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Approximate solutions of integro-differential equations using different polynomials

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Abstract

This work focus on the application of the polynomial touchard and gegenbauer with the Galerkin method, to a specific type of integro-differential equations, as discussed for example in [1, 2, 4, 6, 7]. In addition, we considered second order integro-differential equations. The results were different depending on whether the Galerkin-Touchard method was used or the Galerkin Gegenbauer method, our research shows that the method is speed, stable, effective and provides satisfactory results, as demonstrated by the examples presented.

Subject Classification: *Integro-differential equations, Touchard polynomials, Gegenbauer polynomials, Galerkin method, Approximate solution.*

Keywords (2020): *47Gxx, 11Cxx, 65M60.*

1. Introduction

In both pure and applied mathematics, integral and integro-differential equations are regarded as some of the most relevant equations. In cutting edge fields of science and technology like heat transfer, physics, mechanics, fluid mechanics, and biology, integro-differential equations are crucial. Integro-differential equations are discussed in a number of authors (see [3, 5]). To solve these kinds of problems, numerous researchers have created numerical techniques [2, 10, 12, 13]. In this study, we solved the Fredholm and Volterra Integro-differential equation of the second type using an approximation method. This method is a method based on applying the Galerkin method with different polynomials of the Touchard type, which always gives good results; see [1, 2, 11]. Because of its good properties, especially the derivation, where when we derive Touchard's polynomials, we find that they have a relationship to the

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original Touchard polynomials, meaning that. $T' = nT$. This property makes it very suitable for obtaining good results in approximate solutions of Integro-differential equations, on the other hand. Applying Gegenbauer polynomials, which are considered somewhat complicated considering that they are orthogonal $\langle G_i, G_j \rangle = 0$ only for every, $j = i + 1$ see [7]. However, we obtain excellent approximate solutions. Numerical examples and results were presented in the fifth paragraph, where we presented several examples where we obtained good results, knowing that whenever the higher the degree of polynomials, the better the results and the more effective the method.

2. *P*-Galerkin method

We consider the following integro-differential equation

$$u'(x) = f(x) + \int_{a(x)}^{b(x)} k(x,t)\, u(t)dt \tag{1}$$

$$u(a) = \alpha. \tag{2}$$

Integro-differential equations can also be defined in the following general form

$$\sum_{i=0}^{n} g u^{(i)}(x) = f(x) + \int_{a(x)}^{b(x)} k(x,t)\, u(t)dt \quad u^{(i)}(a) = \alpha_i.$$

$k(x,t)$ And $f(x)$ are knowns functions and u is the unknown function.

The P-Galerkin polynomial method (P It represents the Touchard or Gegenbauer polynomials) is proposed to find the approximate solution.

$$u(x) \simeq u_n(x) = \sum_{i=0}^{n} a_i P_i\,(x), a \le x \le b, \tag{3}$$

Where $P(x)$, is the Touchard or Gegenbauer polynomial constructed and valid in the interval $[a, b]$ Thus, by differentiating (3) with respect to x, we have

$$u'(x) \simeq u_n'(x) = \sum_{i=0}^{n} a_i P_i'\,(x) \tag{4}$$

By replacing (3) and (4) in (1), we obtain

$$\sum_{i=0}^{n} a_i\, P_i'(x) = f(x) + \int_a^x k(x,t) \sum_{i=0}^{N} a_i\, P_i(t)dt \tag{5}$$

We determined the unknown coefficients a_i using the P-Galerkin idea by multiplying (5) by $P_j(x)$ and then integrating from a to b with respect to x. Consequently, we get

$$\sum_{i=0}^{n} a_i \int_a^b P_i'\,(x) P_j(x)dx$$
$$= \int_a^b P_j\,(x) f(x)dx + \int_a^b \int_a^x k(x,t) \sum_{i=0}^{n} a_i\, P_i(t) P_j(x)dtdx, j = 0,1, \dots, n \tag{6}$$

The system of algebraic equations for the unknown a_i with conditions (7) is given by Equation (6).

$$\sum_{i=0}^{n} a_i\, P_i(a) = \alpha. \tag{7}$$

The unknown a_i are determined by solving systems (6) and (7). The values of the constants obtained are then substituted into (3) to obtain the approximate solution.

3. Touchard polynomials

Touchard polynomials, studied by J. Touchard, composed of (a polynomial sequence of binomial type). The touchard polynomials [8, 9] are given as

$$T_n(x) = \sum_{k=0}^{n} s(n,k)\, x^k = \sum_{k=0}^{n} \binom{n}{k} x^k, \binom{n}{k} = \frac{n!}{ki(n-k)!}$$

We have: $T_0(x) = 1, T_1(x) = 1 + x, \dots$

Derivatives of Touchard Polynomials

Are

$$\frac{d}{dx} T_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=1}^{n} \binom{n}{k-1} x^{k-1}$$

4. Gegenbauer polynomials [7]

So we have the following recursion.

$$G_n(\alpha, x) = \frac{\Gamma(2\alpha+n).\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)\Gamma\left(\alpha+n+\frac{1}{2}\right)}.J\left(n, \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, x\right).$$

The Gegenbauer polynomials G_n are polynomials with rational coefficients

$G_0 = 1$	$G_1 = 2x$
$G_2 = 2x^2 - 1$	$G_3 = \frac{8}{3}x^3 - 2x$

5. Numerical results

In this section, we will present six examples that demonstrate the approximate solutions obtained using different polynomials, where the error is defined by the following formula:

$$E(x) = |u(x) - u_n(x)|.$$

Example 1: Consider Volterra's linear integro-differential equation.

$$u'(x) = -1 - \frac{1}{2}x^2 - xe^x \int_0^x tu(t)dt,$$

With the initial condition $u(0) = 0$.

The exact solution is as follows: $u(x) = 1 - exp(x)$.

Table 1

Present the exact and approximate solutions.

x	*Ex. sol u*	*App. sol, T_{10}*	*Error G_{10}*	*Error T_{10}*
0.0	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	-0.1052	-0.1052	1.181224e-07	4.018680e-10
0.2	-0.2214	-0.2214	1.240865e-07	2.001042e-10
0.3	-0.3499	-0.3499	4.428027e-08	2.824012e-10
0.4	-0.4918	-0.4918	8.055553e-08	3.386343e-10

0.5	-0.6487	-0.6487	9.539796e-08	1.475158e-10
0.6	-0.8221	-0.8221	7.641807e-09	3.267855e-10
0.7	-1.0138	-1.0138	7.999178e-08	5.701661e-12
0.8	-1.2255	-1.2255	6.585608e-08	2.418297e-10
0.9	-1.4596	-1.4596	3.825747e-08	2.088929e-10
1	-1.7183	-1.7183	2.750349e-10	3.486100e-14

Example 2: Consider the linear IDV.

$$u'(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)dt,$$

With the initial condition $u(0) = 0$.

The exact solution $u(x) = 2x$.

$u_n(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 2

Presents the results obtained for Example 2 for N = 10.

x	*Ex. sol u*	*App. sol, G_n*	*Error G_n*	*Error T_n*
0.0	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	0.2000	0.2000	5.551115e-17	4.546100e-11
0.2	0.4000	0.4000	5.551115e-17	3.050771e-11
0.3	0.6000	0.6000	1.110223e-16	2.620604e-11
0.4	0.8000	0.8000	3.330669e-16	4.714906e-11
0.5	1.0000	1.0000	2.220446e-16	9.988899e-12
0.6	1.2000	1.2000	2.220446e-16	4.605494e-11
0.7	1.4000	1.4000	2.220446e-16	6.851408e-12
0.8	1.6000	1.6000	2.220446e-16	3.263234e-11
0.9	1.8000	1.8000	2.220446e-16	3.237899e-11
1	2.0000	2.0000	4.440892e-16	8.881784e-16

Example 3: Consider Fredholm IDE.

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t)\,dt, u(0) = 0$$

The exact solution $u(x) = xe^x$

Table 3

Presents the results obtained for Example 3 for N = 10.

x	*Ex. sol u*	*App. sol, G_{10}*	*Error G_{10}*	*Error T_{10}*
0.0	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	0.1105	0.1105	2.510347e-07	1.684015e-10
0.2	0.2443	0.2443	2.635946e-07	5.280915e-12
0.3	0.4050	0.4050	9.387603e-08	1.925111e-10
0.4	0.5967	0.5967	1.711263e-07	5.011702e-11

0.5	0.8244	0.8244	2.025128e-07	1.482066e-10
0.6	1.0933	1.0933	1.631099e-08	4.893796e-11
0.7	1.4096	1.4096	1.696869e-07	9.171108e-11
0.8	1.7804	1.7804	1.399741e-07	5.911982e-11
0.9	2.2136	2.2136	8.117024e-08	4.014122e-12
1	2.7183	2.7183	7.874119e-10	5.684342e-14

Example 4: Consider Fredholm's linear integro-differential equation

$$u'(x) = 3e^{3x} - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xt\, u(t)dt,$$

With the initial condition $u(0) = 1$.

Where $u(x) = e^{3x}$ is the exacte solution

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 4

Presents the results obtained for Example 4.

x	*Ex.sol u*	*App.sol,* T_5	*Error* G_5	*Error* T_5	*Error* G_8	*Error* T_8
0.0	1.0000	1.0000	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00
0.1	1.3499	1.3368	9.728684e-04	1.308929e-02	4.471927e-05	5.441795e-06
0.2	1.8221	1.8170	4.047615e-03	5.076524e-03	4.403741e-05	7.147240e-06
0.3	2.4596	2.4669	2.609232e-03	7.319362e-03	7.340294e-05	1.043610e-05
0.4	3.3201	3.3330	1.202357e-03	1.291152e-02	3.432782e-05	6.601162e-06
0.5	4.4817	4.4897	2.912363e-03	8.021251e-03	5.914745e-05	9.599944e-06
0.6	6.0496	6.0463	2.973748e-04	3.379836e-03	3.149093e-05	7.162753e-06
0.7	8.1662	8.1542	4.503803e-03	1.200298e-02	5.265307e-05	8.843568e-06
0.8	11.0232	11.0143	6.220743e-03	8.916935e-03	1.717965e-05	3.774162e-06
0.9	14.8797	14.8840	2.192459e-03	4.266947e-03	3.549856e-05	8.219492e-06
1	20.0855	20.0847	4.196979e-03	8.789094e-04	4.763927e-06	9.277066e-09

Example 5: Consider FLIDE.

$$u'(x) = 3 + 6x + \int_0^1 xtu(t)\, dt$$

With the initial condition $u(0) = 0$.

The exact solution is

$$u(x) = 3x + 4x^2$$

The solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

Table 5

Presents the results obtained for Example 5.

x	*Ex. sol u*	*App. sol,* G_{10}	*Error* G_{10}	*Error* T_{10}
0.0	0.00000	0.00000	0.000000e+00	0.000000e+00
0.1	0.34000	0.34000	8.883658e-07	0.000000e+00
0.2	0.76000	0.76000	9.327980e-07	0.000000e+00
0.3	1.26000	1.26000	3.322368e-07	0.000000e+00
0.4	1.84000	1.84000	6.056290e-07	0.000000e+00
0.5	2.50000	2.50000	7.165786e-07	0.000000e+00
0.6	3.24000	3.24000	5.761295e-08	0.000000e+00
0.7	4.06000	4.06000	6.006336e-07	0.000000e+00
0.8	4.96000	4.96000	4.951495e-07	0.000000e+00
0.9	5.94000	5.94000	2.874869e-07	0.000000e+00
1	7.00000	7.00000	2.488216e-09	0.000000e+00

Example 6: Consider linear VIDE

$$u''(x) = x + \int_0^x (x-t)u(t)\,dt$$

With the initial condition $u(0) = 0, u'(0) = 1$

The exact solution is given as follows

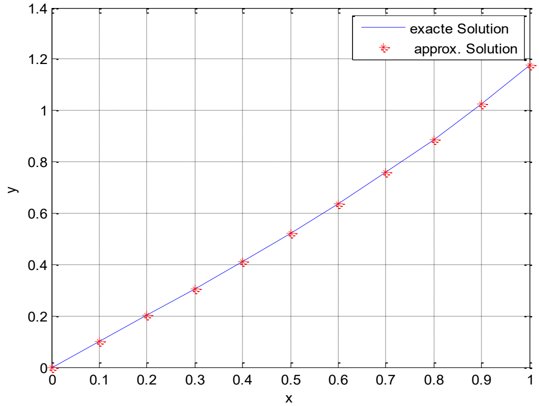
$$u(x) = sinh(x)$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Touchard and Gegenbauer polynomial method.

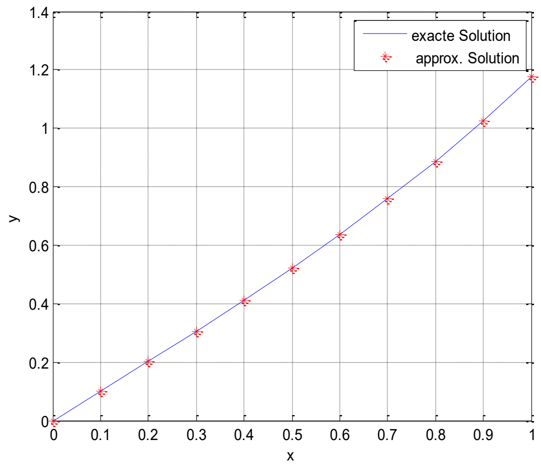
Table 6

Shows the results obtained in Example 6 for N = 10.

x	*Ex. sol u*	*App. sol,* G_{10}	*App. sol,* T_{10}	*Error* G_{10}	*Error* T_{10}
0.0	0.0000	0.0000	0.0000	0.000000e+00	0.000000e+00
0.1	0.1002	0.1002	0.1002	1.484785e-13	3.461069e-08
0.2	0.2013	0.2013	0.2013	2.196854e-13	1.709290e-07
0.3	0.3045	0.3045	0.3045	1.260103e-13	1.162768e-07
0.4	0.4108	0.4108	0.4108	3.160805e-13	1.272785e-07
0.5	0.5211	0.5211	0.5211	1.110223e-15	1.270703e-07
0.6	0.6367	0.6367	0.6367	1.942890e-14	6.612520e-08
0.7	0.7586	0.7586	0.7586	3.588241e-13	1.112242e-07
0.8	0.8881	0.8881	0.8881	3.059775e-13	8.854962e-09
0.9	1.0265	1.0265	1.0265	3.417266e-13	2.446322e-08
1	1.1752	1.1752	1.1752	5.226930e-13	3.258046e-09



Graph for example 6 using. Gegenbauer-Galerkin



Graph for example 6 using. Touchard-Galerkin

Conclusion

In this work, we presented and studied a numerical solution method for integro-differential equations based on Touchard and Gegenbauer polynomials. At the end of our work, we illustrated programming examples using the MATLAB numerical calculation program, where we estimated method errors and compared the approximate solutions obtained using Touchard and Gegenbauer separately, while also comparing with the exact solution.

To test the effectiveness of the method and estimate the accuracy through our results. We noticed that the results were good, that is to say that the greater the degree of N, the closer the results was to precision, and the more efficient the method.

It is pointed out here that the method can be applied to both linear and nonlinear integro-differential equations of higher orders. It was also pointed out here that the method was applied to higher order integro-differential equations, noting that the results were better for Volterra type integro-differential equations when using Gegenbauer-Galerkin (see Table (2, 6)), and exactly the opposite when using Touchard-Galerkin Fredholm type differential integral equations (see Table (1, 3, 4, 5)).

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