots,q$ is given by the series

$${}_p\Psi_q\left[\begin{matrix}(d_1,D_1) & \cdots & (d_p,D_p);\\ (h_1,H_1) & \cdots & (h_q,H_q);\end{matrix} z\right] = \sum_{j=0}^{\infty} \frac{\Gamma(d_1+D_1 j)\cdots\Gamma(d_p+D_p j)}{\Gamma(h_1+H_1 j)\cdots\Gamma(h_q+H_q j)} \frac{z^j}{j!} \tag{1.0.2}$$

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Taking $H_1 = \xi$, $h_1 = \tau$, $d_i = 1$, $D_i = 0$, $h_r = 1$, and $H_r = 0$; $i = 1,2,\dots,p$; $r = 2,3,\dots,q$ in equation (1.0.2), we obtain equation (1.0.1).

The works of Gehlot, Prajapati, and Patel [5], Luchko [6], Mehrez [7], Mehrez and Sitnik [8], and many others contain numerous results connected to the Wright function. Motivated by their efforts, we have derived several results on the Wight function. For more details, refer to Mainardi and Consiglio [9] and Gorenflo, Luchko, and Mainardi [10].

Some functions (viz. generalize k-Mittag-Leffler function, generalized k-Wright function, extended k-generalized Mittag-Leffler function) also allied with Wright function and the detailed study of these allied functions mentioned by Singh and Dubey [11], Dubey *et. al* [12], D'Costa and Menaria [13].

The following well-known definitions and results are useful to investigate several properties of the function $W_{\xi,\tau}(z)$.

- **Gamma function** [14]:

  The Gamma function is defined by

  $$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt, \tag{1.0.3}$$

  where $\Re(s) > 0$.

  The Gamma function satisfies

  $$\Gamma(s+1) = s\Gamma(s). \tag{1.0.1}$$

- **Beta function** [14]:

  The beta function is defined by

  $$B(\eta,\omega) = \int_0^1 t^{\eta-1}(1-t)^{\omega-1}dt, \tag{1.0.5}$$

  where $\Re(\eta) > 0$, $\Re(\omega) > 0$.

  A relation between the Beta function and the Gamma function is given by

  $$B(\eta,\omega) = \frac{\Gamma(\eta)\Gamma(\omega)}{\Gamma(\eta+\omega)} \tag{1.0.6}$$

- **The Mittag-Leffler function** [15, 16]:

  The Mittag-Leffler function is defined by

  $$E_{\xi,\tau}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(\xi j+\tau)}, \tag{1.0.7}$$

  where $\xi,\ \tau \in \mathbb{C}$; $\Re(\xi) > 0$.

- **The modified Bessel functions of the first and second kind [17]:**

  The first-kind modified Bessel function $I_\nu(z)$ with $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ is given by the power series

  $$I_\nu(z) = \sum_{n=0}^\infty \frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\Gamma(\nu+n+1)} \tag{1.0.8}$$

  We define the second-kind modified Bessel function $K_\nu(z)$ as,

  $$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}(I_{-\nu}(z) - I_\nu(z)) \tag{1.0.9}$$

  Furthermore, $K_\nu(z) = K_{-\nu}(z)$ for all $z \neq 0$, with Taylor series

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\left(\left(\frac{z}{2}\right)^{-\nu}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^n}{n!\Gamma(1-\nu+n)} - \left(\frac{z}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^n}{n!\Gamma(1+\nu+n)}\right). \quad (1.0.10)$$

- **Legendre's duplication formula** [14]:
  For $z \in \mathbb{C}; \Re(z) > 0$

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad (1.0.11)$$

- **Gauss's multiplication formula** [14]:
  For $z \in \mathbb{C}; \Re(z) > 0$ and $k \geq 2$,

$$\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{k}\right) = (2\pi)^{\frac{1}{2}(k-1)}k^{\frac{1}{2}-kz}\Gamma(kz) \quad (1.0.12)$$

- **Mellin transform of product of Gamma functions** [18]:
  For $\Re(z) > 0$,

$$\prod_{s=1}^{k-1}\Gamma\left(z+\frac{s}{k}\right) = \int_0^\infty u^{z-1}h_k(u)du \quad (1.0.13)$$

  where $h_k(u)$ is given by

$$h_k(u) = (\pi)^{k-2}\sum_{j=1}^{k-1}\frac{(-1)^{j-1}u^{\frac{j}{k}}}{\prod_{s=1}^{j-1}\sin\left(\frac{\pi s}{k}\right)\prod_{s=1}^{k-j-1}\sin\left(\frac{\pi s}{k}\right)}$$
$$\times\sum_{q_j=0}^{\infty}\frac{\left((-1)^{k-1}\right)^{q_j}u^{q_j}}{(q_j)!\prod_{s=1}^{j-1}\Gamma\left(1+q_j+\frac{s}{k}\right)\prod_{m=1}^{k-j-1}\Gamma\left(1+q_j-\frac{m}{k}\right)} \quad (1.0.14)$$

  In particular for $k = 3$,

$$h_3(u) = \pi\frac{u^{\frac{1}{3}}}{\sin\left(\frac{\pi}{3}\right)}\sum_{q_1=0}^{\infty}\frac{u^{q_1}}{(q_1)!\Gamma\left(1+q_1-\frac{1}{3}\right)} - \pi\frac{u^{\frac{2}{3}}}{\sin\left(\frac{\pi}{3}\right)}\sum_{q_2=0}^{\infty}\frac{u^{q_2}}{(q_2)!\Gamma\left(1+q_2+\frac{1}{3}\right)} \quad (1.0.15)$$

  The Taylor series (1.0.10) shows that

$$h_3(u) = 2u^{\frac{1}{2}}K_{\frac{1}{3}}\left(2\sqrt{u}\right) \quad (1.0.16)$$

- **Some special values of $W_{\xi,\tau}(z)$** [19]:

$$W_{-\frac{1}{2},\frac{1}{2}}(x) = \frac{e^{-\frac{x^2}{4}}}{\sqrt{\pi}} \quad (1.0.17)$$

$$W_{1,\frac{1}{2}}(x) = \frac{\cosh\left(2\sqrt{x}\right)}{\sqrt{\pi}} \quad (1.0.18)$$

$$W_{1,\frac{3}{2}}(x) = \frac{\sinh\left(2\sqrt{x}\right)}{\sqrt{\pi x}} \quad (1.0.19)$$

$$W_{1,\frac{5}{2}}(x) = \frac{2\sqrt{x}\cosh\left(2\sqrt{x}\right)-\sinh\left(2\sqrt{x}\right)}{2\sqrt{\pi}x^{\frac{3}{2}}} \quad (1.0.20)$$

$$W_{-\frac{1}{2},1}(x) = erf\left(\frac{x}{2}\right) + 1, \quad (1.0.21)$$

  where $erf(x)$ is the Error function [20] defined by,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)dt$$

- **Binomial theorem** [21]:
  For $x,\ y \in \mathbb{C}$,

$$(x+y)^n = \sum_{k=0}^{n} n_k\, x^{n-k}y^k. \tag{1.0.22}$$

- **Series manipulations** [14]:

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A(l,k) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} A(l,k-l) \tag{1.0.23}$$

and

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k} B(l,k) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} B(l,k+l) \tag{1.0.24}$$

Note: Since $W_{\xi,\tau}(z)$ is an entire function for all $z,\ \tau \in \mathbb{C};\ \xi > -1$, we can interchange the summation and integral.

2. Derivative and Differential Recurrence relation of $W_{\xi,\tau}(z)$

2.1 *Derivative*

For $z,\ \tau \in \mathbb{C},\ \xi \in \mathbb{R};\ \xi > -1$,

$$\begin{aligned}\frac{d}{dz}W_{\xi,\tau}(z) &= \sum_{j=1}^{\infty}\frac{z^{j-1}}{(j-1)!\Gamma(\xi j+\tau)}\\ &= \sum_{j=0}^{\infty}\frac{z^{j}}{j!\Gamma(\xi(j+1)+\tau)}\\ &= W_{\xi,\tau+\xi}(z)\end{aligned}$$

Again taking differentiation with respect to z

$$\begin{aligned}\frac{d^2}{dz^2}W_{\xi,\tau}(z) &= \frac{d}{dz}W_{\xi,\tau+\xi}(z)\\ &= \sum_{j=1}^{\infty}\frac{z^{j-1}}{(j-1)!\Gamma(\xi j+\tau+\xi)}\\ &= \sum_{j=0}^{\infty}\frac{z^{j}}{j!\Gamma(\xi(j+1)+\tau+\xi)}\\ &= W_{\xi,\tau+2\xi}(z)\end{aligned}$$

Repeating this process $m(\in \mathbb{N})$ times, gives

$$\frac{d^m}{dz^m}W_{\xi,\tau}(z) = W_{\xi,\tau+m\xi}(z) \tag{2.0.1}$$

2.2 *Differential recurrence relation*

Using equation (1.0.4) to equation (1.0.1), we can write

$$W_{\xi,\tau}(xz) = \sum_{j=0}^{\infty}\frac{x^j z^j}{j!(\xi j+\tau-1)\Gamma(\xi j+\tau-1)}$$

thus,

$$z^{\tau-1}W_{\xi,\tau}\left(xz^{\xi}\right) = \sum_{j=0}^{\infty}\frac{x^j z^{\xi j+\tau-1}}{j!(\xi j+\tau-1)\Gamma(\xi j+\tau-1)} \tag{2.0.2}$$

differentiation of (2.0.2) with respect to z gives

$$\frac{d}{dz}\left\{z^{\tau-1}W_{\xi,\tau}\left(xz^{\xi}\right)\right\} = \sum_{j=0}^{\infty}\frac{x^j z^{\xi j+\tau-2}}{j!\Gamma(\xi j+\tau-1)}$$

i.e.

$$\frac{d}{dz}\{z^{\tau-1}W_{\xi,\tau}(xz^{\xi})\} = z^{\tau-2}W_{\xi,\tau-1}(xz^{\xi}) \tag{2.0.3}$$

Again differentiating (2.0.3) with respect to z,

$$\begin{aligned}\frac{d^2}{dz^2}\{z^{\tau-1}W_{\xi,\tau}(xz^{\xi})\} &= \frac{d}{dz}\{z^{\tau-2}W_{\xi,\tau-1}(xz^{\xi})\}\\ &= z^{\tau-3}W_{\xi,\tau-2}(xz^{\xi})\end{aligned}$$

Repeating this process m-times yields

$$\frac{d^m}{dz^m}\{z^{\tau-1}W_{\xi,\tau}(xz^{\xi})\} = z^{\tau-m-1}W_{\xi,\tau-m}(xz^{\xi}). \tag{2.0.4}$$

3. Recurrence Relations

Couple of recurrence relations are obtained in this section.

Lemma 3.1: *For complex vales of z and τ and real $\xi > -1$, then*

$$W_{\xi,\tau}(z) = \tau W_{\xi,\tau+1}(z) + \xi z\frac{d}{dz}W_{\xi,\tau+1}(z) \tag{3.1.1}$$

Proof: From the right-hand side of the above equation, we get

$$\begin{aligned}\tau W_{\xi,\tau+1}(z) + \xi z\frac{d}{dz}W_{\xi,\tau+1}(z) &= \tau W_{\xi,\tau+1}(z) + \xi z\sum_{j=0}^{\infty}\frac{jz^{j-1}}{\Gamma(\xi j+\tau+1)j!}\\ &= \sum_{j=0}^{\infty}\frac{(\xi j+\tau)z^{j}}{\Gamma(\xi j+\tau+1)j!}\end{aligned}$$

by using (1.0.4), we get

$$W_{\xi,\tau}(z) = \tau W_{\xi,\tau+1}(z) + \xi z\frac{d}{dz}W_{\xi,\tau+1}(z)$$

Lemma 3.2: *Let z and τ be complex numbers, $\xi > -1$ and $m \in \mathbb{N}$. Then*

$$W_{\xi,\tau+m\xi}(z) = (\tau + m\xi)W_{\xi,\tau+m\xi+1}(z) + \xi z\frac{d}{dz}W_{\xi,\tau+m\xi+1}(z) \tag{3.2.1}$$

Proof: Consider,

$$\begin{aligned}(\tau + m\xi)W_{\xi,\tau+m\xi+1}(z) + \xi z\frac{d}{dz}W_{\xi,\tau+m\xi+1}(z) &= (\tau + m\xi)W_{\xi,\tau+m\xi+1}(z)\\ &\quad + \sum_{j=0}^{\infty}\frac{\xi jz^{j}}{\Gamma(\xi j+\tau+m\xi+1)j!}\\ &= \sum_{j=0}^{\infty}\frac{(\xi j+\tau+m\xi)z^{j}}{\Gamma(\xi j+\tau+m\xi+1)j!}\end{aligned}$$

this can be written as

$$W_{\xi,\tau+m\xi}(z) = (\tau + m\xi)W_{\xi,\tau+m\xi+1}(z) + \xi z\frac{d}{dz}W_{\xi,\tau+m\xi+1}(z)$$

Lemma 3.3: *For complex vales of z and τ and real $\xi > -1$ and $m \in \mathbb{N}$, we have*

$$W_{\xi,\tau+m\xi-1}(z) = (\tau + m\xi - 1)W_{\xi,\tau+m\xi}(z) + \xi zW_{\xi,\tau+m\xi+\xi}(z) \tag{3.3.1}$$

Proof: Consider,

$$\begin{aligned}(\tau+m\xi-1)W_{\xi,\tau+m\xi}(z)+\xi zW_{\xi,\tau+m\xi+\xi}(z)&=\sum_{j=0}^{\infty}\frac{(\tau+m\xi-1)z^j}{\Gamma(\xi j+\tau+m\xi)j!}+\sum_{j=0}^{\infty}\frac{\xi z^{j+1}}{\Gamma(\xi j+\xi+\tau+m\xi)j!}\\&=\frac{(\tau+m\xi-1)}{\Gamma(\tau+m\xi)}+\sum_{j=1}^{\infty}\frac{(\tau+m\xi-1)z^j}{\Gamma(\xi j+\tau+m\xi)j!}+\sum_{j=1}^{\infty}\frac{\xi jz^j}{\Gamma(\xi j+\tau+m\xi)j!}\\&=\frac{1}{\Gamma(\tau+m\xi-1)}+\sum_{j=1}^{\infty}\frac{(\xi j+\tau+m\xi-1)z^j}{\Gamma(\xi j+\tau+m\xi)j!}\\&=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\xi j+\tau+m\xi-1)j!}\end{aligned}$$

therefore,

$$(\tau+m\xi-1)W_{\xi,\tau+m\xi}(z)+\xi zW_{\xi,\tau+m\xi+\xi}(z)=W_{\xi,\tau+m\xi-1}(z)$$

Theorem 3.4: *For complex vales of* z *and* τ, $\xi>-1$, *and* $m\in\mathbb{N}$, *we have*

$$\begin{aligned}W_{\xi,\tau+m\xi+1}(z)-W_{\xi,\tau+m\xi+2}(z)&=\xi^2z^2\frac{d^2}{dz^2}W_{\xi,\tau+m\xi+3}(z)\\&+\xi(\xi+2+2(\tau+m\xi))z\frac{d}{dz}W_{\xi,\tau+m\xi+3}(z)\\&+(\tau+m\xi)(\tau+m\xi+2)W_{\xi,\tau+m\xi+3}(z)\end{aligned}\tag{3.4.1}$$

Proof: Equation (1.0.4) and equation (1.0.1) gives

$$W_{\xi,\tau+m\xi+1}(z)=\sum_{j=0}^{\infty}\frac{z^j}{(\xi(j+m)+\tau)\Gamma(\xi(j+m)+\tau)j!}\tag{3.4.2}$$

and

$$W_{\xi,\tau+m\xi+2}(z)=\sum_{j=0}^{\infty}\frac{z^j}{(\xi(j+m)+\tau+1)(\xi(j+m)+\tau)\Gamma(\xi(j+m)+\tau)j!}\tag{3.4.3}$$

equation (3.4.3) can be written as,

$$W_{\xi,\tau+m\xi+2}(z)=\sum_{j=0}^{\infty}\left\{\frac{1}{\xi(j+m)+\tau}-\frac{1}{\xi(j+m)+\tau+1}\right\}\frac{z^j}{\Gamma(\xi(j+m)+\tau)j!}\tag{3.4.4}$$

$$W_{\xi,\tau+m\xi+2}(z)=W_{\xi,\tau+m\xi+1}(z)-\sum_{j=0}^{\infty}\frac{z^j}{(\xi(j+m)+\tau+1)\Gamma(\xi(j+m)+\tau)j!}\tag{3.4.5}$$

Let $S=\sum_{j=0}^{\infty}\frac{z^j}{(\xi(j+m)+\tau+1)\Gamma(\xi(j+m)+\tau)j!}$. Then, we have

$$S=W_{\xi,\tau+m\xi+1}(z)-W_{\xi,\tau+m\xi+2}(z)\tag{3.4.6}$$

Using the basic identity

$\frac{1}{H}=\frac{1}{H(H+1)}+\frac{1}{H+1}$ $(H=\xi(j+m)+\tau+1)$ to (3.4.6), we get

$$\begin{aligned}S&=\sum_{j=0}^{\infty}\frac{z^j}{(\xi(j+m)+\tau+1)(\xi(j+m)+\tau+2)\Gamma(\xi(j+m)+\tau)j!}\\&+\sum_{j=0}^{\infty}\frac{z^j}{(\xi(j+m)+\tau+2)\Gamma(\xi(j+m)+\tau)j!}\\&=\sum_{j=0}^{\infty}\frac{(\xi(j+m)+\tau)z^j}{\Gamma(\xi(j+m)+\tau+3)j!}+\sum_{j=0}^{\infty}\frac{(\xi(j+m)+\tau)(\xi(j+m)+\tau+1)z^j}{\Gamma(\xi(j+m)+\tau+3)j!}\end{aligned}$$

$$= \xi \sum_{j=0}^{\infty} \frac{jz^j}{\Gamma(\xi(j+m)+\tau+3)j!} + (\tau+\xi m)\sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi(j+m)+\tau+3)j!} + \sum_{j=0}^{\infty} \frac{\{\xi^2(j+m)^2+(2\xi\tau+\xi)(j+m)+\tau(\tau+1)\}z^j}{\Gamma(\xi(j+m)+\tau+3)j!}$$

hence,

$$\begin{aligned} S = {} & \xi \sum_{j=0}^{\infty} \frac{jz^j}{\Gamma(\xi(j+m)+\tau+3)j!} + (\tau+\xi m)\sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi(j+m)+\tau+3)j!} \\ & + \xi^2 \sum_{j=0}^{\infty} \frac{j^2 z^j}{\Gamma(\xi(j+m)+\tau+3)j!} + 2\xi^2 m \sum_{j=0}^{\infty} \frac{jz^j}{\Gamma(\xi(j+m)+\tau+3)j!} \\ & + \xi^2 m^2 \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi(j+m)+\tau+3)j!} + (2\xi\tau+\xi)\sum_{j=0}^{\infty} \frac{jz^j}{\Gamma(\xi(j+m)+\tau+3)j!} \\ & + (2\xi\tau+\xi)m \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi(j+m)+\tau+3)j!} + \tau(\tau+1)\sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi(j+m)+\tau+3)j!} \end{aligned} \tag{3.4.7}$$

The summation on the right side of (3.4.7) is now expressed as follows:

$$\frac{d}{dz}\left\{zW_{\xi,\tau+3\xi+3}(z)\right\} = \sum_{j=0}^{\infty} \frac{(j+1)z^j}{\Gamma(\xi(j+m)+\tau+3)j!}$$

then we have

$$\sum_{j=0}^{\infty} \frac{jz^j}{\Gamma(\xi(j+m)+\tau+3)j!} = z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) \tag{3.4.8}$$

also

$$\frac{d^2}{dz^2}\left\{z^2 W_{\xi,\tau+3\xi+3}(z)\right\} = \sum_{j=0}^{\infty} \frac{(j+2)(j+1)z^j}{\Gamma(\xi(j+m)+\tau+3)j!} \tag{3.4.9}$$

from (3.4.9), we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{j^2 z^j}{\Gamma(\xi(j+m)+\tau+3)j!} = {} & z^2 \frac{d^2}{dz^2}W_{\xi,\tau+3\xi+3}(z) + 4z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) \\ & + 2W_{\xi,\tau+3\xi+3}(z) - 3\sum_{j=0}^{\infty} \frac{jz^j}{\Gamma(\xi(j+m)+\tau+3)j!} - 2\sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi(j+m)+\tau+3)j!} \end{aligned}$$

using (3.4.8) above equation yields

$$\sum_{j=0}^{\infty} \frac{j^2 z^j}{\Gamma(\xi(j+m)+\tau+3)j!} = z^2 \frac{d^2}{dz^2}W_{\xi,\tau+3\xi+3}(z) + z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) \tag{3.4.10}$$

Applying (3.4.8) and (3.4.10) to (3.4.7), we get

$$\begin{aligned} S = {} & \xi z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) + (\tau+\xi m)W_{\xi,\tau+3\xi+3}(z) \\ & + \xi^2\left\{z^2\frac{d^2}{dz^2}W_{\xi,\tau+3\xi+3}(z) + z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z)\right\} + 2\xi^2 mz\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) \\ & + \xi^2 m^2 W_{\xi,\tau+3\xi+3}(z) + (2\xi\tau+\xi)z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) \\ & + (2\xi\tau+\xi)mW_{\xi,\tau+3\xi+3}(z) + \tau(\tau+1)W_{\xi,\tau+3\xi+3}(z) \end{aligned}$$

after rearranging, the previous equation reduces to

$$\begin{aligned} S = {} & \xi^2 z^2 \frac{d^2}{dz^2}W_{\xi,\tau+3\xi+3}(z) + \xi\{\xi+2+2(\tau+\xi m)\}z\frac{d}{dz}W_{\xi,\tau+3\xi+3}(z) \\ & + (\tau+\xi m)(\tau+\xi m+2)W_{\xi,\tau+3\xi+3}(z) \end{aligned} \tag{3.4.11}$$

equation (3.4.6) gives us

$$W_{\xi,\tau+m\xi+1}(z) - W_{\xi,\tau+m\xi+2}(z) = \xi^2 z^2 \frac{d^2}{dz^2} W_{\xi,\tau+m\xi+3}(z) + \xi\{\xi + 2 + 2(\tau+\xi m)\} z \frac{d}{dz} W_{\xi,\tau+m\xi+3}(z) + (\tau+m\xi)(\tau+m\xi+2) W_{\xi,\tau+m\xi+3}(z) \quad (3.4.12)$$

Corollary 3.4.1

$$\int_0^1 t^{\tau+m\xi} W_{\xi,\tau+m\xi}(t^\xi) dt = W_{\xi,\tau+m\xi+1}(1) - W_{\xi,\tau+m\xi+2}(1), \quad (3.4.13)$$

where z *and* τ *be complex numbers,* $\xi > -1$*, and* $m \in \mathbb{N}$.

Proof: Put $z = 1$ in (3.4.6) to obtain

$$\sum_{j=0}^{\infty} \frac{1}{(\xi(j+m)+\tau+1)\Gamma(\xi(j+m)+\tau)j!} = W_{\xi,\tau+m\xi+1}(1) - W_{\xi,\tau+m\xi+2}(1) \quad (3.4.14)$$

One can easily derive that

$$\int_0^z t^{\tau+m\xi} W_{\xi,\tau+m\xi}(t^\xi) dt = \sum_{j=0}^{\infty} \frac{z^j}{(\xi(j+m)+\tau+1)\Gamma(\xi(j+m)+\tau)j!} \quad (3.4.15)$$

Comparison of (3.4.14) with the identity obtained by setting $z = 1$ in (3.4.15) yields the required eq. (3.4.13).

4. Integral Representation of $W_{\xi,\tau}(z)$

The following outcomes are derived from the aforementioned results:

Lemma 4.1: *Let* z *and* τ *be complex numbers, and* $\xi > -1$*; then*

$$\int_0^z W_{\xi,\tau}(\lambda t^\xi) t^{\tau-1} dt = z^\tau W_{\xi,\tau+1}(\lambda z^\xi) \quad (4.1.1)$$

Proof: Consider,

$$\begin{aligned}\int_0^z W_{\xi,\tau}(\lambda t^\xi) t^{\tau-1} dt &= \int_0^z \sum_{j=0}^{\infty} \frac{(\lambda t^\xi)^j}{\Gamma(\xi j+\tau)j!} t^{\tau-1} dt \\ &= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\xi j+\tau)j!} \int_0^z t^{\xi j+\tau-1} dt \\ &= \sum_{j=0}^{\infty} \frac{\lambda^j z^{\xi j+\tau}}{(\xi j+\tau)\Gamma(\xi j+\tau)j!} \\ &= z^\tau \sum_{j=0}^{\infty} \frac{\lambda^j z^{\xi j}}{\Gamma(\xi j+\tau+1)j!} \\ &= z^\tau W_{\xi,\tau+1}(\lambda z^\xi)\end{aligned}$$

Corollary 4.1.1: *For* $z,\ \tau \in \mathbb{C}$*;* $\xi > -1$ *and* $m \in \mathbb{N}$*, we have*

$$\int_0^z W_{\xi,\tau+m\xi}(\lambda t^\xi) t^{\tau+m\xi-1} dt = z^{\tau+m\xi} W_{\xi,\tau+m\xi+1}(\lambda z^\xi) \quad (4.1.2)$$

Proof: Replacing τ by $\tau + m\xi$ in Lemma 4.1, we obtain the required eq. (4.1.2).

Lemma 4.2: *If* $\tau,\ \eta \in \mathbb{C}$*;* $\Re(\eta) > 0$*,* $\xi > -1$ *then*

$$\frac{1}{\Gamma(\eta)} \int_0^z (z-t)^{\eta-1} W_{\xi,\tau}(\lambda t^\xi) t^{\tau-1} dt = z^{\tau+\eta-1} W_{\xi,\tau+\eta}(\lambda z^\xi) \quad (4.2.1)$$

Proof: Consider,

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}W_{\xi,\tau}\left(\lambda t^{\xi}\right)t^{\tau-1}dt = \frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}\sum_{j=0}^{\infty}\frac{\left(\lambda t^{\xi}\right)^j}{\Gamma(\xi j+\tau)j!}t^{\tau-1}dt$$
$$= \frac{1}{\Gamma(\eta)}\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(\xi j+\tau)j!}\int_0^z (z-t)^{\eta-1}t^{\xi j+\tau-1}dt$$

on simplification, we get

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}W_{\xi,\tau}\left(\lambda t^{\xi}\right)t^{\tau-1}dt = \frac{1}{\Gamma(\eta)}\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(\xi j+\tau)j!}B(\eta,\xi j+\tau)z^{\xi j+\tau+\eta-1}$$

using (1.0.6), we can write

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}W_{\xi,\tau}\left(\lambda t^{\xi}\right)t^{\tau-1}dt = \sum_{j=0}^{\infty}\frac{\lambda^j z^{\xi j+\tau+\eta-1}}{\Gamma(\xi j+\tau+\eta)j!}$$
$$= z^{\tau+\eta-1}\sum_{j=0}^{\infty}\frac{\lambda^j z^{\xi j}}{\Gamma(\xi j+\tau+\eta)j!}$$
$$= z^{\tau+\eta-1}W_{\xi,\tau+\eta}\left(\lambda z^{\xi}\right)$$

Note: If we take $\eta = 1$, then Lemma 4.2 reduces to Lemma 4.1.

Special cases of Lemma 4.2: For $\Re(\eta) > 0$, eq. (1.0.17) to eq. (1.0.21) are applied to eq (4.2.1) to obtain several particular cases in terms of Wright function as listed below:

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}e^{\frac{-\lambda^2}{4t}}t^{-\frac{1}{2}}dt = \sqrt{\pi}z^{\eta-\frac{1}{2}}W_{-\frac{1}{2},\eta+\frac{1}{2}}\left(\frac{\lambda}{\sqrt{z}}\right) \tag{4.2.2}$$

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\xi-1}t^{-\frac{1}{2}}\cosh\left(2\sqrt{\lambda t}\right)dt = \sqrt{\pi}z^{\eta-\frac{1}{2}}W_{1,\eta+\frac{1}{2}}(\lambda z) \tag{4.2.3}$$

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}\sinh\left(2\sqrt{\lambda t}\right)dt = \sqrt{\lambda\pi}z^{\eta+\frac{1}{2}}W_{1,\eta+\frac{3}{2}}(\lambda z) \tag{4.2.4}$$

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}t^{\frac{1}{2}}\cosh\left(2\sqrt{\lambda t}\right)dt = \frac{\sqrt{\pi}z^{\eta+\frac{1}{2}}}{2}\left\{2\lambda z W_{1,\eta+\frac{5}{2}}(\lambda z) + W_{1,\eta+\frac{3}{2}}(\lambda z)\right\} \tag{4.2.5}$$

$$\frac{1}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}erf\left(\frac{\lambda}{2\sqrt{t}}\right)dt = z^{\eta}\left\{W_{-\frac{1}{2},\eta+1}\left(\frac{\lambda}{\sqrt{z}}\right) - \frac{1}{\Gamma(\eta+1)}\right\} \tag{4.2.6}$$

Theorem 4.3: *If* $\tau,\ \gamma,\ y,\ z \in \mathbb{C};\ \Re(\tau) > 0,\ \Re(\gamma) > 0$ *and* $\xi > -1$, *then*

$$\int_0^t u^{\gamma-1}W_{\xi,\gamma}\left(yu^{\xi}\right)(t-u)^{\tau-1}W_{\xi,\tau}\left\{z(t-u)^{\xi}\right\}du = t^{\gamma+\tau-1}W_{\xi,\tau+\gamma}\left\{t^{\xi}(z+y)\right\} \tag{4.3.1}$$

Proof: The left-hand side of (4.3.1) reduces to

$$\int_0^t u^{\gamma-1}W_{\xi,\gamma}\left(yu^{\xi}\right)(t-u)^{\tau-1}W_{\xi,\tau}\left\{z(t-u)^{\xi}\right\}du$$
$$= \int_0^t u^{\gamma-1}\sum_{j=0}^{\infty}\frac{(yu^{\xi})^j}{\Gamma(\xi j+\gamma)j!}(t-u)^{\tau-1}\sum_{m=0}^{\infty}\frac{\left\{z(t-u)^{\xi}\right\}^m}{\Gamma(\xi m+\tau)m!}du$$
$$= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{y^j z^m}{\Gamma(\xi j+\gamma)\Gamma(\xi m+\tau)j!m!}\int_0^t u^{\xi j+\gamma-1}(t-u)^{\xi m+\tau-1}du$$

by applying the definition of the beta function, the previous equation becomes

$$\int_0^t u^{\gamma-1}W_{\xi,\gamma}\left(yu^{\xi}\right)(t-u)^{\tau-1}W_{\xi,\tau}\left\{z(t-u)^{\xi}\right\}du$$
$$= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{y^j z^m}{\Gamma(\xi j+\gamma)\Gamma(\xi m+\tau)j!m!}t^{\xi j+\xi m+\tau+\gamma-1}B(\xi j+\gamma,\xi m+\tau)$$

using (1.0.6), we have

$$\int_0^t u^{\gamma-1}W_{\xi,\gamma}\left(yu^{\xi}\right)(t-u)^{\tau-1}W_{\xi,\tau}\left\{z(t-u)^{\xi}\right\}du = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{y^j z^m t^{\xi j+\xi m+\tau+\gamma-1}}{\Gamma(\xi j+\xi m+\gamma+\tau)j!m!}$$
$$= t^{\gamma+\tau-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{y^j z^m t^{\xi(j+m)}}{\Gamma(\xi(j+m)+\gamma+\tau)j!m!}$$

applying (1.0.23), we get

$$\int_0^t u^{\gamma-1}W_{\xi,\gamma}\left(yu^{\xi}\right)(t-u)^{\tau-1}W_{\xi,\tau}\left\{z(t-u)^{\xi}\right\}du$$
$$= t^{\gamma+\tau-1}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{y^j z^{m-j}t^{\xi m}}{\Gamma(\xi m+\gamma+\tau)j!(m-j)!}$$
$$= t^{\gamma+\tau-1}\sum_{m=0}^{\infty}\frac{z^m t^{\xi m}}{\Gamma(\xi m+\gamma+\tau)m!}\sum_{j=0}^{m}\frac{m!\left(\frac{y}{z}\right)^j}{j!(m-j)!}$$
$$= t^{\gamma+\tau-1}\sum_{m=0}^{\infty}\frac{z^m t^{\xi m}}{\Gamma(\xi m+\gamma+\tau)m!}\sum_{j=0}^{m} m_j \left(\frac{y}{z}\right)^j$$

by using the binomial theorem (1.0.22), the above equation leads to

$$\int_0^t u^{\gamma-1}W_{\xi,\gamma}\left(yu^{\xi}\right)(t-u)^{\tau-1}W_{\xi,\tau}\left\{z(t-u)^{\xi}\right\}du$$
$$= t^{\gamma+\tau-1}\sum_{m=0}^{\infty}\frac{z^m t^{\xi m}}{\Gamma(\xi m+\gamma+\tau)m!}\left(1+\frac{y}{z}\right)^m$$
$$= t^{\gamma+\tau-1}\sum_{m=0}^{\infty}\frac{t^{\xi m}(z+y)^m}{\Gamma(\xi m+\gamma+\tau)m!}$$
$$= t^{\gamma+\tau-1}W_{\xi,\tau+\gamma}\left\{t^{\xi}(z+y)\right\}$$

Theorem 4.4: *For $\tau \in \mathbb{C}$; $\xi > -1$ and $t \neq 0$, we have*

$$\int_0^{\infty} e^{-\frac{x^2}{4t}}W_{\xi,\tau}\left(x^{\xi}\right)x^{\tau-1}dx = \sqrt{\pi}t^{\frac{\tau}{2}}W_{\frac{\xi}{2},\frac{\tau}{2}+\frac{1}{2}}\left(t^{\frac{\xi}{2}}\right) \tag{4.4.1}$$

Proof: Consider,

$$\int_0^{\infty} e^{-\frac{x^2}{4t}}W_{\xi,\tau}\left(x^{\xi}\right)x^{\tau-1}dx = \int_0^{\infty} e^{-\frac{x^2}{4t}}\sum_{j=0}^{\infty}\frac{x^{\xi j}}{\Gamma(\xi j+\tau)j!}x^{\tau-1}dx$$
$$= \sum_{j=0}^{\infty}\frac{1}{\Gamma(\xi j+\tau)j!}\int_0^{\infty} e^{-\frac{x^2}{4t}}x^{\xi j+\tau-1}dx$$

on substitution of $y = \frac{x^2}{4t}$, the last equation becomes

$$\int_0^{\infty} e^{-\frac{x^2}{4t}}W_{\xi,\tau}\left(x^{\xi}\right)x^{\tau-1}dx = \sum_{j=0}^{\infty}\frac{1}{\Gamma(\xi j+\tau)j!}\int_0^{\infty} e^{-y}(4yt)^{\frac{\xi j+\tau-1}{2}}(4yt)^{-\frac{1}{2}}(2t)dy$$
$$= \sum_{j=0}^{\infty}\frac{2t(4t)^{\frac{\xi j+\tau}{2}-1}}{\Gamma(\xi j+\tau)j!}\int_0^{\infty} e^{-y}y^{\frac{\xi j+\tau}{2}-1}dy$$

$$= \sum_{j=0}^{\infty} \frac{2^{\xi j+\tau-1} t^{\frac{\xi j+\tau}{2}}}{\Gamma(\xi j+\tau) j!} \int_0^{\infty} e^{-y} y^{\frac{\xi j+\tau}{2}-1} dy$$

using (1.0.3), the previous equation gives us

$$\int_0^{\infty} e^{-\frac{x^2}{4t}} W_{\xi,\tau}\left(x^{\xi}\right) x^{\tau-1} dx = \sum_{j=0}^{\infty} \frac{2^{\xi j+\tau-1} t^{\frac{\xi j+\tau}{2}}}{\Gamma(\xi j+\tau) j!} \Gamma\left(\frac{\xi j+\tau}{2}\right)$$

taking $z = \frac{\xi j+\tau}{2}$ into Legendre's duplication formula (1.0.11) yields

$$\Gamma\left(\frac{\xi j+\tau}{2}\right) \Gamma\left(\frac{\xi j+\tau+1}{2}\right) = \sqrt{\pi} 2^{1-\xi j-\tau} \Gamma(\xi j+\tau) \tag{4.4.2}$$

using (4.4.2), we get

$$\begin{aligned} \int_0^{\infty} e^{-\frac{x^2}{4t}} W_{\xi,\tau}\left(x^{\xi}\right) x^{\tau-1} dx &= \sum_{j=0}^{\infty} \frac{2^{\xi j+\tau-1} t^{\frac{\xi j+\tau}{2}} \sqrt{\pi} 2^{1-\xi j-\tau}}{\Gamma\left(\frac{\xi j+\tau+1}{2}\right) j!} \\ &= \sqrt{\pi} t^{\frac{\tau}{2}} \sum_{j=0}^{\infty} \frac{t^{\frac{\xi j}{2}}}{\Gamma\left(\frac{\xi j+\tau+1}{2}\right) j!} \\ &= \sqrt{\pi} t^{\frac{\tau}{2}} W_{\frac{\xi}{2}, \frac{\tau+1}{2}}\left(t^{\frac{\xi}{2}}\right) \end{aligned}$$

Theorem 4.5: *For $\eta \in \mathbb{C}$; $\Re(\eta) > 0$ and $\xi > -1$, we have*

$$\frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} W_{\xi,1}\left(\lambda t^{\xi}\right) dt = x^{\eta} W_{\xi,\eta+1}\left(\lambda x^{\xi}\right) \tag{4.5.1}$$

Proof: Consider the integral,

$$\begin{aligned} \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} W_{\xi,1}\left(\lambda t^{\xi}\right) dt &= \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} \sum_{j=0}^{\infty} \frac{\lambda^j t^{\xi j}}{\Gamma(\xi j+1) j!} dt \\ &= \frac{1}{\Gamma(\eta)} \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\xi j+1) j!} \int_0^x (x-t)^{\eta-1} t^{\xi j} dt \end{aligned}$$

on simplification, the above equation leads to

$$\begin{aligned} \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} W_{\xi,1}\left(\lambda t^{\xi}\right) dt &= \frac{1}{\Gamma(\eta)} \sum_{j=0}^{\infty} \frac{\lambda^j x^{\xi j+\eta}}{\Gamma(\xi j+1) j!} B(\eta, \xi j+1) \\ &= \sum_{j=0}^{\infty} \frac{\lambda^j x^{\xi j+\eta}}{\Gamma(\xi j+\eta+1) j!} \\ &= x^{\eta} W_{\xi,\eta+1}\left(\lambda x^{\xi}\right) \end{aligned}$$

In view of Theorem 4.5, we have

Corollary 4.5.1: *For $\xi \in \mathbb{R}$ and $\xi > 0$, we have*

$$\frac{\lambda}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} W_{\xi,1}\left(\lambda t^{\xi}\right) dt = \frac{1}{\xi} W_{\xi,0}\left(\lambda x^{\xi}\right) \tag{4.5.2}$$

Proof: Taking $\eta = \xi$ and multiplying (4.5.1) with λ, it reduces to

$$\frac{\lambda}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} W_{\xi,1}\left(\lambda t^{\xi}\right) dt = \lambda x^{\xi} W_{\xi,\xi+1}\left(\lambda x^{\xi}\right)$$

$$= \sum_{j=0}^{\infty} \frac{\lambda^{j+1} x^{\xi j+\xi}}{\Gamma(\xi j+\xi+1) j!}$$
$$= \frac{1}{\xi} \sum_{j=1}^{\infty} \frac{\lambda^{j} x^{\xi j}}{\Gamma(\xi j) j!}$$

as $\Gamma(z)$ has poles at $0, -1, -2, \ldots$ the above equation reduces to

$$\frac{\lambda}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} W_{\xi,1}(\lambda t^{\xi}) dt = \frac{1}{\xi} \sum_{j=0}^{\infty} \frac{\lambda^{j} x^{\xi j}}{\Gamma(\xi j) j!}$$
$$= \frac{1}{\xi} W_{\xi,0}(\lambda x^{\xi})$$

Theorem 4.6: *Let* $\tau \in \mathbb{C}$*;* $\xi > 0$ *and* $\Re(\tau) > \xi$*. Then*

$$W_{\xi,\tau}(-x) = \frac{1}{\xi \Gamma(\tau-\xi)} \int_0^1 \left(1 - t^{\frac{1}{\xi}}\right)^{\tau-\xi-1} W_{\xi,\xi}(-tx) dt \tag{4.6.1}$$

Proof: Consider the integral,

$$\frac{1}{\xi \Gamma(\tau-\xi)} \int_0^1 \left(1 - t^{\frac{1}{\xi}}\right)^{\tau-\xi-1} W_{\xi,\xi}(-tx) dt$$
$$= \frac{1}{\xi \Gamma(\tau-\xi)} \int_0^1 \left(1 - t^{\frac{1}{\xi}}\right)^{\tau-\xi-1} \sum_{j=0}^{\infty} \frac{(-tx)^j}{\Gamma(\xi j+\xi) j!} dt$$
$$= \frac{1}{\xi \Gamma(\tau-\xi)} \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(\xi j+\xi) j!} \int_0^1 \left(1 - t^{\frac{1}{\xi}}\right)^{\tau-\xi-1} t^j dt$$

taking substitution $t^{\frac{1}{\xi}} = y$,

$$\frac{1}{\xi \Gamma(\tau-\xi)} \int_0^1 \left(1 - t^{\frac{1}{\xi}}\right)^{\tau-\xi-1} W_{\xi,\xi}(-tx) dt$$
$$= \frac{\xi}{\xi \Gamma(\tau-\xi)} \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(\xi j+\xi) j!} \int_0^1 (1-y)^{\tau-\xi-1} y^{\xi j} y^{\xi-1} dy$$

applying (1.0.5), we have

$$\frac{1}{\xi \Gamma(\tau-\xi)} \int_0^1 \left(1 - t^{\frac{1}{\xi}}\right)^{\tau-\xi-1} W_{\xi,\xi}(-tx) dt = \frac{1}{\Gamma(\tau-\xi)} \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(\xi j+\xi) j!} B(\xi j+\xi, \tau-\xi)$$
$$= \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(\xi j+\tau) j!}$$
$$= W_{\xi,\tau}(-x)$$

5. Duplication Formula for $W_{\xi,\tau}(z)$

Two forms of Duplication formulae have been obtained in this section.

First form:

Legendre's duplication formula is given in eq. (1.0.11). When we take $z = \xi j + \tau$, it reduces to

$$\Gamma(\xi j + \tau) = \frac{\sqrt{\pi} \Gamma(2\xi j + 2\tau)}{2^{2\xi j+2\tau-1} \Gamma\left(\xi j+\tau+\frac{1}{2}\right)} \tag{5.0.1}$$

using definition of the Wright function,

$$W_{\xi,\tau}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi j+\tau)j!}$$
$$= \sum_{j=0}^{\infty} \frac{2^{2\xi j+2\tau-1}\Gamma\left(\xi j+\tau+\frac{1}{2}\right)z^j}{\sqrt{\pi}\Gamma(2\xi j+2\tau)j!}$$
$$= \frac{2^{2\tau-1}}{\sqrt{\pi}}\sum_{j=0}^{\infty} \frac{2^{2\xi j}\Gamma\left(\xi j+\tau+\frac{1}{2}\right)z^j}{\Gamma(2\xi j+2\tau)j!}$$

applying (1.0.3), leads to

$$W_{\xi,\tau}(-z) = \frac{2^{2\tau-1}}{\sqrt{\pi}}\sum_{j=0}^{\infty} \frac{2^{2\xi j}(-1)^j z^j}{\Gamma(2\xi j+2\tau)j!}\int_0^{\infty} t^{\xi j+\tau+\frac{1}{2}-1}e^{-t}dt$$
$$= \frac{2^{2\tau-1}}{\sqrt{\pi}}\int_0^{\infty} t^{\tau-\frac{1}{2}}e^{-t}\sum_{j=0}^{\infty} \frac{2^{2\xi j}(-1)^j z^j t^{\xi j}}{\Gamma(2\xi j+2\tau)j!}dt$$

For $\xi > -\frac{1}{2}$, we have

$$W_{\xi,\tau}(-z) = \frac{2^{2\tau-1}}{\sqrt{\pi}}\int_0^{\infty} t^{\tau-\frac{1}{2}}e^{-t}W_{2\xi,2\tau}\left(-z(4t)^{\xi}\right)dt \tag{5.0.2}$$

Equation (5.0.2) is the first form of duplication formulas for $W_{\xi,\tau}(z)$.

Second form: Eq. (5.0.1) can be written as

$$(\xi j+\tau-1)\Gamma(\xi j+\tau-1) = \frac{\sqrt{\pi}(2\xi j+2\tau-1)\Gamma(2\xi j+2\tau-1)}{2^{2\xi j+2\tau-1}(\xi j+\tau-\frac{1}{2})\Gamma\left(\xi j+\tau-\frac{1}{2}\right)}$$
$$= \frac{\sqrt{\pi}\Gamma(2\xi j+2\tau-1)}{2^{2\xi j+2\tau-2}\Gamma\left(\xi j+\tau-\frac{1}{2}\right)}, \tag{5.0.3}$$

by definition of the Wright function,

$$W_{\xi,\tau}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\xi j+\tau)j!}$$
$$= \sum_{j=0}^{\infty} \frac{z^j}{(\xi j+\tau-1)\Gamma(\xi j+\tau-1)j!}$$

using (5.0.3), we can write

$$W_{\xi,\tau}(z) = \sum_{j=0}^{\infty} \frac{2^{2\xi j+2\tau-2}\Gamma\left(\xi j+\tau-\frac{1}{2}\right)z^j}{\sqrt{\pi}\Gamma(2\xi j+2\tau-1)j!}$$
$$= \frac{2^{2\tau-2}}{\sqrt{\pi}}\sum_{j=0}^{\infty} \frac{2^{2\xi j}\Gamma\left(\xi j+\tau-\frac{1}{2}\right)z^j}{\Gamma(2\xi j+2\tau-1)j!}$$

therefore,

$$W_{\xi,\tau}(-z) = \frac{2^{2(\tau-1)}}{\sqrt{\pi}}\sum_{j=0}^{\infty} \frac{2^{2\xi j}(-1)^j z^j}{\Gamma(2\xi j+2\tau-1)j!}\int_0^{\infty} t^{\xi j+\tau-\frac{1}{2}-1}e^{-t}dt$$
$$= \frac{2^{2(\tau-1)}}{\sqrt{\pi}}\int_0^{\infty} t^{\tau-\frac{3}{2}}e^{-t}\sum_{j=0}^{\infty} \frac{2^{2\xi j}(-1)^j z^j t^{\xi j}}{\Gamma(2\xi j+2\tau-1)j!}dt$$

For $\xi > -\frac{1}{2}$ and $\Re(\tau) > \frac{1}{2}$, we get

$$W_{\xi,\tau}(-z) = \frac{2^{2(\tau-1)}}{\sqrt{\pi}}\int_0^{\infty} t^{\tau-\frac{3}{2}}e^{-t}W_{2\xi,2\tau-1}\left(-z(4t)^{\xi}\right)dt \tag{5.0.4}$$

Equation (5.0.4) is the second form of duplication formula for $W_{\xi,\tau}(z)$.

6. Multiplication Theorem for $W_{\xi,\tau}(z)$

In this section, authors have obtained two Multiplication theorems.

Theorem 6.1: *Let z and τ be complex numbers, $\xi > -1$; we have*

$$W_{\xi,\tau}(z) = (2\pi)^{\frac{1}{2}(1-k)} k^{k\tau-\frac{1}{2}} \int_0^\infty h_k(u) u^{\tau-1} W_{\xi k,\tau k}\left(z k^{k\xi} u^{\xi}\right) du, \qquad (6.1.1)$$

where $h_k(u)$ is given by equation (1.0.14).

Proof: Taking $z = \xi r + \tau$ in Gauss's multiplication formula (1.0.12), it reduces to

$$\prod_{s=1}^{k} \Gamma\left(\xi r + \tau + \frac{s-1}{k}\right) = (2\pi)^{\frac{1}{2}(k-1)} k^{\frac{1}{2}-k\xi r-k\tau} \Gamma(k\xi r + k\tau)$$

therefore,

$$\Gamma(\xi r + \tau) = \frac{(2\pi)^{\frac{1}{2}(k-1)} k^{\frac{1}{2}-k\xi r-k\tau} \Gamma(k\xi r+k\tau)}{\prod_{s=1}^{k-1} \Gamma\left(\xi r+\tau+\frac{s}{k}\right)}, \qquad (6.1.2)$$

by equation (1.0.1),

$$\begin{aligned} W_{\xi,\tau}(z) &= \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\xi r+\tau) r!} \\ &= \sum_{r=0}^{\infty} \frac{\prod_{s=1}^{k-1} \Gamma\left(\xi r+\tau+\frac{s}{k}\right) z^r}{(2\pi)^{\frac{1}{2}(k-1)} k^{\frac{1}{2}-k\xi r-k\tau} \Gamma(k\xi r+k\tau) r!} \\ &= (2\pi)^{\frac{1}{2}(1-k)} k^{k\tau-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{\prod_{s=1}^{k-1} \Gamma\left(\xi r+\tau+\frac{s}{k}\right) z^r k^{k\xi r}}{\Gamma(k\xi r+k\tau) r!} \end{aligned}$$

using, Mellin transform of product of the Gamma functions (1.0.13),

$$\begin{aligned} W_{\xi,\tau}(z) &= (2\pi)^{\frac{1}{2}(1-k)} k^{k\tau-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{z^r k^{k\xi r} \int_0^\infty u^{\xi r+\tau-1} h_k(u) du}{\Gamma(k\xi r+k\tau) r!} \\ &= (2\pi)^{\frac{1}{2}(1-k)} k^{k\tau-\frac{1}{2}} \int_0^\infty u^{\tau-1} h_k(u) \sum_{r=0}^{\infty} \frac{z^r k^{k\xi r} u^{\xi r}}{\Gamma(k\xi r+k\tau) r!} du \end{aligned}$$

therefore,

$$W_{\xi,\tau}(z) = (2\pi)^{\frac{1}{2}(1-k)} k^{k\tau-\frac{1}{2}} \int_0^\infty h_k(u) u^{\tau-1} W_{\xi k,\tau k}\left(z k^{k\xi} u^{\xi}\right) du$$

Equation (6.1.1) is the first Multiplication theorem for $W_{\xi,\tau}(z)$.

Theorem 6.2: *For complex-valued z and τ with $\xi > 0$, we have*

$$W_{\xi,\tau}(z) = (2\pi)^{\frac{1}{2}(1-k)} k^{k(\tau-1)+\frac{1}{2}} \int_0^\infty h_k(u) u^{\tau-2} W_{\xi k,(\tau-1)k+1}\left(z k^{k\xi} u^{\xi}\right) du, \quad (6.2.1)$$

where $h_k(u)$ is given by equation (1.0.14).

Proof: Equation (6.1.2) can be expressed as

$$(\xi r + \tau - 1)\Gamma(\xi r + \tau - 1) = \frac{(2\pi)^{\frac{1}{2}(k-1)} (\xi r+\tau-1) k^{\frac{1}{2}-k\xi r-k\tau+k} \Gamma(k\xi r+k\tau-k)}{\prod_{s=1}^{k-1} \Gamma\left(\xi r+\tau-1+\frac{s}{k}\right)}, \quad (6.2.2)$$

Now, by equation (1.0.1),

$$W_{\xi,\tau}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(\xi r+\tau)r!}$$
$$= \sum_{r=0}^{\infty} \frac{z^r}{(\xi r+\tau-1)\Gamma(\xi r+\tau-1)r!}$$

using (6.2.2), we can write

$$W_{\xi,\tau}(z) = \sum_{r=0}^{\infty} \frac{\prod_{s=1}^{k-1}\Gamma\left(\xi r+\tau-1+\frac{s}{k}\right)z^r}{(2\pi)^{\frac{1}{2}(k-1)}k^{\frac{1}{2}-k\xi r-k\tau+k}(\xi r+\tau-1)\Gamma(k\xi r+k\tau-k)r!}$$
$$= (2\pi)^{\frac{1}{2}(1-k)}k^{k(\tau-1)-\frac{1}{2}}\sum_{r=0}^{\infty} \frac{k\prod_{s=1}^{k-1}\Gamma\left(\xi r+\tau-1+\frac{s}{k}\right)z^r k^{k\xi r}}{(k\xi r+k\tau-k)\Gamma(k\xi r+k\tau-k)r!}$$
$$= (2\pi)^{\frac{1}{2}(1-k)}k^{k(\tau-1)+\frac{1}{2}}\sum_{r=0}^{\infty} \frac{\prod_{s=1}^{k-1}\Gamma\left(\xi r+\tau-1+\frac{s}{k}\right)z^r k^{k\xi r}}{\Gamma(k\xi r+k\tau-k+1)r!}$$

using the Mellin transform of product of Gamma functions,

$$W_{\xi,\tau}(z) = (2\pi)^{\frac{1}{2}(1-k)}k^{k(\tau-1)+\frac{1}{2}}\sum_{r=0}^{\infty} \frac{z^r k^{k\xi r}\int_0^{\infty} u^{\xi r+\tau-2}h_k(u)du}{\Gamma(k\xi r+k\tau-k+1)r!}$$
$$= (2\pi)^{\frac{1}{2}(1-k)}k^{k(\tau-1)+\frac{1}{2}}\int_0^{\infty} u^{\tau-2}h_k(u)\sum_{r=0}^{\infty} \frac{z^r k^{k\xi r}u^{\xi r}}{\Gamma(k\xi r+k\tau-k+1)r!}du$$

therefore,

$$W_{\xi,\tau}(z) = (2\pi)^{\frac{1}{2}(1-k)}k^{k(\tau-1)+\frac{1}{2}}\int_0^{\infty} h_k(u)u^{\tau-2}W_{\xi k,(\tau-1)k+1}\left(zk^{k\xi}u^{\xi}\right)du$$

Eq. (6.2.1) is the second multiplication theorem for $W_{\xi,\tau}(z)$.

Special Cases:

Case 1: Taking $k = 3$ in (6.1.1), it reduces to

$$W_{\xi,\tau}(z) = (2\pi)^{\frac{1}{2}(1-3)}3^{3\tau-\frac{1}{2}}\int_0^{\infty} h_3(u)u^{\tau-1}W_{3\xi,3\tau}\left(z3^{3\xi}u^{\xi}\right)du$$

using equation (1.0.16), the previous equation becomes

$$W_{\xi,\tau}(z) = \frac{3^{3\tau-\frac{1}{2}}}{\pi}\int_0^{\infty} u^{\tau-\frac{1}{2}}K_{\frac{1}{3}}\left(2\sqrt{u}\right)W_{3\xi,3\tau}\left(z3^{3\xi}u^{\xi}\right)du. \quad (6.2.3)$$

Case 2: Using (1.0.16) and putting $k = 3$ in (6.2.1), it reduces to

$$W_{\xi,\tau}(z) = \frac{3^{3\tau-\frac{5}{2}}}{\pi}\int_0^{\infty} u^{\tau-\frac{3}{2}}K_{\frac{1}{3}}\left(2\sqrt{u}\right)W_{3\xi,3\tau-2}\left(z3^{3\xi}u^{\xi}\right)du \quad (6.2.4)$$

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