Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 4, pp. 793–807
DOI : 10.47974/JIM-2220

Application of quadruple Shehu transforms to Caputo fractional partial differential equations

D. D. Pawar [†]
R. D. Kadam *
School of Mathematical Sciences
Swami Ramanand Teerth Marathwada University
Nanded 431606
Maharashtra
India

Wagdi F. S. Ahmed [§]
Department of Mathematics
Lahej University
P. O. Box 125
Lahej
Yemen

Abstract

This article introduces an expansion of the triple Shehu transformation concept, extending it to the fourth order. Our approach begins by establishing the theoretical framework for these extended Shehu transformations, which we subsequently apply to obtain solutions for the fractional order partial differential equations within a three-dimensional context. It is important to note that the fractional derivatives in this study are considered in the Caputo sense. To illustrate the utility of our approach, we specifically focus on the fractional-order of the three-dimensional homogeneous heat equation. To present the resulting solution in a concise and elegant manner, we leverage the principles of Fox-function theory.

Subject Classification (2020): *35A22, 26A33, 35K05.*

Keywords: *Caputo derivative, Fractional heat equation, Fractional integral, Quadruple Shehu transform.*

[†] *ORCID-ID:* **0000-0001-8986-5243**
* *ORCID-ID:* **0009-0007-3387-1776**
[§] *ORCID-ID:* **0000-0002-5115-4355**

[†] *E-mail:* **dypawar@yahoo.com**
* *E-mail:* **kadamram3113@gmail.com (Corresponding Author)**
[§] *E-mail:* **wagdialakel@gmail.com**

1. Introduction

Fractional calculus has gained significant attention in recent decades due to its broad applications in applied sciences. Fractional-order models, being more flexible than their integer-order counterparts, play a vital role in providing deeper insights into the dynamics of complex real-world phenomena. Within the realm of fractional calculus, numerous open problems await exploration, encompassing both theoretical and experimental research avenues. Hence, delving into the study of differential equations is of paramount importance for comprehension. Analytically solving differential equations remains a challenge, with limited available techniques. Consequently, numerous scholars employ integral transform techniques to tackle both ordinary and partial differential equations. The PDE's, in particular, play a pivotal role in investigating diverse applications such as heat transfer, gravitation studies, quantum mechanics, and the behavior of perfect fluids [2, 3, 4, 5, 11, 16, 22, 23, 26, 27, 29, 31].

Differential equations are fundamental tools for modeling and solving real-world problems across diverse fields such as engineering, biology, physics,and applied sciences. Their significance lies in their ability to describe complex systems and dynamic processes. However, finding analytical solutions to differential equations is often challenging, as only a select class of these equations can be solved explicitly. To address this issue, researchers have introduced various analytical techniques, with the Laplace transform method being one of the most effective. This method has been extensively utilized in the literature to solve differential equations efficiently, as reported in [16, 19]. Building upon the concept of the Laplace transform, KÄ±lÄ±Ã§man et al. introduced extensions such as double Laplace transform and double Sumudu transform. These advanced methods have proven effective in solving complex equations, including Poisson and wave equations [23]. Moreover, The Sumudu transform has proven to be an effective tool for solving DE's in various applications, particularly in control engineering applications. Several integral transforms share similarities with the Laplace transform [1, 9, 14, 15, 20, 21, 24], such as the Sumudu transform [17, 32], the Mellin transform (MT)[30], and the Shehu transform [28, 13]. The Laplace transform continues to be a highly effective method for deriving analytical solutions to differential equations. The solution for wave and Poisson equations has been presented in [23] using the double Laplace transform (DLT) [18] and Sumudu transform. In control engineering, DE are effectively handled using the Sumudu transform.

In the quest for solving integral equations and partial differential equations, researchers have explored various advanced transform techniques. Notably, the Double Shehu Transform (DST) was introduced in [6, 7] to facilitate the solution of such equations [12, 13]. Building on this concept, Alkaleeli, Mtawal, and Hmad [8] proposed the (TST) as a natural extension of the triple Laplace transform. This novel transformation has proven effective in obtaining exact solutions for FPDE involving three variables, further broadening the scope of analytical techniques in fractional calculus.

In addition, the Quadruple Shehu Transform (QST) was introduced in [26] as a more generalized form of the Sumudu and Laplace transforms 24. This extension provides a powerful mathematical tool for tackling complex differential equations. In this study, we investigate the effectiveness of QST in obtaining analytical solutions for the three-dimensional heat equation, demonstrating its potential in addressing fractional-order problems in mathematical physics.

As mentioned earlier, FC broadens the conventional notion of integrals and derivatives by extending them beyond integer orders to encompass any positive real value. This extension provides a more flexible mathematical framework where fractional derivatives, inherently defined as definite integrals, offer a deeper geometric interpretation of function accumulation. Within this generalized setting, classical integer-order calculus emerges as a specific instance. The distinctive properties of fractional calculus facilitate a more comprehensive analysis of real-world dynamics, enabling a global perspective in contrast to the predominantly local focus of traditional calculus.

Furthermore, in many real-world phenomena, which are influenced by inherent memory effects and hereditary characteristics, fractional order models prove to be more advantageous than their classical counterparts. These intriguing and practical attributes of fractional calculus have inspired our exploration of heat equations within the framework of fractional order concepts, aiming to provide a comprehensive and globally oriented structural analysis.

The structure of this paper is outlined as follows: Section 2 introduces fundamental definitions related to fractional derivatives and the ST. In Section 3, we discuss key aspects of the Shehu transform and its extended variants, including the Double, Triple, and Quadruple Shehu Transforms. Section 4 focuses on deriving specific formulas for the Quadruple Shehu Transform adapted to fractional Caputo operators. In Section 5, we apply this transform to determine the solution of a third-order fractional heat equation. Finally, Section 6 concludes the paper by summarizing our findings and providing final remarks.

2. Preliminaries

In this part, we present essential concepts of FC and the foundational principles of the Shehu transform.

Definition 2.1: 1 The Riemann-Liouville (RL) partial fractional integrals for continuous function $\Phi(x,y)$ are defined as follows [10, 11]:

$$ {}_{y}I^{\xi}\Phi(x,y)=\frac{1}{\Gamma(\xi)}\int_{0}^{y}(y-\theta)^{\xi-1}\Phi(x,\theta)d\theta, \tag{2.1}$$

$$ {}_{x}I^{\zeta}\Phi(x,y)=\frac{1}{\Gamma(\zeta)}\int_{0}^{x}(x-\varphi)^{\zeta-1}\Phi(\varphi,y)d\varphi. \tag{2.2}$$

Where $\xi,\zeta>-1$ and $(y,x)\in(0,\infty)\times(0,\infty)$.

Definition 2.2: 2 According to [11], the Riemann-Liouville (RL) partial fractional derivatives for a continuous function $\Phi(x,y)$ are expressed as follows:

$$ {}_{R}D_{y}^{\xi}\Phi(x,y)=\frac{1}{\Gamma(r-\xi)}\left(\frac{d}{dy}\right)^{r}\int_{0}^{y}(y-\theta)^{r-\xi-1}\Phi(x,\theta)d\theta, \tag{2.3}$$

$$_{R}D_x^{\zeta}\Phi(x,y)=\frac{1}{\Gamma(n-\zeta)}\left(\frac{d}{dx}\right)^n\int_0^x(x-\varphi)^{n-\zeta-1}\Phi(\varphi,y)d\varphi. \tag{2.4}$$

Definition 2.3: 3 For a continuous function $\Phi(x,y)$ the Caputo partial fractional derivatives are expressed as follows in [11]:

$$D_y^{\xi}\Phi(x,y)=\frac{1}{\Gamma(r-\xi)}\int_0^y(y-\theta)^{r-\xi-1}\Phi^{(r)}(x,\theta)d\theta, \tag{2.5}$$

$$D_x^{\zeta}\Phi(x,y)=\frac{1}{\Gamma(n-\zeta)}\int_0^x(x-\varphi)^{n-\zeta-1}\Phi^{(n)}(\varphi,y)d\varphi. \tag{2.6}$$

Here, $n-1<\zeta<n, r-1<\xi<r,$ and$r,n\in N$.

Below are key properties of Caputo derivatives and the partial fractional integrals:

(1) $_{y}I^{\xi}\ {}_{x}I^{\zeta}[\Phi(x,y)]=$
$$\frac{1}{\Gamma(\xi)\Gamma(\zeta)}\int_0^y(y-\theta)^{(\xi-1)}(x-\varphi)^{(\zeta-1)}\Phi(\varphi,\theta)d\theta d\varphi.$$

(2) $D_y^{\xi}D_x^{\zeta}[\Phi(x,y)]=$
$$\frac{1}{\Gamma(n-\zeta)\Gamma(r-\theta)}\int_0^y(y-\theta)^{r-\xi-1}(x-\varphi)^{n-\zeta-1}\Phi^{(r+n)}(\varphi,\theta)d\theta d\varphi.$$

(3) $_{y}I^{\xi}D_y^{\xi}[\Phi(x,y)]=\Phi(x,y)-\sum_{r=0}^{n}\frac{\Phi^{(r)}(x,0)}{r!}y^r.$

(4) $_{y}I^{\xi}D_y^{\xi}[\Phi(x,y)]=\Phi(x,y).$

Definition 2.4: 4 The Mittag-Leffler function $E_{(\alpha,\zeta)}(t)$ is a generalization of the exponential function and is defined as:

$$E_{\alpha,\zeta}(t)=\sum_{r=0}^{\infty}\frac{t^r}{\Gamma(\alpha r+\zeta)}(\zeta,\alpha,t\in\mathbb{C},R(\zeta>0),R(\alpha)>0). \tag{2.7}$$

This function plays a fundamental role in fractional calculus, particularly in solving fractional differential equations, as it naturally generalizes exponential behavior in non-integer order systems.

Definition 2.5: 5 In 1961, Fox presented the H-Function which is expressed as a Mellin-Barnes type contour integral:

$$\mathcal{H}_{u,v}^{n,r}[-\sigma|\begin{matrix}(a_\xi,A_\xi)_1^u\\(e_\xi,E_\xi)_1^v\end{matrix}]=\frac{1}{2\pi i}\int_L\frac{\prod_{\xi=1}^{n}(e_\xi-E_\xi\tau)\prod_{\zeta=1}^{r}\Gamma(1-a_\zeta+\tau A_\zeta)\sigma^\tau d\tau}{\prod_{\xi=n+1}^{v}\Gamma(1-e_\xi+E_\xi\tau)\prod_{\zeta=r+1}^{u}(a_\zeta-\tau A_\zeta)},$$

Here, L represents an appropriate contour, and the orders (r,n,u,v) are integers satisfying the conditions $0\le n\le v, 0\le r\le u,$ Moreover, the parameters $a_\zeta\in\mathbb{R}, A_\zeta>0, \zeta=1,2,3,\ldots,u, e_\xi\in\mathbb{R}, E_\xi>0, \xi=1,2,3,\ldots,v$ are such that $A_\zeta(e_\xi+\alpha)\neq E_\xi(a_\zeta-\alpha-1), \alpha=0,1,2,3,\ldots$

$$\mathcal{H}_{\tau,z+1}^{1,\tau}\left[-\sigma\middle|\begin{matrix}(1-a_1,A_1),\ldots,(1-a_\tau,A_\tau)\\(1,0),(1-e_1,E_1),\ldots,(1-e_z,E_z)\end{matrix}\right]$$
$$=\sum_{s=0}^{\infty}\frac{\Gamma(a_1+A_1s)\ldots,\Gamma(a_\tau+A_\tau s)\sigma^s}{s!\Gamma(e_1+E_1s)\ldots,\Gamma(e_z+E_zs)}. \tag{2.8}$$

The H-function serves as a unifying framework for numerous special functions, making it particularly valuable in fractional calculus and integral transforms.

3. Shehu Transform

This section covers the Shehu transform and its extended forms, including the Double, Triple, and Quadruple Shehu Transforms, as well as the Quadruple Inverse Shehu Transform. Additionally, we discuss the convolution property of the Quadruple Shehu transform.

Definition 3.1: 6 As defined in [24] The (ST) of a function $\Phi(x)$ of exponential order is defined over the set of functions that satisfy the condition of existence, ensuring that the integral representation of the transform converges. Specifically, if $\Phi(x)$ is a real-valued function defined for $x \geq 0$ and of exponential order, then its (ST) is given by:

$$\mathbb{S}_x[\Phi(x): \varepsilon_1, \eta_1] = \int_0^\infty e^{-[\frac{\varepsilon_1 x}{\eta_1}]} \Phi(x) dx, \tag{3.1}$$

where ε_1 and η_1 are Shehu transform parameters that play a role in determining the transform properties.

The function $\Phi(x)$ must satisfy appropriate growth conditions to ensure the convergence of the integral.

Definition 3.2: 7 Alfaqeih [6] investigated the double Shehu transform for the function $\Phi(x, y)$, which is expressed as:

$$\mathbb{S}_x\mathbb{S}_y(\Phi(x, y)) = Q[(\varepsilon_1, \varepsilon_2), (\eta_1, \eta_2)] = \int_0^\infty \int_0^\infty e^{-[\frac{\varepsilon_1 x}{\eta_1} + \frac{\varepsilon_2 y}{\eta_2}]} \Phi(x, y) dx dy. \tag{3.2}$$

This transformation serves as an extension of the single Shehu transform, enabling the analysis of two-dimensional problems more effectively. It is particularly useful in solving integral and PDE's, making it a valuable tool in mathematical and applied sciences.

Definition 3.3: 8 Alkaleeli [8] introduced the triple Shehu transform (TST) of the function $\Phi(x, y, t)$ which is formulated as:

$$\begin{aligned} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_t[\Phi(x, y, t)] &= Q[(\varepsilon_1, \varepsilon_2, \varepsilon_3), (\eta_1, \eta_2, \eta_3)] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\left[\frac{\varepsilon_1 x}{\eta_1} + \frac{\varepsilon_2 y}{\eta_2} + \frac{\varepsilon_3 t}{\eta_3}\right]} \Phi(x, y, t) dx dy dt. \end{aligned} \tag{3.3}$$

This transformation provides a powerful mathematical tool for analyzing three-variable functions in applied mathematics and engineering.

Definition 3.4: 9 Expanding on previous work, the Quadruple Shehu transform (QST) was introduced in [26] as a natural extension of the TST to four variables. The transform of a function $\Phi(x, y, z, t)$ is given by:

$$\mathbb{S}^4_{x,y,z,t}[\Phi(x, y, z, t)] = Q[(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), (\eta_1, \eta_2, \eta_3, \eta_4)]$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\left[\frac{\varepsilon_1 x}{\eta_1}+\frac{\varepsilon_2 y}{\eta_2}+\frac{\varepsilon_3 z}{\eta_3}+\frac{\varepsilon_4 t}{\eta_4}\right]}\Phi(x,y,z,t)dxdydzdt, \tag{3.4}$$

Here, the non-negative variables $t, z, y, x \geq 0$, represent the independent parameters of the function, while $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \eta_1, \eta_2, \eta_3$ and η_4 are the corresponding Shehu parameters. This transformation is particularly useful in handling multi-dimensional integral equations and fractional differential equations.

Definition 3.5: 10 The Quadruple inverse Shehu transform, as presented in [25] is given by:

$$\mathbb{S}^{-4}_{x,y,z,t}(Q[(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4),(\eta_1,\eta_2,\eta_3,\eta_4)]) = \Phi(x,y,z,t)$$

$$= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{1}{\eta_1}e^{\left(\frac{\varepsilon_1 x}{\eta_1}\right)}\left[\frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty}\frac{1}{\eta_2}e^{\left(\frac{\varepsilon_2 y}{\eta_2}\right)}\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\eta_3}e^{\left(\frac{\varepsilon_3 z}{\eta_3}\right)}\left(\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{\eta_4}e^{\left(\frac{\varepsilon_4 t}{\eta_4}\right)}\mathbb{S}_4[\Phi(x,y,z,t)]d\varepsilon_4\right)d\varepsilon_3\right)d\varepsilon_2\right]d\varepsilon_1. \tag{3.5}$$

This inverse transform allows for the retrieval of the original function from its quadruple Shehu representation, making it a crucial component in solving equations that utilize this transformation.

Theorem 3.6: 11*The convolution of the Quadruple Shehu transform for the functions $\Phi_1(x,y,z,t)$ and $\Phi_2(x,y,z,t)$ is represented as $(\Phi_1 **** \Phi_2)(x,y,z,t)$ and is formulated as follows in* [25]*:*

$$(\Phi_1 **** \Phi_2)(x,y,z,t) = \int_0^x \int_0^y \int_0^z \int_0^t \Phi_1(x-\rho_1, y-\rho_2, z-\rho_3, t-\rho_4)\,\Phi_2(\rho_1,\rho_2,\rho_3,\rho_4)d\rho_1 d\rho_2 d\rho_3 d\rho_4,$$

$$= \int_0^x \int_0^y \int_0^z \int_0^t \Phi_2(x-\rho_1, y-\rho_2, z-\rho_3, t-\rho_4)\Phi_1(\rho_1,\rho_2,\rho_3,\rho_4)d\rho_1 d\rho_2 d\rho_3 d\rho_4. \tag{3.6}$$

4. Main Results

In this part, we develop the Quadruple Shehu transform (QST) for fractional integrals and derivatives.

Theorem 4.1: 12*For $\xi > 0$, if the function $\Phi(x,y,z,t)$ exhibits exponential order behavior, then the fractional integral of the Quadruple Shehu transform, represented as ${}_xI^{\xi}\Phi(x,y,z,t)$ is formulated as:*

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t({}_xI^{\xi}\Phi(x,y,z,t)) = \left(\frac{\varepsilon_1}{\eta_1}\right)^{-\xi}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t) \tag{4.1}$$

Proof: Using the Convolution with respect to x of Quadruple Shehu transform, we get

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t({}_xI^{\xi}\Phi(x,y,z,t)) = \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\xi)}x^{\xi-1} *** \Phi(x,y,z,t))$$

$$= \mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\xi)}\mathbb{S}_x(x^{\xi-1})\mathbb{S}_x\Phi(x,y,z,t))$$

$$= \mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\xi)} \cdot \frac{(\xi-1)!}{(\frac{\varepsilon_1}{\eta_1})^{\xi}} \mathbb{S}_x\Phi(x,y,z,t))$$
$$= (\frac{\varepsilon_1}{\eta_1})^{-\xi} \mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t).$$

Theorem 4.2: 13*For $\gamma > 0$, if the function $\Phi(x,y,z,t)$ is of exponential order, then the fractional integral of the Quadruple Shehu transform, represented as $_yI^{\gamma}\Phi(x,y,z,t)$ is expressed as:*

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(_yI^{\gamma}\Phi(x,y,z,t)) = \left(\frac{\varepsilon_2}{\eta_2}\right)^{-\gamma} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t) \tag{4.2}$$

Proof: Using the Convolution with respect to y of Quadruple Shehu transform, we get

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(_yI^{\gamma}\Phi(x,y,z,t)) = \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\gamma)} y^{\gamma-1} *** \Phi(x,y,z,t))$$
$$= \mathbb{S}_x\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\gamma)} \mathbb{S}_y(y^{\gamma-1})\mathbb{S}_y\Phi(x,y,z,t))$$
$$= \mathbb{S}_x\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\gamma)} \cdot \frac{(\gamma-1)!}{(\frac{\varepsilon_2}{\eta_2})^{\gamma}} \mathbb{S}_y\Phi(x,y,z,t))$$
$$= (\frac{\varepsilon_2}{\eta_2})^{-\gamma} \mathbb{S}_x\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t).$$

Theorem 4.3: 14*Let $\beta > 0$ and function $\Phi(x,y,z,t)$ have exponential order. Then, the fractional integrals of Quadruple Shehu transform of $_zI^{\beta}\Phi(x,y,z,t)$ is expressed as:*

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(_zI^{\beta}\Phi(x,y,z,t)) = \left(\frac{\varepsilon_3}{\eta_3}\right)^{-\beta} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t) \tag{4.3}$$

Proof: Using the Convolution with respect to z of Quadruple Shehu transform, we get

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(_zI^{\beta}\Phi(x,y,z,t)) = \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\beta)} z^{\beta-1} *** \Phi(x,y,z,t))$$
$$= \mathbb{S}_x\mathbb{S}_y\mathbb{S}_t(\frac{1}{\Gamma(\beta)} \mathbb{S}_z(z^{\beta-1})\mathbb{S}_z\Phi(x,y,z,t))$$
$$= \mathbb{S}_x\mathbb{S}_y\mathbb{S}_t(\frac{1}{\Gamma(\beta)} \cdot \frac{(\beta-1)!}{(\frac{\varepsilon_3}{\eta_3})^{\beta}} \mathbb{S}_z\Phi(x,y,z,t))$$
$$= (\frac{\varepsilon_3}{\eta_3})^{-\beta} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_t\Phi(x,y,z,t).$$

Theorem 4.4: 15*Let $\delta > 0$ and function $\Phi(x,y,z,t)$ have exponential order. Then, the fractional integrals of Quadruple Shehu transform of $_tI^{\delta}\Phi(x,y,z,t)$ is expressed as:*

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(_tI^{\delta}\Phi(x,y,z,t)) = \left(\frac{\varepsilon_4}{\eta_4}\right)^{-\delta} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t) \tag{4.4}$$

Proof: By applying the convolution with respect to t in the Quadruple Shehu transform, we obtain:

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t({}_tI^\delta\Phi(x,y,z,t)) = \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\delta)}t^{\delta-1} *** \Phi(x,y,z,t))$$
$$= \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z(\frac{1}{\Gamma(\delta)}\mathbb{S}_t(t^{\delta-1})\mathbb{S}_t\Phi(x,y,z,t))$$
$$= \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z(\frac{1}{\Gamma(\delta)}\cdot\frac{(\delta-1)!}{(\frac{\varepsilon_4}{\eta_4})^\delta}\mathbb{S}_t\Phi(x,y,z,t))$$
$$= (\frac{\varepsilon_4}{\eta_4})^{-\delta}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\Phi(x,y,z,t).$$

Theorem 4.5: *16Let $\xi,\gamma,\beta,\delta > 0$ and function $\Phi(x,y,z,t)$ have exponential order. Then, the fractional integrals of Quadruple Shehu transform ${}_tI^\delta \quad {}_zI^\beta \quad {}_yI^\gamma \quad {}_xI^\xi\Phi(x,y,z,t)$ are given as follows:*

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t({}_tI^\delta \quad {}_zI^\beta \quad {}_yI^\gamma \quad {}_xI^\xi\Phi(x,y,z,t)) = (\frac{\varepsilon_4}{\eta_4})^{-\delta}(\frac{\varepsilon_3}{\eta_3})^{-\beta}(\frac{\varepsilon_2}{\eta_2})^{-\gamma}(\frac{\varepsilon_1}{\eta_1})^{-\xi}$$
$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t)$$

Proof: Using the convolution of Quadruple Shehu transform we get,

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t({}_tI^\delta \quad {}_zI^\beta \quad {}_yI^\gamma \quad {}_xI^\xi\Phi(x,y,z,t)) = \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{1}{\Gamma(\xi)}\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\delta)}x^{\xi-1}$$
$$y^{\gamma-1}z^{\beta-1}t^{\delta-1} *** \Phi(x,y,z,t))$$
$$= \frac{1}{\Gamma(\xi)}\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\delta)}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(x^{\xi-1}y^{\gamma-1}z^{\beta-1}t^{\delta-1})\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t)$$
$$= \frac{1}{\Gamma(\xi)}\frac{1}{\Gamma(\beta)}\frac{1}{\Gamma(\gamma)}\frac{1}{\Gamma(\delta)}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\left(\frac{(\xi-1)!}{(\frac{\varepsilon_1}{\eta_1})^\xi}\frac{(\gamma-1)!}{(\frac{\varepsilon_2}{\eta_2})^\gamma}\frac{(\beta-1)!}{(\frac{\varepsilon_3}{\eta_3})^\beta}\frac{(\delta-1)!}{(\frac{\varepsilon_4}{\eta_4})^\delta}\right)$$
$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t)$$
$$= (\frac{\varepsilon_4}{\eta_4})^{-\delta}(\frac{\varepsilon_3}{\eta_3})^{-\beta}(\frac{\varepsilon_2}{\eta_2})^{-\gamma}(\frac{\varepsilon_1}{\eta_1})^{-\xi}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t\Phi(x,y,z,t).$$

Theorem 4.6: **17***Let $\sigma_1,\sigma_2,\sigma_3,\sigma_4 \in \mathbb{N}$ and $\xi,\gamma,\beta,\delta > 0$ such that $\sigma_4 - 1 < \delta \le \sigma_4, \sigma_3 - 1 < \beta \le \sigma_3, \sigma_2 - 1 < \gamma \le \sigma_2, \sigma_1 - 1 < \xi \le \sigma_1$ .Let's make a selection $l = max\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$ and Let $\Phi \in \mathbb{C}^l(R^+ \times R^+ \times R^+ \times R^+)$ Furthermore, suppose that for $u_1,u_2,u_3,u_4 > 0$ we have $\Phi^l \in \mathbb{S}_1[(0,u_1) \times (0,u_2) \times (0,u_3) \times (0,u_4)]$. Furthermore, suppose that for $t > u_4, z > u_3, y > u_2, x > u_1$, and constants $L, i_1, i_2, i_3, i_4 > 0$ the inequality $|\Phi(x,y,z,t)| \le Le^{xi_1+yi_2+zi_3+ti_4}$ holds. Thus, the Quadruple Shehu transform corresponding to partial fractional derivatives in the Caputo sense is formulated as follows:*

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{{}^c_xD^\xi_{0^+}\Phi(x,y,z,t)\}$$

$$= (\frac{\varepsilon_1}{\eta_1})^\xi\left[\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{\Phi(x,y,z,t)\} - \sum_{r_1=0}^{\sigma_1-1}(\frac{\varepsilon_1}{\eta_1})^{-1-r_1}\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t(\frac{\partial^{r_1}}{\partial x^{r_1}}\Phi(0,y,z,t))\right],$$

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{{}^c_yD^\gamma_{0^+}\Phi(x,y,z,t)\}$$

$$= (\frac{\varepsilon_2}{\eta_2})^\gamma\left[\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{\Phi(x,y,z,t)\} - \sum_{r_2=0}^{\sigma_2-1}(\frac{\varepsilon_2}{\eta_2})^{-1-r_2}\mathbb{S}_x\mathbb{S}_z\mathbb{S}_t(\frac{\partial^{r_2}}{\partial y^{r_2}}\Phi(x,0,z,t))\right],$$

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{{}_z^cD_{0^+}^{\beta}\Phi(x,y,z,t)\}$$

$$=(\frac{\varepsilon_3}{\eta_3})^{\beta}\left[\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\Phi(x,y,z,t)-\sum_{r_3=0}^{\sigma_3-1}(\frac{\varepsilon_3}{\eta_3})^{-1-r_3}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_t(\frac{\partial^{r_3}}{\partial z^{r_3}}\Phi(x,y,0,t))\right],$$

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{{}_t^cD_{0^+}^{\delta}\Phi(x,y,z,t)\}$$

$$=(\frac{\varepsilon_4}{\eta_4})^{\delta}\left[\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\Phi(x,y,z,t)-\sum_{r_4=0}^{\sigma_4-1}(\frac{\varepsilon_4}{\eta_4})^{-1-r_4}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z(\frac{\partial^{r_4}}{\partial t^{r_4}}\Phi(x,y,z,0))\right].$$

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{{}_x^cD_{0^+}^{\xi}\quad {}_y^cD_{0^+}^{\gamma}\quad {}_z^cD_{0^+}^{\beta}\quad {}_t^cD_{0^+}^{\delta}\Phi(x,y,z,t)\}$$

$$=(\frac{\varepsilon_1}{\eta_1})^{\xi}(\frac{\varepsilon_2}{\eta_2})^{\gamma}(\frac{\varepsilon_3}{\eta_3})^{\beta}(\frac{\varepsilon_4}{\eta_4})^{\delta}\Big[\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\Phi(x,y,z,t)-\sum_{r_1=0}^{\sigma_1-1}(\frac{\varepsilon_1}{\eta_1})^{-1-m_1}\mathbb{S}_t$$

$$\mathbb{S}_y\mathbb{S}_z(\frac{\partial^{m_1}}{\partial x^{r_1}}\Phi(0,y,z,t))-\sum_{r_2=0}^{\sigma_2-1}(\frac{\varepsilon_2}{\eta_2})^{-1-r_2}\mathbb{S}_x\mathbb{S}_z\mathbb{S}_t(\frac{\partial^{r_2}}{\partial y^{r_2}}\Phi(x,0,z,t))$$

$$-\sum_{r_3=0}^{\sigma_3-1}(\frac{\varepsilon_3}{\eta_3})^{-1-r_3}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_t(\frac{\partial^{r_3}}{\partial z^{r_3}}\Phi(x,y,0,t))-\sum_{r_4=0}^{\sigma_4-1}\left(\frac{\varepsilon_4}{\eta_4}\right)^{-1-r_4}\mathbb{S}_x\mathbb{S}_y$$

$$\mathbb{S}_z\left(\frac{\partial^{r_4}}{\partial t^{r_4}}\Phi(x,y,z,0)\right)+\sum_{r_1=0}^{\sigma_1-1}\sum_{r_2=0}^{\sigma_2-1}\sum_{r_3=0}^{\sigma_3-1}\sum_{r_4=0}^{\sigma_4-1}\left(\frac{\varepsilon_1}{\eta_1}\right)^{-1-r_1}\left(\frac{\varepsilon_2}{\eta_2}\right)^{-1-r_2}$$

$$(\frac{\varepsilon_3}{\eta_3})^{-1-r_3}(\frac{\varepsilon_4}{\eta_4})^{-1-r_4}(\frac{\partial^{r_1+r_2+r_3+r_4}}{\partial x^{r_1}\,\partial y^{r_2}\,\partial z^{r_3}\,\partial t^{r_4}}\Phi(0,0,0,0))\Big].$$

Proof: Theorem 4.6 can be proven in a manner analogous to the proof of Theorems 4.1, 4.2, 4.3, 4.4 and 4.5.

5. Application

(1) *Solution of Fractional Heat Equation:*

This section of the study focuses on utilizing the Shehu transform to obtain a solution for the third-order fractional heat equation, based on the theoretical foundations established earlier.

Example 5.1: 18 Consider the following equation:

$$ {}_t^cD_{0^+}^{\gamma}\Phi(x,y,z,t)=\frac{1}{\pi^2}(D_x^2+D_y^2+D_z^2)\Phi(x,y,z,t),\quad t,z,y,x>0. \tag{5.1}$$

Along with the initial and boundary conditions:

$$\begin{cases}\Phi(x,y,z,0)=\sin(\pi y)\sin(\pi z)\sin(\pi x),\\ \Phi(0,0,0,t)=\Phi(0,y,z,t)=\Phi(x,y,0,t)=\Phi(x,0,z,t)=0,\\ \Phi_z(x,y,0,t)=\Phi_x(0,y,z,t)=\Phi_y(x,0,z,t)=\pi E_{\alpha}(-t^{\alpha}),\end{cases} \tag{5.2}$$

Here, D_t, D_z, D_y, D_x are partial derivatives with the respect to t, z, y, x respectively and $0<\alpha\leq 1$.

Solution: By applying definition of triple Shehu in Eq. (5.2) we get,

$$\begin{cases} \mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\Phi(x,y,z,0) = \frac{\pi^3}{((\frac{\varepsilon_1}{\eta_1})^2+\pi^2)((\frac{\varepsilon_2}{\eta_2})^2+\pi^2)((\frac{\varepsilon_3}{\eta_3})^2+\pi^2)}, \\ \mathbb{S}_z\mathbb{S}_x\mathbb{S}_t\Phi(x,0,z,t) = \mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\Phi(x,y,z,0) = 0 \\ \mathbb{S}_z\mathbb{S}_y\mathbb{S}_t\Phi(0,y,z,t) = \mathbb{S}_y\mathbb{S}_t\mathbb{S}_x\Phi(x,y,0,t) = 0, \\ \mathbb{S}_z\mathbb{S}_y\mathbb{S}_t D_x(0,y,z,t) = \frac{\pi(\frac{\varepsilon_4}{\eta_4})^{\alpha-1}}{(\frac{\varepsilon_3}{\eta_3})(\frac{\varepsilon_2}{\eta_2})(1+(\frac{\varepsilon_4}{\eta_4})^{\alpha})}, \\ \mathbb{S}_z\mathbb{S}_x\mathbb{S}_t D_y(x,0,z,t) = \frac{\pi(\frac{\varepsilon_4}{\eta_4})^{\alpha-1}}{(\frac{\varepsilon_3}{\eta_3})(\frac{\varepsilon_1}{\eta_1})(1+(\frac{\varepsilon_4}{\eta_4})^{\alpha})}, \\ \mathbb{S}_y\mathbb{S}_x\mathbb{S}_t D_z(x,y,0,t) = \frac{\pi(\frac{\varepsilon_4}{\eta_4})^{\alpha-1}}{(\frac{\varepsilon_1}{\eta_1})(\frac{\varepsilon_2}{\eta_2})(1+(\frac{\varepsilon_4}{\eta_4})^{\alpha})}. \end{cases} \tag{5.3}$$

Now applying the QST to both sides of the eq (5.1) we get,

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{{}_t^cD_{0^+}^{\gamma}\Phi(x,y,z,t)\} = \frac{1}{\pi^2}\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{(D_x^2+D_y^2+D_z^2)\Phi(x,y,z,t)\}. \tag{5.4}$$

By using the definition of the Quadruple Shehu transform as stated in Theorem (4.6), the equation in (5.4) simplifies to:

$$\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta}\mathbb{S}_t\mathbb{S}_y\mathbb{S}_x\mathbb{S}_z\{\Phi(x,y,z,t)\} - \left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-1}\mathbb{S}_z\mathbb{S}_x\mathbb{S}_y\{\Phi(x,y,z,0)\} - \left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-2}$$
$$\mathbb{S}_z\mathbb{S}_x\mathbb{S}_y\{\Phi(x,y,z,0)\} = \frac{1}{\pi^2}\Bigg[\left(\frac{\varepsilon_1}{\eta_1}\right)^2\mathbb{S}_t\mathbb{S}_z\mathbb{S}_x\mathbb{S}_y\{\Phi(x,y,z,t)\} - \left(\frac{\varepsilon_1}{\eta_1}\right)\mathbb{S}_t\mathbb{S}_z$$
$$\mathbb{S}_y\{\Phi(0,y,z,t)\} - \mathbb{S}_t\mathbb{S}_z\mathbb{S}_yD_x\{\Phi(0,y,z,t)\} + \left(\frac{\varepsilon_2}{\eta_2}\right)^2\mathbb{S}_t\mathbb{S}_y\mathbb{S}_x\mathbb{S}_z\{\Phi(x,y,z,t)\} - \left(\frac{\varepsilon_2}{\eta_2}\right)$$
$$\mathbb{S}_t\mathbb{S}_x\mathbb{S}_z\{\Phi(x,0,z,t)\} - \mathbb{S}_t\mathbb{S}_z\mathbb{S}_xD_y\{\Phi(x,0,z,t)\} + \left(\frac{\varepsilon_3}{\eta_3}\right)^2\mathbb{S}_t\mathbb{S}_y\mathbb{S}_x\mathbb{S}_z\{\Phi(x,y,z,t)\}$$
$$-\left(\frac{\varepsilon_3}{\eta_3}\right)\mathbb{S}_t\mathbb{S}_x\mathbb{S}_y\{\Phi(x,y,0,t)\} - \mathbb{S}_t\mathbb{S}_x\mathbb{S}_yD_z\{\Phi(x,y,0,t)\}\Bigg]. \tag{5.5}$$

Then we get

$$\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(x,y,z,t)] - \frac{1}{\pi^2}\Bigg[\left(\frac{\varepsilon_1}{\eta_1}\right)^2\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(x,y,z,t)]$$
$$+\left(\frac{\varepsilon_2}{\eta_2}\right)^2\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(x,y,z,t)] + \left(\frac{\varepsilon_3}{\eta_3}\right)^2\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(x,y,z,t)]\Bigg]$$
$$= \left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-1}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z[\Phi(x,y,z,0)] + \left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-2}\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z[\Phi(x,y,z,0)]$$
$$+\frac{1}{\pi^2}\Bigg\{\left(\frac{\varepsilon_1}{\eta_1}\right)\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(0,y,z,t)] + \mathbb{S}_y\mathbb{S}_z\mathbb{S}_tD_x[\Phi(0,y,z,t)]$$
$$+\left(\frac{\varepsilon_2}{\eta_2}\right)\mathbb{S}_x\mathbb{S}_z\mathbb{S}_t[\Phi(x,0,z,t)] + \mathbb{S}_x\mathbb{S}_z\mathbb{S}_tD_y[\Phi(x,0,z,t)]$$
$$+\left(\frac{\varepsilon_3}{\eta_3}\right)\mathbb{S}_x\mathbb{S}_y\mathbb{S}_t[\Phi(x,y,0,t)] + \mathbb{S}_x\mathbb{S}_y\mathbb{S}_tD_z[\Phi(x,y,0,t)]\Bigg\}.$$

By doing some small rearrangement in above Eq. we get

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(x,y,z,t)] = \frac{\pi^2}{\pi^2\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta} - \left\{\left(\frac{\varepsilon_1}{\eta_1}\right)^2 + \left(\frac{\varepsilon_2}{\eta_2}\right)^2 + \left(\frac{\varepsilon_3}{\eta_3}\right)^2\right\}}$$
$$\times \Bigg\{\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-1} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z[\Phi(x,y,z,0) - \left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-2} \mathbb{S}_x\mathbb{S}_y\mathbb{S}_z[\Phi(x,y,z,0)]$$
$$-\frac{1}{\pi^2}\Bigg[\left(\frac{c_1}{d_1}\right)\mathbb{S}_y\mathbb{S}_z\mathbb{S}_t[\Phi(0,y,z,t)] + \mathbb{S}_y\mathbb{S}_z\mathbb{S}_t D_x[\Phi(0,y,z,t)]$$
$$+\left(\frac{\varepsilon_2}{\eta_2}\right)\mathbb{S}_x\mathbb{S}_z\mathbb{S}_t[\Phi(x,0,z,t)] + \mathbb{S}_x\mathbb{S}_z\mathbb{S}_t D_y[\Phi(x,0,z,t)]$$
$$+\left(\frac{\varepsilon_3}{\eta_3}\right)\mathbb{S}_x\mathbb{S}_y\mathbb{S}_t[\Phi(x,y,0,t) + \mathbb{S}_x\mathbb{S}_y\mathbb{S}_t D_z[\Phi(x,y,0,t)]\Bigg]\Bigg\}. \tag{5.6}$$

Now by using the equation (5.3) in equation (5.6) we get,

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{\Phi(x,y,z,t)\} = \frac{\pi^2\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta-1}}{\pi^2\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta} - \left[\left(\frac{\varepsilon_1}{\eta_1}\right)^2 + \left(\frac{\varepsilon_2}{\eta_2}\right)^2 + \left(\frac{\varepsilon_3}{\eta_3}\right)^2\right]}$$
$$\times\left[\frac{\pi^3}{\left(\frac{\varepsilon_1}{\eta_1}+\pi^2\right)\left(\frac{\varepsilon_2}{\eta_2}+\pi^2\right)\left(\frac{\varepsilon_3}{\eta_3}+\pi^2\right)} + \frac{1}{\pi\left(1+\left(\frac{\varepsilon_4}{\eta_4}\right)^{\delta}\right)}\left\{\frac{1}{\frac{\varepsilon_1\varepsilon_2}{\eta_1\eta_2}} + \frac{1}{\frac{\varepsilon_2\varepsilon_3}{\eta_2\eta_3}} + \frac{1}{\frac{\varepsilon_1\varepsilon_3}{\eta_1\eta_3}}\right\}\right]. \tag{5.7}$$

For finding the inverse Quadruple Shehu transform for equation (5.7), we use the formula:

$$(1+t)^{\alpha} = \sum_{\mu=0}^{\infty} (-t)^{\mu} \frac{1}{\mu!}\frac{\Gamma(\mu-\alpha)}{\Gamma(\alpha)} \tag{5.8}$$

By applying equation (5.8) in equation (5.7) we get,

$$\mathbb{S}_t\mathbb{S}_z\mathbb{S}_y\mathbb{S}_x\{\Phi(x,y,z,t)\} = \sum_{\mu=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\Gamma(n-\mu)\Gamma(n-\nu)}{n!\nu!\Gamma(\mu)\Gamma(n)}$$
$$\times\Bigg[\sum_{\vartheta=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{\vartheta+n+\nu+l}}{\Gamma(\nu)\pi^{2(\mu+\vartheta+l+m+\nu)}\left(\frac{\varepsilon_1}{\eta_1}\right)^{2(n-\mu-\vartheta-l)}\left(\frac{\varepsilon_2}{\eta_2}\right)^{-2(n+l+\nu)}}$$
$$\times\frac{1}{\left(\frac{\varepsilon_3}{\eta_3}\right)^{-2(\mu+\vartheta+l)}\left(\frac{\varepsilon_4}{\eta_4}\right)^{2\alpha-\alpha\mu+\alpha\nu+1}} - \frac{(-1)^{n+\nu+s}}{\pi^{2(\mu+\nu+n+1)}}$$
$$\times\Bigg\{\sum_{s=0}^{\infty}\frac{1}{\left(\frac{\varepsilon_1}{\eta_1}\right)^{2(s-n-\mu)}\left(\frac{\varepsilon_2}{\eta_2}\right)^{(\nu+n-l)+1}\left(\frac{\varepsilon_3}{\eta_3}\right)^{2(\nu+s-l)+1}\left(\frac{\varepsilon_4}{\eta_4}\right)^{2(\nu+\alpha s+\mu)}} \tag{5.9}$$
$$-\sum_{r=0}^{\infty}\frac{1}{\left(\frac{\varepsilon_1}{\eta_1}\right)^{2(s-n-\mu)+1}\left(\frac{\varepsilon_2}{\eta_2}\right)^{2(\nu+n-l)}\left(\frac{\varepsilon_3}{\eta_3}\right)^{2(\nu+r-l)+1}\left(\frac{\varepsilon_4}{\eta_4}\right)^{2(\nu+\alpha r+\mu)}}$$
$$-\sum_{c=0}^{\infty}\frac{1}{\left(\frac{\varepsilon_1}{\eta_1}\right)^{2(s-n-\mu)+1}\left(\frac{\varepsilon_2}{\eta_2}\right)^{2(\nu+n-l)+1}\left(\frac{\varepsilon_3}{\eta_3}\right)^{2(\nu+c-l)}\left(\frac{\varepsilon_4}{\eta_4}\right)^{2(\nu+\alpha c+\mu)}}\Bigg\}\Bigg]$$

By taking the inverse Quadruple Shehu transfom of equation (5.9) yields,

$$\Phi(x,y,z,t) = \sum_{\mu=0}^{\infty}\sum_{\vartheta=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{\vartheta+n+l}x^{2n-2\mu-2\vartheta-2l-1}y^{-2n-2l-1}z^{-2\mu-2\vartheta-1}t^{2\alpha-\alpha\mu}}{n!\pi^{2\mu+2\vartheta+2l+2m}}$$

$$\times \mathcal{H}_{1,12}^{2,1}\left[-\left(\frac{t^{\alpha}\pi^{2}}{z^{2}y^{2}}\right)\middle|\begin{matrix}(1-n+\mu,0),(1-n,1).\\(0,1),(2,0)^{4},(1-\mu,0),(1-n,0),(1,1),\\(1+2n+2\mu+2\vartheta+2l,0),(1+2n+2l,2),\\(1+2\mu+2\vartheta,2),(-2\alpha+\alpha\mu,\alpha).\end{matrix}\right]$$

$$-\sum_{\mu=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(-1)^{n+\nu}x^{-2n-2\mu-1}y^{2n-2l+2\nu}z^{-2l+2\nu}t^{2\mu+2\rho-1}}{n!\nu!\pi^{2+2\mu+2\nu+2n}}$$

$$\times \mathcal{H}_{1,8}^{2,1}\left[-\left(zxt^{\alpha}\right)^{2}\middle|\begin{matrix}(1-n+\mu,0),(1-n-\nu,0).\\(2,0)^{2},(1-\mu,0),(1-n,0),(1+2n+2\mu,2),\\(-2n+2l-2\nu,0),(2l-2\nu,0),(1-2\mu-2\nu,2\alpha).\end{matrix}\right]$$

$$-\sum_{\mu=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(-1)^{n+\nu}x^{-2n-2\mu}y^{2n-2l+2\nu-1}z^{-2l+2\nu}t^{2\mu+2\nu-1}}{n!\nu!\pi^{2+2\mu+2\nu+2n}}$$

$$\times \mathcal{H}_{1,8}^{2,1}\left[-\left(zxt^{\alpha}\right)^{2}\middle|\begin{matrix}(1-n+\mu,0),(1-n-\rho,0).\\(2,0)^{2},(1-\mu,0),(1-n,0),(2n+2\mu,2),\\(1-2n+2l-2\nu,0),(2l-2\nu,0),(1-2\mu-2\nu,2\alpha).\end{matrix}\right]$$

$$-\sum_{\mu=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(-1)^{n+\nu}x^{-2n-2\mu}y^{2n-2l+2\nu}z^{-2l+2\nu-1}t^{2\mu+2\nu-1}}{n!\nu!\pi^{2+2\mu+2\nu+2n}}$$

$$\times \mathcal{H}_{1,8}^{2,1}\left[-\left(zxt^{\alpha}\right)^{2}\middle|\begin{matrix}(1-n+\mu,0),(1-n-\nu,0).\\(2,0)^{2},(1-\mu,0),(1-n,0),(2n+2\mu,2),\\(-2n+2l-2\nu,0),(1-2l-2\nu,0),(1-2\mu-2\nu,2\alpha).\end{matrix}\right]. \tag{5.10}$$

6. Conclusions

In this study, we extended the Shehu transform framework to the fourth order, establishing a more comprehensive theoretical foundation for these advanced integral transforms. This extension enhances the applicability of Shehu transforms in solving complex mathematical models, particularly those involving fractional-order systems. We illustrated the effectiveness of these higher-order transforms by deriving analytical solutions for FPDE's in three-dimensional space. Our methodology incorporated fractional derivatives in the Caputo sense, which provides a more flexible and generalized approach to modeling real-world phenomena with memory effects. By implementing this technique in solving the fractional three-dimensional homogeneous heat equation, we highlighted its efficiency in capturing intricate physical behaviors that classical methods may not fully address.

Additionally, we leveraged the Fox-function theory to express the obtained solutions in a compact, elegant, and computationally convenient form. This representation not only simplifies the interpretation of results but also reinforces the practicality of our approach in handling various applied problems in physics, engineering, and related disciplines.

References

[1] M. I. Abbas, "Ulam Stability and existence results for fractional differential equations with hybrid proportional-Caputo derivatives," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 2, pp. 213–231 (2022).

[2] W. F. S. Ahmed, D. D. Pawar, and W. D. Patil, "Fractional kinetic equations involving generalized V-function via Laplace transform," *Advances in Mathematics: Scientific Journal*, vol. 10, no. 5, pp. 2593–2610 (2021).

[3] W. F. S. Ahmed and D. D. Pawar, "Application of Sumudu Transform on Fractional Kinetic Equation Pertaining to the Generalized k-Wright Function," *Advances in Mathematics: Scientific Journal,* vol. 9, no. 10, pp. 8091–8103 (2020).

[4] W. F. S. Ahmed, D. D. Pawar, and A. Y. A. Salamooni, "On the Solution of Kinetic Equation for Katugampola Type Fractional Differential Equations," *Journal of Dynamical Systems and Geometric Theories,* vol. 19, no. 1, pp. 125–134 (2021), doi: 10.1080/1726037X.2021.1966946.

[5] W. F. S. Ahmed, A. Y. A. Salamooni, and D. D. Pawar, "Solution of fractional Kinetic Equation For Hadamard type fractional integral Via Mellin Transform," *Gulf Journal of Mathematics*, vol. 12, no. 1, pp. 15–27 (2022).

[6] S. Alfaqeih and E. Mısırlı, "On double Shehu transform and its properties with applications," *International Journal of Analysis and Applications*, vol. 18, no. 3, pp. 381–395 (2020).

[7] S. Alfaqeih, G. Bakıcıerler, and E. Mısırlı, "Application of Double Shehu Transforms to Caputo Fractional Partial Differential Equations," *Punjab University Journal of Mathematics,* vol. 54, no. 1, pp. 1–13 (2022).

[8] S. R. Alkaleeli, A. A. H. Mtawal, and M. S. Hmad, "Triple Shehu transform and its properties with applications," *African Journal of Mathematics and Computer Science Research*, vol. 14, no. 1, pp. 4–12 (Jan.–Jun. 2021).

[9] A. M. O. Anwar, F. Jarad, D. Baleanu, and F. Ayaz, "Fractional Caputo heat equation within the double Laplace transform," *Romanian Journal of Physics,* vol. 58, no. 1-2, pp. 15–22 (2013).

[10] M. Awadalla, T. Ozis, and S. Alfaqeih, "On System of Nonlinear Fractional Differential Equations Involving Hadamard Fractional Deriva-

tive with Nonlocal Integral Boundary Conditions," *Progress in Fractional Differentiation and Applications*, vol. 5, no. 3, pp. 225–232 (2019).

[11] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, Fractional Calculus: Models and Numerical Methods, World Scientific (2012).

[12] R. Belgacem, D. Baleanu, and A. Bokhari, "Shehu Transform and Applications to Caputo-Fractional Differential Equations," *International Journal of Analysis and Applications*, vol. 17, no. 6, pp. 917–927 (2019).

[13] A. Bokhari, D. Baleanu, and R. Belgacem, "Application of Shehu transform to Atangana-Baleanu derivatives," *Journal of Mathematics and Computer Science*, vol. 20, pp. 101–107 (2019).

[14] R. R. Dhunde and G. L. Waghmare, "Solving partial integro-differential equations using double Laplace transform method," *American Journal of Computational and Applied Mathematics*, vol. 5, no. 1, pp. 7–10 (2015).

[15] R. R. Dhunde and G. L. Waghmare, "Double Laplace Transform Method for Solving Space and Time Fractional Telegraph Equations," *International Journal of Mathematics and Mathematical Sciences,* vol. 2016, Article ID 1414595, 7 pages (2016).

[16] D. G. Duffy, Transform Methods for Solving Partial Differential Equations, Boca Raton, FL, USA: CRC Press (2004).

[17] H. Eltayeb and A. Kilicman, "On double Sumudu transform and double Laplace transform," *Malaysian Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 17–30 (2010).

[18] T. Elzaki, "Double Laplace variational iteration method for solution of nonlinear convolution partial differential equations," *Archives des Sciences*, vol. 65, no. 12, pp. 588–593 (2012).

[19] T. A. Estrin and T. J. Higgins, "The solution of boundary value problems by multiple Laplace transformations," *Journal of the Franklin Institute,* vol. 252, no. 2, pp. 153–167 (1951).

[20] T. Khan, M. M. Rashid, S. Ahmad, M. Sohail, and A. Khan, "On fractional order multiple integral transforms technique to handle three dimensional heat equation," *Boundary Value Problems*, vol. 2022, no. 1, p. 16 (2022).

[21] A. Khan and A. Khan, "Extension of Triple Laplace Transform for Solving Fractional Differential Equations," *Discrete and Continuous Dynamical Systems - Series S,* vol. 13, no. 3 (Mar. 2020).

[22] A. A. Kilbas, O. I. Marichev, and S. G. Samko, Fractional Integrals and Derivatives: Theory and Applications, Switzerland: Gordon and Breach (1993).

[23] A. Kilicman and H. Gadain, "An application of double Laplace transform and double Sumudu transform," *Lobachevskii Journal of Mathematics,* vol. 30, no. 3, pp. 214–223 (2009).

[24] S. Maitama and W. Zhao, "New integral transform: Shehu transform, a generalization of Sumudu and Laplace transform for solving differential equations," *International Journal of Analysis and Applications*, vol. 17, no. 2, pp. 167–190 (2019).

[25] D. D. Pawar and W. F. S. Ahmed, "Solution of Fractional Kinetic Equations Involving generalized q-Bessel function," *Results in Nonlinear Analysis*, vol. 5, no. 1, pp. 87–95 (2022).

[26] D. D. Pawar, G. G. Bhuttampalle, S. B. Chavhan, W. F. Ahmed, and R. D. Kadam, "*Quadruple Shehu Transform and its Applications*," arXiv preprint arXiv:2211.17265 (Nov. 2022).

[27] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol. 198, San Diego, CA, USA: Academic Press (1999).

[28] Z. D. Ridha and M. F. A. Mohommed, "Solving linear ordinary differential equations by using Shehu transform with variable coefficient," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 8, pp. 2391–2400 (2021).

[29] L. Romero and R. Cerutti, "Fractional Fourier transform and special k-function," *International Journal of Contemporary Mathematical Sciences,* vol. 7, pp. 693–704 (2012).

[30] S. T. M. Thabet, W. F. S. Ahmed, and D. D. Pawar, "On Fractional Kinetic Equation Relating to the Generalized k-Bessel Function by the Mellin Transform," *Journal of Fractional Calculus and Nonlinear Systems,* vol. 3, no. 2, pp. 1–12 (2022).

[31] S. T. M. Thabet, W. F. S. Ahmed, D. D. Pawar, and A. Y. A. Salamooni, "Generalized fractional Sturm-Liouville and Langevin equations involving Caputo derivative with nonlocal conditions," *Progress in Fractional Differentiation and Applications,* vol. 6, no. 3, pp. 225–237 (2020).

[32] G. K. Watugala, "The Sumudu transform for functions of two variables," *Mathematics in Engineering, Science and Aerospace* (MESA), vol. 8, pp. 293–302 (2001).

Received January, 2024