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Certain conditions for integral operators associated with generalized distribution series to be in different subclasses of univalent functions

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Abstract

This paper presents a comprehensive investigation of integral operators generated by generalized distribution series. The main aim is to derive sufficient conditions for the parameters of these operators by applying coefficient estimates from a specific subclass of analytic functions. The study also derives inclusion results for these operators within various subclasses of univalent functions, such as the starlike and convex classes. Through a systematic approach, several inclusion results are obtained, some of which extend and generalize previously known findings.

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Keywords: *Convex functions, Starlike functions, Spirallike functions, Generalized distribution series, Integral operators*

1. Introduction

The study of random variable distributions is one of the prominent research areas that has attracted a lot of researchers' attention. These distributions are characterized by their probability density functions in real variables x and complex variables z. Various researchers have used different approaches to determine the parameters of integral operators whose coefficients are probability density functions. The study of generalized distribution series is a growing

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interest in geometric function theory. These series play a crucial role in the analysis of analytic functions, especially in exploring valuable insights into fundamental geometric properties such as univalence, starlikeness, and convexity. Researchers are using various methods to study different classes by integral operators, which are associated with generalized distribution series in an analytic context. For instance, Porwal [11] introduced a power series linked to a generalized distribution, which facilitated the examination of univalent function properties from a geometric perspective. To explore this idea further, the generalized distribution is defined as follows:

The probability mass function corresponding to the generalized distribution is given by

$$P(j)=\frac{c_j}{L}, j\in\mathbb{N}\cup\{0\},$$

where $c_j\geq 0$ and the infinite series $\sum_{j=0}^{\infty} c_j$ is convergent. The constant L is defined by

$$L=\sum_{j=0}^{\infty} c_j. \tag{1}$$

Let further defined a functions as follows

$$\varphi(x)=\sum_{j=0}^{\infty} c_j x^j. \tag{2}$$

From equation (1), it follows that the series in (2) converges for all $|x|\leq 1$. Consequently, the generalized distribution series can be represented as

$$\mathcal{P}_{\varphi}(z)=z+\sum_{j=2}^{\infty}\frac{c_{j-1}}{L}z^j.$$

The generalized distribution framework enhances understanding of transformation behavior, and conformal mappings, proving to be a valuable asset in the geometric analysis of complex functions (refer to [2] and reference therein).

Let $\Psi(z)$ be a complex valued function defined by the power series

$$\Psi(z)=z+\sum_{j=2}^{\infty} x_j z^j. \tag{3}$$

The collection of all such functions that are analytic in $\mathbb{D}=\{z:|z|<1\}$ is denoted by $\mathcal{A}$. The class $\mathcal{S}$ refers to the family of functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$. Let $\mathcal{S}^*$ and $\mathcal{C}$ denote the class of starlike and convex functions, respectively (see [3]).

Let $\Psi\in\mathcal{S}$ is said to be uniformly starlike (respectively, uniformly convex) in $\mathbb{D}$ if it belongs to the class $\mathcal{S}^*$ (respectively, $\mathcal{C}$) and satisfies the condition that the image of every circular arc $\gamma\in\mathbb{D}$, whose center $z_0\in\mathbb{D}$, is a starlike (respectively, convex) arc for $\Psi(z_0)$. The set of all such functions is denoted by $\mathcal{UST}$ for uniformly starlike functions, and by $\mathcal{UCV}$ for uniformly convex functions (see [4, 5] for further details).

Moreover, if a function $\Psi\in\mathcal{C}$ satisfies the above condition only for arcs whose centers $|z_0|<k$, where $0\leq k<\infty$, then Ψ is called $k-$uniformly convex in $\mathbb{D}$. This subclass is denoted by $k-\mathcal{UCV}$, as introduced in [6].

The application of the Alexander transform yields a correlated class $\mathcal{S}_p$, where Ψ belongs to $\mathcal{UCV}$ if and only if $z\Psi'$ belongs to $\mathcal{S}_p$ [7]. Likewise, within the class $k-\mathcal{ST}$, if Ψ is a member of $k-\mathcal{UCV}$, then $z\Psi'$ is a member of $k-\mathcal{ST}$, and vice versa [8].

Definition 1.1: ([9]) Consider a function $\Psi \in \mathcal{A}$. If for parameters $0 \leq \theta < 1$ and $0 \leq k < \infty$, the function Ψ satisfies the inequality

$$\Re\left(1+\frac{z\Psi''}{\Psi'}\right) \geq k\left|\frac{z\Psi''}{\Psi'}\right| + \theta,$$

then $\Psi \in k-\mathcal{UCV}(\theta)$.

Setting $\theta = 0$ in the class $k-\mathcal{UCV}(\theta)$ reduces to $k-\mathcal{UCV}$. Another class $k-\mathcal{S}_p(\theta)$, is introduced as a function $\Psi \in k-\mathcal{UCV}(\theta)$ if and only if $z\Psi' \in k-\mathcal{S}_p(\theta)$. For further discussion and properties of these classes, refer to [9].

Definition 1.2: ([10]) Let $\Psi \in \mathcal{A}$. If for $0 < r < \infty$, the following condition holds

$$\Re\left(1+\frac{z\Psi''}{\Psi'}\right) + r \geq \left|\frac{z\Psi''}{\Psi'} + 1 - r\right|,$$

then the function Ψ is said to belong to the class $\mathcal{CP}(r)$.

Definition 1.3: ([11]) Let $\Psi \in \mathcal{A}$ and $q > 0$. The subclasses $\mathcal{S}_q^*$ and $\mathcal{K}_q$ are defined by the following analytic conditions

$$\mathcal{S}_q^* = \left\{\Psi \in \mathcal{A}: \left|\frac{z\Psi'}{\Psi} - 1\right| < q\right\},$$
$$\mathcal{K}_q = \left\{\Psi \in \mathcal{A}: \left|\frac{z\Psi''}{\Psi'}\right| < q\right\}.$$

Let $\Psi \in \mathcal{S}$, is said to be spirallike if it satisfies a specific condition. Specifically, Ψ is spirallike if, for all z in the unit disk, the following inequality holds

$$\Re\left(e^{-i\eta}\frac{z\Psi'(z)}{\Psi(z)}\right) > 0,$$

where the parameter $|\eta| < \frac{\pi}{2}$.

The concept of spirallike functions, formalized through the class denoted by $\mathcal{SP}(\eta)$, was introduced by Špaček [12] as an extension of the classical starlike functions.

Definition 1.4: ([13, 14]). Consider a function $\Psi \in \mathcal{A}$ with parameters $0 \leq \omega < 1$, $0 \leq \zeta < 1$ and $|\eta| < \frac{\pi}{2}$. The subclasses of spirallike functions $\mathcal{S}(\eta, \omega, \zeta)$ and $\mathcal{K}(\eta, \omega, \zeta)$ are defined as follows

$$\mathcal{S}(\eta, \omega, \zeta) := \left\{\Psi \in \mathcal{S}: \Re\left(e^{i\eta}\frac{z\Psi'}{(1-\zeta)\Psi + \zeta z\Psi'}\right) > \omega\cos\eta\right\},$$
$$\mathcal{K}(\eta, \omega, \zeta) := \left\{\Psi \in \mathcal{S}: \Re\left(e^{i\eta}\frac{z\Psi'' + \Psi'}{\Psi' + \zeta z\Psi''}\right) > \omega\cos\eta\right\}.$$

When $\zeta = 0$, the classes reduce to the previously established classes

$$\mathcal{S}(\eta, \omega, \zeta) \equiv \mathcal{SP}(\eta, \omega) \text{ and } \mathcal{K}(\eta, \omega, \zeta) \equiv \mathcal{KP}(\eta, \omega),$$

as introduced in [15].

Definition 1.5: ([16]). Let $\Psi \in \mathcal{A}$, and consider the parameters $0 \leq \varsigma < 1$ and $\mu \leq 1$. The subclasses $\mathcal{UCSP}(\mu, \varsigma)$ and $\mathcal{SP}_p(\mu, \varsigma)$ are defined by the following conditions

$$\mathcal{UCSP}(\mu, \varsigma) := \left\{\Psi \in \mathcal{A}: \Re\left(e^{-i\mu}\left(1 + \frac{z\Psi''}{\Psi'}\right)\right) \geq \left|\frac{z\Psi''}{\Psi'}\right| + \varsigma\right\},$$
$$\mathcal{SP}_p(\mu, \varsigma) := \left\{\Psi \in \mathcal{A}: \Re\left(e^{-i\mu}\frac{z\Psi'}{\Psi}\right) \geq \left|\frac{z\Psi'}{\Psi} - 1\right| + \varsigma\right\}.$$

Note that when $\varsigma = 0$, then these subclasses reduce to

$$\mathcal{UCSP}(\mu, \varsigma) \equiv \mathcal{UCSP}(\mu) \text{ and } \mathcal{SP}_p(\mu, \varsigma) \equiv \mathcal{SP}_p(\mu),$$

for more details refer to [17].

Definition 1.6: ([18]). Let $\Psi \in \mathcal{A}$, and consider the parameters $0 \leq x < 1$ and $0 \leq \varpi < 1$. The class $\mathcal{S}(x, \varpi)$ consists of functions satisfying the condition

$$\Re\left(\frac{z\Psi' + xz^2\Psi''}{\Psi}\right) > \varpi.$$

Definition 1.7: ([9]). Let $\Psi \in \mathcal{A}$ and consider the parameters $\tau \in \mathbb{C}\backslash\{0\}$, $0 \leq v < 1$, $0 \leq u \leq 1$ and $a < 1$. Then the function $\Psi \in \mathcal{R}_{v,u}^t(a)$ if and only if

$$\left|\frac{(1-u+2v)\frac{\Psi}{z} + (u-2v)\Psi' + vz\Psi'' - 1}{2t(1-a) + (1-u+2v)\frac{\Psi}{z} + (u-2v)\Psi' + vz\Psi'' - 1}\right| < 1.$$

For a function $\Psi \in \mathcal{R}_{v,u}^t(a)$, the coefficients x_j in its series expansion satisfy the estimate

$$|x_j| \leq \frac{2|t|(1-a)}{1 + (u - 2v + jv)(j-1)}, j = 2,3, \cdots.$$

To simplify the denominator, observe that (see [9])

$$1 + (u - 2v + jv)(j - 1) \geq j(u - 3v)$$

Using this bound, the coefficient inequality becomes

$$|x_j| \leq \frac{2|t|(1-a)}{j(u-3v)}, j = 2,3, \cdots. \tag{4}$$

In the field of geometric function theory, special functions such as the hypergeometric function [19, 20, 21], confluent hypergeometric functions [22, 23 24], Wright's function [25], and generalized Bessel functions [26, 27, 24] have been widely utilized in the analysis of various subclasses of univalent functions. Several researchers have focused on deriving analytic properties and coefficient estimates for operators associated with different classes [28, 29]. These works collectively demonstrate the ongoing interest in extending the classes and motivate the present study.

Consider the integral operator $\mathcal{P}_\varphi(\Psi, z)$, proposed by the others in [2], which is defined for functions $\Psi \in \mathcal{A}$, as $\mathcal{P}_\varphi(\Psi, z) = \mathcal{P}_\varphi(z) * \Psi(z)$, where $*$ represent the Hadamard product.

$$\mathcal{P}_{\varphi}(\Psi, z) = z + \sum_{j=2}^{\infty} \frac{c_{j-1}x_j}{L} z^j,$$

the coefficients X_j are defined as

$$X_j = \frac{c_{j-1}}{L} x_j. \tag{5}$$

The Hadamard product of two functions is given as

$$(\Psi * \psi)(z) = z + \sum_{j=2}^{\infty} x_j y_j z^j, \tag{6}$$

where $\Psi(z), \psi(z) \in \mathcal{A}$.

The function $\varphi(x)$, defined in equation (2) satisfies the following properties:

$$\varphi'(x) = \sum_{j=0}^{\infty} (j+1)c_{j+1}x^j, \tag{7}$$

$$\Phi(x) = \int_0^x \varphi(t)dt = \sum_{j=0}^{\infty} \frac{c_j}{j+1} x^{j+1}. \tag{8}$$

In [30], Porwal introduced a series based on the Poisson distribution and demonstrated its application in the study of univalent functions, which opened up new avenues in geometric function theory. Following this, several researchers explored Gaussian hyper-geometric distribution series (refer to [31]), confluent hyper-geometric distribution series (refer to [32]), Poisson distribution series (refer to [2, 33]), and Pascal distribution series (refer to [34, 35]). Inspired by the findings in [1, 36], the present work derives sufficient conditions under which the operators $\mathcal{P}_{\varphi}(\Psi, z)$ and $\mathcal{T}_{\varphi}(\Psi, z)$ belong to the function classes $\mathcal{S}(\eta, \omega, \zeta)$, $\mathcal{K}(\eta, \omega, \zeta)$, $\mathcal{SP}_p(\mu, \varsigma)$, $\mathcal{UCSP}(\mu, \varsigma)$, $k - \mathcal{UCV}(\sigma)$, $k - \mathcal{S}_p(\theta)$, $\mathcal{CP}(r)$, $\mathcal{S}_q^*$, $\mathcal{K}_q$ and $\mathcal{S}(x, \varpi)$.

This study explores the geometric characteristics of integral operators associated with generalized distribution series in the framework of $\mathcal{R}_{v,u}^t(a)$. Our focus on to derive sufficient conditions under which the operators $\mathcal{P}_{\varphi}(\Psi, z)$ and $\mathcal{T}_{\varphi}(\Psi, z)$ are applicable, assuming Ψ belongs to $\mathcal{R}_{v,u}^t(a)$. These results are thoroughly discussed in Section 3, where we also highlight some well-known special cases. To support our main results, Section 2 recalls essential results from previous research, and Section 4 provides final observations.

2. Preliminary results

The following preliminary results will be instrumental in proving our main theorems.

Lemma 2.1: ([13]). *Suppose $\Psi \in \mathcal{A}$ is given as in equation (3). Then Ψ belongs to the class $\mathcal{S}(\eta, \omega, \zeta)$ if and only if the following condition is satisfied*

$$\sum_{j=2}^{\infty} [((1-\zeta)\sec\eta + \zeta(1-\omega))j + (1-\zeta)(1-\omega-\sec\eta)]|x_j| \leq 1-\omega, \tag{9}$$

where the parameters satisfy $|\eta| < \frac{\pi}{2}$, $0 \leq \omega < 1$, *and* $0 \leq \zeta < 1$.

Lemma 2.2 ([14]). Let $\Psi \in \mathcal{A}$ be defined as in equation (3). Then Ψ belongs to the class $\mathcal{K}(\eta, \omega, \zeta)$ if and only if

$$\sum_{j=2}^{\infty} j\big[\big((1-\zeta)\sec\eta+\zeta(1-\omega)\big)j+(1-\zeta)(1-\omega-\sec\eta)\big]\big|x_j\big| \leq 1-\omega, \tag{10}$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\omega<1$ *and* $0\leq\zeta<1$.

Lemma 2.3: ([16]). *Let* $\Psi\in\mathcal{A}$. *If it satisfies the inequality*

$$\sum_{j=2}^{\infty}(2j-\cos\mu-\varsigma)|x_j|\leq\cos\mu-\varsigma, \tag{11}$$

then Ψ *belongs to the class* $\mathcal{SP}_p(\mu,\varsigma)$, *provided that* $\mu\leq 1$ *and* $0\leq\varsigma<1$.

Lemma 2.4: ([16]). *Let* $\Psi\in\mathcal{A}$. *If it satisfies the inequality*

$$\sum_{j=2}^{\infty} j(2j-\cos\mu-\varsigma)|x_j|\leq\cos\mu-\varsigma, \tag{12}$$

then Ψ *belongs to the class* $\mathcal{UCSP}(\mu,\varsigma)$ *for* $\mu\leq 1$ *and* $0\leq\varsigma<1$.

Lemma 2.5: ([9]). *Let* $\Psi\in\mathcal{A}$ *and if it satisfies*

$$\sum_{j=2}^{\infty}[j(1+k)-(k+\theta)]|x_j|\leq 1-\theta, \tag{13}$$

then $\Psi\in k-\mathcal{S}_p(\theta)$, *provided that* $0\leq\sigma<1$ *and* $0\leq k<\infty$.

Lemma 2.6: ([9]). *Let* $\Psi\in\mathcal{A}$, *and if it satisfies*

$$\sum_{j=2}^{\infty} j[j(1+k)-(k+\theta)]|x_j|\leq 1-\theta, \tag{14}$$

then Ψ *belongs to the class* $k-\mathcal{UCV}(\theta)$ *for* $0\leq\theta<1$ *and* $0\leq k<\infty$.

Lemma 2.7: ([10]). *Let* $\Psi\in\mathcal{A}$. *If it satisfies the inequality*

$$\sum_{j=2}^{\infty}[j+2(r-1)]j|x_j|\leq 2r-1, \tag{15}$$

then Ψ *belongs to the class* $\mathcal{CP}(r)$ *for* $0<r<\infty$.

Lemma 2.8: ([11]). *Let* $\Psi\in\mathcal{A}$, *and if it satisfies*

$$\sum_{j=2}^{\infty}(q+j-1)|x_j|\leq q, \tag{16}$$

then Ψ *belongs to the class* $\mathcal{S}_q^*$ *for* $q>0$.

Lemma 2.9: ([11]). *Let* $\Psi\in\mathcal{A}$, *and if it satisfies*

$$\sum_{j=2}^{\infty} j(q+j-1)|x_j|\leq q, \tag{17}$$

then Ψ *belongs to the class* $\mathcal{K}_q$ *for* $q>0$.

Lemma 2.10: ([18]). *Let* $\Psi\in\mathcal{A}$. *If it satisfies the inequality*

$$\sum_{j=2}^{\infty}(j+xj(j-1)-\varpi)|x_j|\leq 1-\varpi, \tag{18}$$

then Ψ *belongs to the class* $\mathcal{S}(x,\varpi)$ *for* $0\leq x<1$ *and* $0\leq\varpi<1$.

3. Main results

In this section, we present sufficient criteria ensuring that the integral operators belong to certain subclasses of univalent functions

Theorem 3.1: *Let* $t \in \mathbb{C}\backslash\{0\}$, *with parameters* $0 \leq v < 1$, $0 \leq u \leq 1$ *and* $a < 1$. *Suppose* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and the following condition satisfied*

$$[(1-\zeta)\sec\eta + \zeta(1-\omega)]\varphi(1) + (1-\zeta)(1-\omega-\sec\eta)\Phi(1) \leq (1-\omega)\left(c_0 + \frac{L(u-3v)}{2|t|(1-a)}\right),$$

then for $|\eta| < \frac{\pi}{2}$, $0 \leq \omega < 1$ *and* $0 \leq \zeta < 1$, $\mathcal{P}_{\varphi}(\Psi, z) \in \mathcal{S}(\eta, \omega, \zeta)$.

Proof: Substituting expression (5) in Lemma 2.1, we obtain

$$\sum_{j=2}^{\infty}\left[((1-\zeta)\sec\eta + \zeta(1-\omega))j + (1-\zeta)(1-\omega-\sec\eta)\right]\frac{c_{j-1}}{L}|x_j| \leq 1-\omega. \quad (19)$$

Now, we apply the bound from (4) in (19), we get

$$\left(((1-\zeta)\sec\eta + \zeta(1-\omega))j\sum_{j=2}^{\infty}\frac{c_{j-1}}{L} + (1-\zeta)(1-\omega-\sec\eta)\sum_{j=2}^{\infty}\frac{c_{j-1}}{L}\right)\frac{2|t|(1-a)}{j(u-3v)} \leq 1-\omega. \quad (20)$$

Using (2) and (8) in the respective sums of (20), we get

$$((1-\zeta)\sec\eta + \zeta(1-\omega))(\varphi(1)-c_0) + (1-\zeta)(1-\omega-\sec\eta)(\Phi(1)-c_0) \leq (1-\omega)\frac{L(u-3v)}{2|t|(1-a)}. \quad (21)$$

Thus, the result is established using (21).

Theorem 3.2: *Let* $t \in \mathbb{C}\backslash\{0\}$, *with parameters* $0 \leq v < 1$, $0 \leq u \leq 1$ *and* $a < 1$. *Assume that* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and the following condition holds*

$$(1-\zeta)(1-\omega-\sec\eta)\varphi(1) + [(1-\zeta)\sec\eta + \zeta(1-\omega)]\varphi'(1) - (1-\zeta)(1-\omega-\sec\eta)c_0 \leq L\frac{(u-3v)}{2|t|(1-a)}(1-\omega).$$

Then for $|\eta| < \frac{\pi}{2}$, $0 \leq \omega < 1$ *and* $0 \leq \zeta < 1$, $\mathcal{P}_{\varphi}(\Psi, z) \in \mathcal{K}(\eta, \omega, \zeta)$.

Proof: Substituting expression (5) and (4) in Lemma 2.3, we get

$$((1-\zeta)\sec\eta + \zeta(1-\omega))\sum_{j=2}^{\infty} j^2\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} + (1-\zeta)(1-\omega-\sec\eta)\sum_{j=2}^{\infty} j\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} \leq 1-\omega. \quad (22)$$

Next, apply (7) and (2) in (22), we get

$$((1-\zeta)\sec\eta + \zeta(1-\omega))\varphi'(1) + (1-\zeta)(1-\omega-\sec\eta)(\varphi(1)-c_0) \leq (1-\omega)\frac{L(u-3v)}{2|t|(1-a)}. \quad (23)$$

Conclude the proof by applying the hypothesis of the theorem and equation (23).

Remark 3.1: It is observed that if $v = 0$ in Theorem 3.2, it reduces to Theorem 2.3 presented in [36].

Corollary 3.3: *Let $t \in \mathbb{C}\backslash\{0\}$, with parameters $0 \le v < 1$, $0 \le u \le 1$ and $a < 1$. Assume that $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ and*

$$(1-\omega-\sec\eta)\Phi(1)+\sec\eta\varphi(1)\le(1-\omega)\left(c_0+\frac{L(u-3v)}{2|t|(1-a)}\right),$$

then for $|\eta| < \frac{\pi}{2}$ and $0 \le \omega < 1$, the operator $\mathcal{P}_{\varphi}(\Psi, z)$ belongs to the class $\mathcal{SP}(\eta, \omega)$.

Corollary 3.4: *Let $t \in \mathbb{C}\backslash\{0\}$, with parameters $0 \le v < 1$, $0 \le u \le 1$ and $a < 1$. Assume that $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ and*

$$(1-\omega-\sec\eta)\varphi(1)+\sec\eta\varphi'(1)\le(1-\omega)\frac{L(u-3v)}{2|t|(1-a)}+(1-\omega-\sec\eta)c_0,$$

then for $|\eta| < \frac{\pi}{2}$ and $0 \le \omega < 1$, the operator $\mathcal{P}_{\varphi}(\Psi, z)$ belongs to the class $\mathcal{KP}(\eta, \omega)$.

Theorem 3.5: *Let $t \in \mathbb{C}\backslash\{0\}$, with $0 \le v < 1$, $0 \le u \le 1$ and $a < 1$. Assume that $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ and the following condition holds*

$$2\varphi(1)-(\cos\mu+\varsigma)\Phi(1)+(\cos\mu+\varsigma-2)c_0\le(\cos\mu-\varsigma)\frac{L(u-3v)}{2|t|(1-a)}.$$

Then for $\mu \le 1$ and $0 \le \varsigma < 1$, $\mathcal{P}_{\varphi}(\Psi, z) \in \mathcal{SP}_p(\mu, \varsigma)$.

Proof: Substituting expression (5) in Lemma 2.3, we obtain

$$\sum_{j=2}^{\infty}(2j-\cos\mu-\varsigma)\frac{c_{j-1}}{L}|x_j|\le\cos\mu-\varsigma. \tag{24}$$

Now, we apply the bound from (4), we have

$$\begin{aligned}&\sum_{j=2}^{\infty}(2j-\cos\mu-\varsigma)\frac{c_{j-1}}{L}|x_j|\\&\le\left(\sum_{j=2}^{\infty}2j\frac{c_{j-1}}{L}-(\cos\mu+\varsigma)\sum_{j=2}^{\infty}\frac{c_{j-1}}{L}\right)\frac{2|t|(1-a)}{j(u-3v)}\\&=\frac{2|t|(1-a)}{L(u-3v)}[2(\varphi(1)-c_0)-(\cos\mu+\varsigma)(\Phi(1)-c_0)].\end{aligned} \tag{25}$$

Substituting the results (25) back into (24), we get

$$2(\varphi(1)-c_0)-(\cos\mu+\varsigma)(\Phi(1)-c_0)\le(\cos\mu-\varsigma)\frac{L(u-3v)}{2|t|(1-a)}. \tag{26}$$

Conclude the proof by applying the hypothesis of the theorem and equation (26).

Corollary 3.6: *Let $t \in \mathbb{C}\backslash\{0\}$, with $0 \le v < 1$, $0 \le u \le 1$ and $a < 1$. Suppose $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ and*

$$2\varphi(1)-\Phi(1)\cos\mu+(\cos\mu-2)c_0\le\cos\mu\frac{L(u-3v)}{2|t|(1-a)},$$

then for $\mu \le 1$, $\mathcal{P}_{\varphi}(\Psi, z) \in \mathcal{SP}_p(\mu)$.

Theorem 3.7: *Let $t \in \mathbb{C}\backslash\{0\}$, with $0 \leq v < 1$, $0 \leq u \leq 1$ and $a < 1$. Assume that $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ and*

$$(2-\cos\mu-\varsigma)\varphi'(1)+2\varphi(1)-(2-\cos\mu-\varsigma)c_0 \leq L\frac{(u-3v)}{2|t|(1-a)}(\cos\mu-\varsigma),$$

then for $\mu \leq 1$ and $0 \leq \varsigma < 1$, $\mathcal{P}_{\varphi}(\Psi, z)$ belongs to the class $\mathcal{UCSP}(\mu, \varsigma)$.

Proof: By substituting (5) in Lemma 2.4, we obtain

$$\sum_{j=2}^{\infty} j(2j-\cos\mu-\varsigma)\frac{c_{j-1}}{L}|x_j| \leq \cos\mu-\varsigma. \tag{27}$$

Now, we apply the bound from (4), we have

$$\begin{aligned}
&\sum_{j=2}^{\infty} j(2j-\cos\mu-\varsigma)\frac{c_{j-1}}{L}|x_j| \\
&\leq \sum_{j=2}^{\infty} j(2j-\cos\mu-\varsigma)\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} \\
&= \frac{2|t|(1-a)}{L(u-3v)}\left(2\sum_{j=0}^{\infty}(j+1)c_{j+1}+(2-\cos\mu-\varsigma)\sum_{j=1}^{\infty}c_j\right).
\end{aligned} \tag{28}$$

Substituting the results (28) back into (27), we get

$$2\varphi(1)+(2-\cos\mu-\varsigma)(\varphi'(1)-c_0) \leq (\cos\mu-\varsigma)\frac{L(u-3v)}{2|t|(1-a)}. \tag{29}$$

Thus, the result is established using (29).

Theorem 3.8: *Let $t \in \mathbb{C}\backslash\{0\}$, with parameters $0 \leq v < 1$, $0 \leq u \leq 1$ and $a < 1$. Suppose $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ and*

$$(k+\theta)\Phi(1)-(1+k)\varphi(1) \leq \left(c_0+\frac{L(u-3v)}{2|t|(1-a)}\right)(\theta-1),$$

then for $0 \leq \theta < 1$ and $0 \leq k < \infty$, $\mathcal{P}_{\varphi}(\Psi, z)$ belongs to the class $k-\mathcal{S}_p(\theta)$.

Proof: Substituting (5) in Lemma 2.5, we obtain

$$\sum_{j=2}^{\infty}(j(1+k)-(k+\theta))\frac{c_{j-1}}{L}|x_j| \leq 1-\theta. \tag{30}$$

Now, we apply the bound from (4), we have

$$\begin{aligned}
&\sum_{j=2}^{\infty}(j(1+k)-(k+\theta))\frac{c_{j-1}}{L}|x_j| \\
&\quad \leq \frac{2|t|(1-a)}{L(u-3v)}\left(\sum_{j=2}^{\infty}(1+k)c_{j-1}-\sum_{j=2}^{\infty}(k+\theta)\frac{c_{j-1}}{j}\right).
\end{aligned} \tag{31}$$

Applying (31) in (30), we get

$$(1+k)\varphi(1)-(1+k)c_0-(k+\theta)\Phi(1)+(k+\theta)c_0 \leq (1-\theta)\frac{L(u-3v)}{2|t|(1-a)}. \tag{32}$$

Conclude the proof by applying the hypothesis of the theorem and equation (32).

Theorem 3.9: *Let $t \in \mathbb{C}\backslash\{0\}$, with parameters $0 \leq v < 1$, $0 \leq u \leq 1$ and $a < 1$. Assume that $\Psi \in \mathcal{R}_{v,u}^{t}(a)$, and the following condition holds*

$$(1+k)\varphi'(1)+(1-\theta)\varphi(1)\leq(1-\theta)\left(c_0+L\frac{(u-3v)}{2|t|(1-a)}\right).$$

Then for $0\leq\theta<1$ *and* $0\leq k<\infty$, $\mathcal{P}_\varphi(\Psi,z)\in k-\mathcal{UCV}(\theta)$.

Proof: Substituting inequality (5) and (4) in (14), we obtain

$$\sum_{j=2}^{\infty} j^2(1+k)\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)}-\sum_{j=2}^{\infty} j(k+\theta)\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)}\leq 1-\theta. \tag{33}$$

Next apply (7) in (33), we get

$$(1+k)\varphi'(1)+(1-\theta)(\varphi(1)-c_0)\leq(1-\theta)\frac{L(u-3v)}{2|t|(1-a)}. \tag{34}$$

Finally, applying the theorem's hypothesis with inequality (34) completes the proof.

Corollary 3.10: Let $t\in\mathbb{C}\backslash\{0\}$, with $0\leq v<1$, $0\leq u\leq 1$ and $a<1$. Assume that $\Psi\in\mathcal{R}_{v,u}^t(a)$ and inequality below holds

$$(1+k)\varphi(1)-k\Phi(1)-c_0\leq\frac{L(u-3v)}{2|t|(1-a)},$$

then for $0\leq k<\infty$, $\mathcal{P}_\varphi(\Psi,z)\in k-\mathcal{ST}$.

Corollary 3.11: *Let* $t\in\mathbb{C}\backslash\{0\}$, *with* $0\leq v<1$, $0\leq u\leq 1$ *and* $a<1$. *Assume that* $\Psi\in\mathcal{R}_{v,u}^t(a)$ *and the inequality below holds*

$$(1+k)\varphi'(1)+\varphi(1)-c_0\leq\frac{L(u-3v)}{2|t|(1-a)},$$

then for $0\leq k<\infty$, $\mathcal{P}_\varphi(\Psi,z)\in k-\mathcal{UCV}$.

Theorem 3.12: *Let* $t\in\mathbb{C}\backslash\{0\}$, *with* $0\leq v<1$, $0\leq u\leq 1$ *and* $a<1$. *Assume that* $\Psi\in\mathcal{R}_{v,u}^t(a)$ *and the inequality below holds*

$$(2r-1)\varphi(1)+\varphi'(1)\leq\left(c_0+\frac{L(u-3v)}{2|t|(1-a)}\right)(2r-1),$$

then for $0<r\leq\infty$, *the operator* $\mathcal{P}_\varphi(\Psi,z)$ *belongs to the class* $\mathcal{CP}(r)$.

Proof: By substituting (5) and (4) in (15), we get

$$\left(\sum_{j=2}^{\infty}(j-1)j\frac{c_{j-1}}{L}+(2r-1)\sum_{j=2}^{\infty} j\frac{c_{j-1}}{L}\right)\frac{2|\tau|(1-\gamma)}{j(u-3v)}\leq(2r-1). \tag{35}$$

Simplifying (35), we have

$$\sum_{j=0}^{\infty}(j+1)c_{j+1}+(2r-1)\sum_{j=1}^{\infty} c_j\leq L\frac{(u-3v)}{2|\tau|(1-\gamma)}(2r-1). \tag{36}$$

By Applying (7) and (2) in (36), the desired result is obtained.

Theorem 3.13: *Let* $t\in\mathbb{C}\backslash\{0\}$, *with* $0\leq v<1$, $0\leq u\leq 1$ *and* $a<1$. *Assume that* $\Psi\in\mathcal{R}_{v,u}^t(a)$ *and the inequality below holds*

$$\varphi(1)+(q-1)\Phi(1)-qc_0\leq qL\frac{(u-3v)}{2|t|(1-a)},$$

then for $q>0$, $\mathcal{P}_\varphi(\Psi,z)\in\mathcal{S}_q^*$.

Proof: By substituting (5) in Lemma 2.8, we obtain

$$\sum_{j=2}^{\infty} (q+j-1)\frac{c_{j-1}}{L}|x_j| \leq q. \tag{37}$$

By applying (4), we get

$$\sum_{j=2}^{\infty} j\frac{c_{j-1}}{j(u-3v)} + (q-1)\sum_{j=2}^{\infty} \frac{c_{j-1}}{j(u-3v)} \leq q\frac{L}{2|t|(1-a)}. \tag{38}$$

Using (38) back into (37), we get

$$(\varphi(1)-c_0) + (q-1)(\Phi(1)-c_0) \leq qL\frac{(u-3v)}{2|t|(1-a)}. \tag{39}$$

Conclude the proof by applying the hypothesis of the theorem and equation (39).

Theorem 3.14: *Let* $t \in \mathbb{C}\backslash\{0\}$*, with* $0 \leq v < 1,\ 0 \leq u \leq 1$ *and* $a < 1$. *Assume that* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and the below inequality holds*

$$q\varphi(1) + \varphi'(1) - qc_0 \leq qL\frac{(u-3v)}{2|t|(1-a)},$$

then for $q > 0,\ \mathcal{P}_{\varphi}(\Psi, z) \in \mathcal{K}_q$.

Proof: By substituting (5) in the sufficient condition for the class $\mathcal{K}_q$ provided in (17), we get

$$\sum_{j=2}^{\infty} j(q+j-1)\frac{c_{j-1}}{L}|x_j| \leq q. \tag{40}$$

By applying (4), we get

$$\sum_{j=2}^{\infty} jq\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} + \sum_{j=2}^{\infty} j(j-1)\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} \leq q. \tag{41}$$

Using (7) in (41), we get

$$q(\varphi(1)-c_0) + \varphi'(1) \leq qL\frac{(u-3v)}{2|t|(1-a)}. \tag{42}$$

Conclude the proof by applying the hypothesis of the theorem and equation (42).

Theorem 3.15: *Let* $t \in \mathbb{C}\backslash\{0\}$*, with* $0 \leq v < 1,\ 0 \leq u \leq 1$ *and* $a < 1$. *Assume that* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and the below inequality holds*

$$\varphi(1) - c_0 + \varkappa\varphi'(1) - \varpi\Phi(1) \leq (1-\varpi)L\frac{(u-3v)}{2|t|(1-a)},$$

then for $\varkappa > 0$ *and* $0 \leq \varpi < 1$*, the operator* $\mathcal{P}_{\varphi}(\Psi, z)$ *belongs to the class* $\mathcal{S}(\varkappa, \varpi)$.

Proof: Substituting (5) in (18), we obtain

$$\sum_{j=2}^{\infty} (j + \varkappa j(j-1) - \varpi)\frac{c_{j-1}}{L}|x_j| \leq 1 - \varpi. \tag{43}$$

Applying (4) in (43), we get

$$\sum_{j=2}^{\infty} (j + \varkappa j(j-1) - \varpi)\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} \leq 1 - \varpi. \tag{44}$$

Simplifying (44), we have

$$\sum_{j=2}^{\infty} \frac{c_{j-1}}{(u-3v)} + \sum_{j=2}^{\infty} \varkappa(j-1)\frac{c_{j-1}}{(u-3v)} - \sum_{j=2}^{\infty} \varpi \frac{c_{j-1}}{j(u-3v)} \leq (1-\varpi)\frac{L}{2|t|(1-a)}. \quad (45)$$

Using (7) and (8) in (45), we get

$$\varphi(1) - c_0 + \varkappa\varphi'(1) - \varpi\Phi(1) \leq (1-\varpi)L\frac{(u-3v)}{2|t|(1-a)}. \quad (46)$$

Thus, by applying the given assumptions along with (46), the proof is complete.

Let us consider a linear operator $\mathcal{T}_\varphi(z)$, which is given by

$$\mathcal{T}_\varphi(z) = \int_0^z \frac{\mathcal{P}_\varphi(t)}{t}dt.$$

Now, consider $\Psi \in \mathcal{A}$. In that case, the transformed function under $\mathcal{T}_\varphi$ becomes

$$\mathcal{T}_\varphi(\Psi, z) = z + \sum_{j=2}^{\infty} \frac{c_{j-1}}{jL} x_j z^j = z + \sum_{j=2}^{\infty} Y_j z^j,$$

where each coefficient Y_j is defined as

$$Y_j = \frac{c_{j-1}}{jL} x_j. \quad (47)$$

Theorem 3.16: *Let* $t \in \mathbb{C}\backslash\{0\}$, $0 \leq v < 1$, $0 \leq u \leq 1$ *and* $a < 1$. *Suppose* $\Psi \in \mathcal{R}_{v,u}^t(a)$ *and the following condition satisfied*

$$[(1-\zeta)\sec\eta + \zeta(1-\omega)]\varphi(1) + (1-\zeta)(1-\omega-\sec\eta)\Phi(1) \leq (1-\omega)\left(c_0 + \frac{L(u-3v)}{2|t|(1-a)}\right),$$

then for $|\eta| < \frac{\pi}{2}$, $0 \leq \omega < 1$ *and* $0 \leq \zeta < 1$, $\mathcal{T}_\varphi(\Psi, z) \in \mathcal{K}(\eta, \omega, \zeta)$.

Proof: Substituting (47) and (4) in (10), we obtain

$$\left(\sum_{j=2}^{\infty} j[((1-\zeta)\sec\eta + \zeta(1-\omega))j]\frac{c_{j-1}}{jL} + \sum_{j=2}^{\infty} j[(1-\zeta)(1-\omega-\sec\eta)]\frac{c_{j-1}}{jL}\right)\frac{2|t|(1-a)}{j(u-3v)} \leq 1-\omega. \quad (48)$$

Simplifying (48), we get

$$((1-\zeta)\sec\eta + \zeta(1-\omega))\sum_{j=2}^{\infty} c_{j-1} + (1-\zeta)(1-\omega-\sec\eta)\sum_{j=2}^{\infty}\frac{c_{j-1}}{j} \leq L\frac{(u-3v)}{2|t|(1-a)}(1-\omega). \quad (49)$$

Next, applying the results (2) and (8) in the first and second sum of (49) respectively, and substituting into (48), we obtain

$$((1-\zeta)\sec\eta + \zeta(1-\omega))(\varphi(1) - c_0) + (1-\zeta)(1-\omega-\sec\eta)(\Phi(1) - c_0) \leq L\frac{(u-3v)}{2|t|(1-a)}(1-\omega). \quad (50)$$

Conclude the proof by applying the hypothesis of the theorem and equation (50).

Theorem 3.17: *Let* $t \in \mathbb{C}\backslash\{0\}$*, with* $0 \le v < 1,\ 0 \le u \le 1$ *and* $a < 1$*. Assume that* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and the following inequality is satisfied*

$$(\cos\mu + \varsigma - 2)c_0 - (\cos\mu + \varsigma)\Phi(1) + 2\varphi(1) \le \frac{L(u-3v)}{2|t|(1-a)}(\cos\mu - \varsigma).$$

Then for $\mu \le 1$ *and* $0 \le \varsigma < 1$*,* $\mathcal{T}_\varphi(\Psi, z) \in \mathcal{UCSP}(\mu, \varsigma)$*.*

Proof: Substituting (47) in (12), we obtain

$$\sum_{j=2}^{\infty} j(2j - \cos\mu - \varsigma)\frac{c_{j-1}}{jL}|x_j| \le \cos\mu - \varsigma. \tag{51}$$

Now, we apply the bound from (4), we have

$$\begin{aligned}&\sum_{j=2}^{\infty}(2j - \cos\mu - \varsigma)\frac{c_{j-1}}{L}|x_j| \\ &\le \sum_{j=2}^{\infty} 2j\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)} - \sum_{j=2}^{\infty}(\cos\mu + \varsigma)\frac{c_{j-1}}{L}\frac{2|t|(1-a)}{j(u-3v)}.\end{aligned} \tag{52}$$

Using (52) in (51), we get

$$2(\varphi(1) - c_0) - (\cos\mu + \varsigma)(\Phi(1) - c_0) \le (\cos\mu - \varsigma)\frac{L(u-3v)}{2|t|(1-a)}. \tag{53}$$

Conclude the proof by applying the hypothesis of the theorem and equation (53).

Theorem 3.18: *Let* $t \in \mathbb{C}\backslash\{0\}$*,* $0 \le v < 1$*,* $0 \le u \le 1$ *and* $a < 1$*. Suppose* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and*

$$(k + \theta)\Phi(1) - (1 + k)\varphi(1) \le \left(c_0 + \frac{L(u-3v)}{2|t|(1-a)}\right)(\theta - 1),$$

then for $0 \le \theta < 1$ *and* $0 \le k < \infty$*,* $\mathcal{T}_\varphi(h)(z) \in k - \mathcal{UCV}(\theta)$*.*

Proof: Applying (47) into Lemma 2.6 and then substituting expression (4), we obtain

$$\sum_{j=2}^{\infty} j^2(1 + k)\frac{c_{j-1}}{jL}\frac{2|t|(1-a)}{j(u-3v)} - \sum_{j=2}^{\infty} j(k + \theta)\frac{c_{j-1}}{jL}\frac{2|t|(1-a)}{j(u-3v)} \le 1 - \theta. \tag{54}$$

Simplifying the expression (54), we get

$$(1 + k)\sum_{j=1}^{\infty} c_j - (k + \theta)\sum_{j=1}^{\infty}\frac{c_j}{j+1} \le L\frac{(u-3v)}{2|t|(1-a)}(1 - \theta). \tag{55}$$

The desired outcome follows upon inserting (2) and (8) into equation (55).

Theorem 3.19: *Let* $t \in \mathbb{C}\backslash\{0\}$*, with* $0 \le v < 1,\ 0 \le u \le 1$ *and* $a < 1$*. Assume that* $\Psi \in \mathcal{R}_{v,u}^{t}(a)$ *and the condition below holds*

$$\varphi(1) + (q - 1)\Phi(1) - qc_0 \le q\frac{L(u-3v)}{2|t|(1-a)},$$

then for $q > 0$*,* $\mathcal{T}_\varphi(\Psi, z) \in \mathcal{K}_q$*.*

Proof: Substituting (47) in Lemma 2.9 and then substituting expression (4), we obtain

$$\left(\sum_{j=2}^{\infty} j^2\frac{c_{j-1}}{jL} + \sum_{j=2}^{\infty} j(q - 1)\frac{c_{j-1}}{jL}\right)\frac{2|t|(1-a)}{j(u-3v)} \le q. \tag{56}$$

Simplifying the expression (56), we get

$$\sum_{j=1}^{\infty} c_j + (q-1)\sum_{j=1}^{\infty} \frac{c_j}{j+1} \leq L\frac{(u-3v)}{2|t|(1-a)}q. \tag{57}$$

By applying (2) and (8) in (57), we get

$$\varphi(1) - c_0 + (q-1)(\Phi(1) - c_0) \leq q\frac{L(u-3v)}{2|t|(1-a)}. \tag{58}$$

Thus, by applying the given assumptions along with (58), the proof is complete.

Remark 3.2: It is important to observe that the sufficient conditions outlined in Theorem 3.16, 3.17, 3.18 and 3.19 for the operator $\mathcal{T}_{\varphi}(\Psi, z)$ to belong to the respective classes $\mathcal{K}(\eta, \omega, \zeta)$, $\mathcal{UCSP}(\mu, \varsigma)$, $k - \mathcal{UCV}(\theta)$ and $\mathcal{K}_q$ are identical to those presented in Theorem 3.1, 3.5, 3.8 and 3.13 which are the inclusion results of $\mathcal{P}_{\varphi}(\Psi, z)$ in the respective classes $\mathcal{S}(\eta, \omega, \zeta)$, $\mathcal{SP}_p(\sigma, \delta)$, $k - \mathcal{S}_p(\sigma)$ and $\mathcal{S}_{\lambda}^*$.

4. Conclusion

This work investigates the geometric properties of integral operators defined by a generalized distribution series. By utilizing coefficient estimates from a specific subclass of analytic functions, we examine how these operators behave geometrically. Additionally, we establish sufficient conditions that ensure the inclusion of the generalized distribution series in various subclasses of univalent functions. The findings are important relationships among integral operators, generalized distribution series, and univalent functions theory.

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