

A new variable step-size hybrid block method for the numerical solution of ordinary differential equations

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Abstract

This paper introduced a new variable step-size hybrid block scheme for the numerical solution of ordinary differential equations (ODEs). Some classes of differential equations encounter reduced accuracy or stability issues when fixed step-size methods are used. The proposed approach introduces a scaling factor that changes the step size as the numerical iteration progresses. Stability properties of the proposed scheme are also analyzed. The method is discovered to be zero stable, A-stable, and convergent, which serves as a good alternate approach for solving ODEs.

Subject Classification: 65L05, 65L06, 65L20.

Keywords: Ordinary differential equations, ODEs, Numerical method, Backward differentiation formula.

1. Introduction

Many of the real-world challenges that occur in science and engineering like spring and damping systems, electric circuit, chemical kinetics and much more can be represented through ordinary differential equations (ODEs). Stiff ODEs are differential equations that pose difficulties in computation due to their solutions containing elements that

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decay at vastly disparate speeds [3]. These characteristics require numerical techniques that are accurate and efficient.

For solving stiff initial value problems implicit numerical methods are more effective, while explicit methods often have severe stability issues that require the step length to be very small. The backward differentiation formula (BDF) is a family of implicit methods that have gained recognition for its favorable stability properties and is particularly helpful for solving stiff ODEs. The backward differentiation formula (BBDF) approach is known for its excellent accuracy and stability especially when dealing with stiffness. Due to their implicit nature, larger step sizes can be used without sacrificing numerical stability [6]. This reduces computational effort compared to explicit methods, which would otherwise need unreasonably small step sizes. BBDF techniques are therefore a more effective and reliable method for solving stiff ODEs, which enhances their importance in modern numerical analysis.

In 1952, [2] formally introduced the idea of stiffness. They discussed the challenges involved when working with systems that have different time scales. [3] discussed the application of BDF methods, which became essential for the numerical solutions of stiff ODEs. In recent decades, considerable research attempts have been made by researchers to propose efficient methods for solving ODEs. These include the work of [9-14]. Researchers later found out that block methods provided increased computational efficiency by computing more than one solution point at a time, unlike non-block methods that only compute one solution at a time.

Block backward differentiation formula methods (BBDF), combine the effectiveness of block methods with the stability of backward differentiation formulas. [4] proposed the r -point BBDF, the scheme improves CPU time and decrease the number of steps required. The two-point superclass BBDF method proposed by Suleiman et al. [7] expands on the conventional BBDF by adding a parameter which allows generation of different sets of the formula.

[8] proposed a diagonally implicit BBDF method, which improved efficiency by reducing the computation time involved in solving stiff

ODEs. Most researchers have developed variable step-size schemes that adjust the step sizes based on solution behavior. The variable step-size methods have been shown to enhance both accuracy and efficiency. [1] and [5] are among the earliest researchers to explore variable step-size BBDF methods. Such methods can change the step-size and also have the stability of BBDF methods to obtain more accurate and efficient solutions.

This study improves on existing numerical techniques for stiff ODEs by combining the flexibility of variable step sizes with the stability of BBDF methods.

Derivation of the Method

The scheme associated with the new hybrid block method is defined as;

$k = \frac{1}{2}, 1, \frac{3}{2}, 2$ we get

$$\gamma_{0,\frac{1}{2}}W_{\eta-1} + \gamma_{1,\frac{1}{2}}W_{\eta} + \gamma_{\frac{3}{2},\frac{1}{2}}W_{\eta+\frac{1}{2}} - h\psi_{\frac{1}{2}}\left(\phi_{\eta+\frac{1}{2}} - \rho\phi_{\eta-\frac{3}{2}}\right) = 0 \quad (1)$$

$k = 1,$

$$\gamma_{0,1}W_{\eta-1} + \gamma_{1,1}W_{\eta} + \gamma_{\frac{3}{2},1}W_{\eta+\frac{1}{2}} + \gamma_{2,1}W_{\eta+1} - h\psi_1(\phi_{\eta+1} - \rho\phi_{\eta-1}) = 0 \quad (2)$$

$k = \frac{3}{2},$

$$\gamma_{0,\frac{3}{2}}W_{\eta-1} + \gamma_{1,\frac{3}{2}}W_{\eta} + \gamma_{\frac{3}{2},\frac{3}{2}}W_{\eta+\frac{1}{2}} + \gamma_{2,\frac{3}{2}}W_{\eta+1} + \gamma_{\frac{5}{2},\frac{3}{2}}W_{\eta+\frac{3}{2}} - h\psi_{\frac{3}{2}}\left(\phi_{\eta+\frac{3}{2}} - \rho\phi_{\eta-\frac{1}{2}}\right) = 0 \quad (3)$$

$k = 2,$

$$\gamma_{0,2}W_{\eta-1} + \gamma_{1,2}W_{\eta} + \gamma_{\frac{3}{2},2}W_{\eta+\frac{1}{2}} + \gamma_{2,2}W_{\eta+1} + \gamma_{\frac{5}{2},2}W_{\eta+\frac{3}{2}} + W_{3,2}y_{\eta+2} - h\psi_2(\phi_{\eta+2} - \rho\phi_{\eta}) = 0 \quad (4)$$

We derive the formula using Taylor's series expansion about x_n as presented.

The linear difference operator related with the new method is;

$$\left. \begin{aligned} L_{\frac{1}{2}}[W(x_\eta), h] &= \gamma_{0, \frac{1}{2}} W(x_\eta - rh) + \gamma_{1, \frac{1}{2}} W(x_\eta) + \gamma_{2, \frac{1}{2}} W\left(x_\eta + \frac{h}{2}\right) - h\psi_{1,2} \left[\phi\left(x_\eta + \frac{h}{2}\right) - \rho\phi\left(x_\eta - \frac{3rh}{2}\right) \right] \\ L_1[W(x_\eta), h] &= \gamma_{0,1} W(x_\eta - rh) + \gamma_{1,1} W(x_\eta) + \gamma_{\frac{3}{2},1} W\left(x_\eta + \frac{h}{2}\right) + \gamma_{2,1} W(x_\eta + h) \\ &\quad - h\psi_1 [\phi(x_\eta + h) - \rho\phi(x_\eta - rh)] \\ L_{\frac{3}{2}}[W(x_\eta), h] &= \gamma_{0, \frac{3}{2}} W(x_\eta - rh) + \gamma_{1, \frac{3}{2}} W(x_\eta) + \gamma_{\frac{3}{2}, \frac{3}{2}} W\left(x_\eta + \frac{h}{2}\right) + \gamma_{2, \frac{3}{2}} W(x_\eta + h) + \gamma_{\frac{5}{2}, \frac{3}{2}} W\left(x_\eta + \frac{3h}{2}\right) \\ &\quad - h\psi_{\frac{3}{2}} \left[\phi\left(x_\eta + \frac{3h}{2}\right) - \rho\phi\left(x_\eta - \frac{rh}{2}\right) \right] \\ L_2[W(x_\eta), h] &= \gamma_{0,2} W(x_\eta - rh) + \gamma_{1,2} W(x_\eta) + \gamma_{\frac{3}{2},2} W\left(x_\eta + \frac{h}{2}\right) + \gamma_{2,2} W(x_\eta + h) + \gamma_{\frac{5}{2},2} W\left(x_\eta + \frac{3h}{2}\right) \\ &\quad + \gamma_{3,2} W(x_\eta + 2h) - h\psi_2 [\phi(x_\eta + 2h) - \rho\phi(x_\eta)] \end{aligned} \right\} \quad (5)$$

For the first point, we let $k = \frac{1}{2}$

To derive the first point $W_{\eta+\frac{1}{2}}$. Define the linear difference operator as

$$\gamma_{0, \frac{1}{2}} W_{\eta-1} + \gamma_{1, \frac{1}{2}} W_\eta + \gamma_{\frac{3}{2}, \frac{1}{2}} W_{\eta+\frac{1}{2}} - h\psi_{\frac{1}{2}} \left(\phi_{\eta+\frac{1}{2}} - \rho\phi_{\eta-\frac{3}{2}} \right) = 0 \quad (6)$$

We use Taylor's series to expand equation (6) about x_n and collect like terms to gives

$$P_{0, \frac{1}{2}} W(x_\eta) + P_{1, \frac{1}{2}} h W(x_\eta) + P_{2, \frac{1}{2}} h^2 W(x_\eta) + P_{3, \frac{1}{2}} h^3 W(x_\eta) + \dots = 0 \quad (7)$$

Where,

$$\left. \begin{aligned} P_{0, \frac{1}{2}} &= \gamma_{0, \frac{1}{2}} + \gamma_{1, \frac{1}{2}} + \gamma_{\frac{3}{2}, \frac{1}{2}} = 0 \\ P_{1, \frac{1}{2}} &= -r\gamma_{0, \frac{1}{2}} + \frac{1}{2}\gamma_{\frac{3}{2}, \frac{1}{2}} - \beta_{\frac{1}{2}} + \rho\psi_{\frac{1}{2}} = 0 \\ P_{2, \frac{1}{2}} &= \frac{r^2}{2}\gamma_{0, \frac{1}{2}} + \frac{1}{8}\gamma_{\frac{3}{2}, \frac{1}{2}} - \frac{1}{2}\beta_{\frac{1}{2}} - \frac{3}{2}r\rho\psi_{\frac{1}{2}} = 0 \\ P_{3, \frac{1}{2}} &= -\frac{r^3}{6}\gamma_{0, \frac{1}{2}} + \frac{1}{48}\gamma_{\frac{3}{2}, \frac{1}{2}} - \frac{1}{48}\beta_{\frac{1}{2}} - \frac{r^3}{2}\rho\psi_{\frac{1}{2}} = 0 \end{aligned} \right\} \quad (8)$$

To derive the first point $W_{\eta+\frac{1}{2}}$ we normalize the coefficient $\gamma_{\frac{3}{2}, \frac{1}{2}}$ 1. Solving the simultaneous equations for $\gamma_{j, \frac{1}{2}}$ and $\psi_{j, \frac{1}{2}}$ gives

$$\gamma_{0,\frac{1}{2}} = \frac{1}{4} \frac{6pr + \rho + 1}{r(2pr + r + 1)}, \gamma_{1,1} = -\frac{1}{4} \frac{8pr^2 + 6pr + 4r^2 + \rho + 4r + 1}{r(2pr + r + 1)},$$

$$\psi_{\frac{1}{2}} = \frac{1}{4} \frac{2r + 1}{2pr + r + 1}$$

Substituting the values into equation (6), we have

$$W_{\eta+\frac{1}{2}} = -\frac{1}{4} \frac{6pr + \rho + 1}{r(2pr + r + 1)} W_{\eta-1} - \frac{1}{4} \frac{8pr^2 + 6pr + 4r^2 + \rho + 4r + 1}{r(2pr + r + 1)} W_{\eta}$$

$$+ \frac{1}{4} \frac{2r + 1}{2pr + r + 1} h\phi_{\eta+\frac{1}{2}} - \frac{1}{4} \frac{2r + 1}{2pr + r + 1} \rho h\phi_{\eta-\frac{3}{2}} \quad (9)$$

We apply the same procedure for $k = i = 1, \frac{3}{2}, 2$.

Therefore, the hybrid block method is obtained as:

$$W_{\eta+\frac{1}{2}} = -\frac{1}{4} \frac{6pr + \rho + 1}{r(2pr + r + 1)} W_{\eta-1} - \frac{1}{4} \frac{8pr^2 + 6pr + 4r^2 + \rho + 4r + 1}{r(2pr + r + 1)} W_{\eta} +$$

$$\frac{1}{4} \frac{2r + 1}{2pr + r + 1} h\phi_{\eta+\frac{1}{2}} - \frac{1}{4} \frac{2r + 1}{2pr + r + 1} \rho h\phi_{\eta-\frac{3}{2}} \quad (10)$$

$$W_{\eta+1} = \frac{6pr^2 + 6pr + \rho - 1}{r(2pr^2 + pr - 3r - 4)(2r + 1)} W_{\eta-1} - \frac{2pr^3 + 5pr^2 + 4pr - r^2 + \rho - 2r - 1}{r(2pr^2 + pr - 3r - 4)} W_{\eta}$$

$$+ \frac{8(pr^3 + 2pr^2 + pr - r^2 - 2r - 1)}{(2pr^2 + pr - 3r - 4)(2r + 1)} W_{\eta+\frac{1}{2}} - \frac{r + 1}{2pr^2 + pr - 3r - 4} h\phi_{\eta+1}$$

$$+ \frac{r + 1}{2pr^2 + pr - 3r - 4} \rho h\phi_{\eta-1} \quad (11)$$

$$W_{\eta+\frac{3}{2}} = \frac{3(2pr^3 + 9pr^2 + 11pr + 3\rho + 3)}{r(2pr^3 + 3pr^2 - 22r - 39)(2r + 1)(r + 1)} W_{\eta-1}$$

$$+ \frac{2pr^4 + 9pr^3 + 9pr^2 - 6pr - 4r^2 - 9\rho - 12r - 9}{r(2pr^3 + 3pr^2 - 22r - 39)} W_{\eta}$$

$$+ \frac{3(4pr^4 + 16pr^3 + 15pr^2 - 12r^2 - 36r - 27)}{(2r + 1)(2pr^3 + 3pr^2 - 22r - 39)} W_{\eta+\frac{1}{2}}$$

$$+ \frac{3(2pr^4 + 7pr^3 + 6pr^2 - 12r^2 - 36r - 27)}{(2pr^3 + 3pr^2 - 22r - 39)(r + 1)} W_{\eta+1}$$

$$- \frac{3(2r + 3)}{2pr^3 + 3pr^2 - 22r - 39} h\phi_{\eta+1} + \frac{3(2r + 3)}{2pr^3 + 3pr^2 - 22r - 39} \rho h\phi_{\eta-1} \quad (12)$$

$$\begin{aligned}
W_{n+2} = & -\frac{36(\rho-1)}{(6\rho^2+9\rho r+50r^2+187r+168)(r+1)(2r+1)r}W_{\eta-1} \\
& -\frac{25\rho^2+44\rho r+3r^2-12\rho+12r+12}{r(3\rho r+25r+56)}W_{\eta} \\
& +\frac{32(3\rho^2+6\rho r+r^2+4r+4)}{(2r+1)(3\rho r+25r+56)}W_{\eta+\frac{1}{2}}-\frac{36(\rho^2+2\rho r+r^2+4r+4)}{(r+1)(3\rho r+25r+56)}W_{\eta+1} \\
& +\frac{32(\rho^2+2\rho r+3r^2+12r+12)}{6\rho^2+9\rho r+50r^2+187r+168}W_{\eta+\frac{3}{2}} \\
& +\frac{6(r+2)}{3\rho r+25r+56}h\phi_{\eta+2}-\frac{6(r+2)}{3\rho r+25r+56}\rho h\phi_{\eta}
\end{aligned} \tag{13}$$

For stability purposes, ρ is chosen in the interval $(-1, 1)$ as in [7]. By choosing $\rho = \frac{2}{5}$, and for different values of r in equation (10), (11), (12) and (13), the new method becomes:

For $r = 1$

$$\left. \begin{aligned}
W_{\eta+\frac{1}{2}} &= -\frac{19}{56}W_{\eta-1} + \frac{75}{76}W_{\eta} + \frac{15}{56}h\phi_{\eta+\frac{1}{2}} - \frac{6}{56}h\phi_{\eta-\frac{3}{2}} \\
W_{\eta+1} &= -\frac{7}{29}W_{\eta-1} + \frac{4}{29}W_{\eta} + \frac{32}{29}W_{\eta+\frac{1}{2}} + \frac{10}{29}h\phi_{\eta+1} - \frac{4}{29}h\phi_{\eta} \\
W_{\eta+\frac{3}{2}} &= -\frac{13}{118}W_{\eta-1} + \frac{23}{59}W_{\eta} - \frac{61}{59}W_{\eta+\frac{1}{2}} + \frac{207}{118}W_{\eta+1} + \frac{15}{59}h\phi_{\eta+\frac{3}{2}} - \frac{6}{59}h\phi_{\eta} \\
W_{\eta+2} &= \frac{6}{685}W_{\eta-1} - \frac{83}{137}W_{\eta} + \frac{224}{137}W_{\eta+\frac{1}{2}} - \frac{306}{137}W_{\eta+1} + \frac{1504}{685}W_{\eta+\frac{3}{2}} + \frac{30}{137}h\phi_{\eta+2} - \frac{12}{137}h\phi_{\eta}
\end{aligned} \right\} \tag{14}$$

For $r = 2$

$$\left. \begin{aligned}
W_{\eta+\frac{1}{2}} &= -\frac{31}{184}W_{\eta-1} + \frac{215}{184}W_{\eta} + \frac{25}{92}h\phi_{\eta+\frac{1}{2}} - \frac{10}{92}h\phi_{\eta-\frac{3}{2}} \\
W_{\eta+1} &= -\frac{23}{100}W_{\eta-1} + \frac{3}{4}W_{\eta} + \frac{12}{25}W_{\eta+\frac{1}{2}} + \frac{1}{2}h\phi_{\eta+1} - \frac{1}{5}h\phi_{\eta-1} \\
W_{\eta+\frac{3}{2}} &= -\frac{169}{3590}W_{\eta-1} + \frac{7}{718}W_{\eta} - \frac{693}{1795}W_{\eta+\frac{1}{2}} + \frac{511}{359}W_{\eta+1} + \frac{105}{359}h\phi_{\eta+\frac{3}{2}} - \frac{42}{359}h\phi_{\eta-\frac{1}{2}} \\
W_{\eta+2} &= \frac{9}{9485}W_{\eta-1} - \frac{148}{271}W_{\eta} + \frac{2048}{1355}W_{\eta+\frac{1}{2}} - \frac{576}{271}W_{\eta+1} + \frac{4096}{1897}W_{\eta+\frac{3}{2}} + \frac{60}{271}h\phi_{\eta+2} - \frac{24}{271}h\phi_{\eta}
\end{aligned} \right\} \tag{15}$$

For $r = 5/8$

$$\left. \begin{aligned}
 W_{\eta+\frac{1}{2}} &= -\frac{232}{425}W_{\eta-1} + \frac{657}{425}W_{\eta} + \frac{9}{34}h\phi_{\eta+\frac{1}{2}} - \frac{9}{85}h\phi_{\eta-\frac{3}{2}} \\
 W_{\eta+1} &= -\frac{7}{29}W_{\eta-1} + \frac{4}{29}W_{\eta} + \frac{32}{29}W_{\eta+\frac{1}{2}} + \frac{10}{29}h\phi_{\eta+1} - \frac{4}{29}h\phi_{\eta-1} \\
 W_{\eta+\frac{3}{2}} &= -\frac{13}{118}W_{\eta-1} + \frac{23}{59}W_{\eta} - \frac{61}{59}W_{\eta+\frac{1}{2}} + \frac{207}{118}W_{\eta+1} + \frac{15}{59}h\phi_{\eta+\frac{3}{2}} - \frac{6}{59}h\phi_{\eta-\frac{1}{2}} \\
 W_{\eta+2} &= \frac{6}{685}W_{\eta-1} - \frac{83}{137}W_{\eta} + \frac{224}{137}W_{\eta+\frac{1}{2}} - \frac{306}{137}W_{\eta+1} + \frac{1504}{685}W_{\eta+\frac{3}{2}} + \frac{30}{137}h\phi_{\eta+2} - \frac{12}{137}h\phi_{\eta}
 \end{aligned} \right\} \quad (16)$$

Order of the Method

This section derives the order for the proposed method corresponding to equation (10), (11), (12), and (13). First, we consider the method when $r = 1$ which can be rearranged as

$$\left. \begin{aligned}
 W_{\eta+\frac{1}{2}} + \frac{19}{56}W_{\eta-1} - \frac{75}{76}W_{\eta} &= \frac{15}{56}h\phi_{\eta+\frac{1}{2}} - \frac{6}{56}h\phi_{\eta-\frac{3}{2}} \\
 W_{\eta+1} + \frac{7}{29}W_{\eta-1} - \frac{4}{29}W_{\eta} - \frac{32}{29}y_{\eta+\frac{1}{2}} &= \frac{10}{29}h\phi_{\eta+1} - \frac{4}{29}h\phi_{\eta} \\
 W_{\eta+\frac{3}{2}} + \frac{13}{118}W_{\eta-1} - \frac{23}{59}W_{\eta} + \frac{61}{59}y_{\eta+\frac{1}{2}} - \frac{207}{118}W_{\eta+1} &= \frac{15}{59}h\phi_{\eta+\frac{3}{2}} - \frac{6}{59}h\phi_{\eta} \\
 W_{\eta+2} - \frac{6}{685}W_{\eta-1} + \frac{83}{137}W_{\eta} - \frac{224}{137}W_{\eta+\frac{1}{2}} + \frac{306}{137}W_{\eta+1} - \frac{1504}{685}W_{\eta+\frac{3}{2}} &= \frac{30}{137}h\phi_{\eta+2} - \frac{12}{137}h\phi_{\eta}
 \end{aligned} \right\} \quad (17)$$

We rewrite the formulae (17) in matrix form as follows

$$\begin{pmatrix} 0 & \frac{19}{56} & 0 & -\frac{75}{56} \\ 0 & \frac{7}{29} & 0 & -\frac{4}{29} \\ 0 & \frac{13}{118} & 0 & -\frac{23}{59} \\ 0 & -\frac{6}{685} & 0 & \frac{83}{137} \end{pmatrix} \begin{pmatrix} W_{\eta-\frac{3}{2}} \\ W_{\eta-1} \\ W_{\eta-\frac{1}{2}} \\ W_{\eta} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{32}{29} & 1 & 0 & 0 \\ \frac{61}{59} & -\frac{207}{118} & 1 & 0 \\ -\frac{224}{137} & \frac{306}{137} & -\frac{1504}{685} & 1 \end{pmatrix} \begin{pmatrix} W_{\eta+\frac{1}{2}} \\ W_{\eta+1} \\ W_{\eta+\frac{3}{2}} \\ W_{\eta+2} \end{pmatrix} =$$

$$h \begin{pmatrix} -\frac{6}{56} & 0 & 0 & 0 \\ 0 & -\frac{4}{29} & 0 & 0 \\ 0 & 0 & -\frac{6}{59} & 0 \\ 0 & 0 & 0 & -\frac{12}{137} \end{pmatrix} \begin{pmatrix} \phi_{\eta-\frac{3}{2}} \\ \phi_{\eta-1} \\ \phi_{\eta-\frac{1}{2}} \\ \phi_{\eta} \end{pmatrix} + h \begin{pmatrix} \frac{15}{56} & 0 & 0 & 0 \\ 0 & \frac{10}{29} & 0 & 0 \\ 0 & 0 & \frac{15}{59} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \phi_{\eta+\frac{1}{2}} \\ \phi_{\eta+1} \\ \phi_{\eta+\frac{3}{2}} \\ \phi_{\eta+2} \end{pmatrix} \quad (18)$$

Let

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \gamma_1 = \begin{pmatrix} \frac{19}{56} \\ \frac{7}{29} \\ \frac{13}{118} \\ -\frac{6}{685} \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} -\frac{75}{56} \\ \frac{4}{29} \\ \frac{23}{59} \\ \frac{83}{137} \end{pmatrix}, \gamma_4 = \begin{pmatrix} 1 \\ -\frac{32}{29} \\ \frac{61}{59} \\ -\frac{224}{137} \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 \\ 1 \\ -\frac{207}{118} \\ \frac{306}{137} \end{pmatrix}, \\ \gamma_6 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1504}{685} \end{pmatrix}, \gamma_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \psi_0 = \begin{pmatrix} -\frac{6}{56} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \psi_1 = \begin{pmatrix} 0 \\ -\frac{4}{29} \\ 0 \\ 0 \end{pmatrix}, \psi_2 = \begin{pmatrix} 0 \\ 0 \\ -\frac{6}{59} \\ 0 \end{pmatrix}, \psi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{12}{137} \end{pmatrix}, \\ \psi_4 &= \begin{pmatrix} \frac{15}{56} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \psi_5 = \begin{pmatrix} 0 \\ \frac{10}{29} \\ 0 \\ 0 \end{pmatrix}, \psi_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{15}{59} \\ 0 \end{pmatrix}, \psi_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{30}{137} \end{pmatrix} \end{aligned}$$

The technique for determining the order of a block method used by [1] is implemented on equation (18) to find the order of (17). The value of E_v is therefore given below

When $v = 0$,

$$E_0 = \sum_{j=0}^7 \gamma_j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (19)$$

When $v = 1$,

$$E_1 = \sum_{j=0}^7 j\gamma_j - \sum_{j=0}^7 2\psi_j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

When $v = 2$,

$$E_2 = \sum_{j=0}^7 j^2 \frac{\gamma_j}{2!} - \sum_{j=0}^7 2j\psi_j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (21)$$

When $v = 3$,

$$E_3 = \sum_{j=0}^7 j^3 \frac{\gamma_j}{3!} - \sum_{j=0}^7 2j^2 \frac{\psi_j}{2!} = \begin{pmatrix} \frac{23}{56} \\ \frac{25}{87} \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (22)$$

Hence, the proposed method is of order 2 with error constant $\begin{pmatrix} \frac{23}{56} \\ \frac{25}{87} \\ 0 \\ 0 \end{pmatrix}$

Applying similar procedure to the method (15) and (16) shows that their orders are also two.

Stability Analysis

This section analyses the stability of the proposed method based on zero-stability. This leads to figuring out where the roots of that stability polynomial end up relative to the unit circle in the complex plane. Therefore, we apply the scalar linear test differential equation, to find the stability region of the proposed method (13):

$W' = \tau w$, where τ is a complex constant and $Re(\tau) < 0$

Taking the method (14) when $r = 1$, we rewrite it as:

$$\left. \begin{aligned} W_{\eta+\frac{1}{2}} - \frac{15}{56}h\phi_{\eta+\frac{1}{2}} &= -\frac{19}{56}W_{\eta-1} + \frac{75}{76}W_{\eta} - \frac{6}{56}h\phi_{\eta-\frac{3}{2}} \\ W_{\eta+1} - \frac{32}{29}\phi_{\eta+\frac{1}{2}} - \frac{10}{29}h\phi_{\eta+1} &= -\frac{7}{29}W_{\eta-1} + \frac{4}{29}W_{\eta} - \frac{4}{29}h\phi_{\eta} \\ W_{\eta+\frac{3}{2}} - \frac{207}{118}W_{\eta+1} + \frac{61}{59}\phi_{\eta+\frac{1}{2}} - \frac{15}{59}h\phi_{\eta+\frac{3}{2}} &= -\frac{13}{118}W_{\eta-1} + \frac{23}{59}W_{\eta} - \frac{6}{59}h\phi_{\eta} \\ W_{\eta+2} + \frac{306}{137}W_{\eta+1} - \frac{224}{137}\phi_{\eta+\frac{1}{2}} - \frac{1504}{685}W_{\eta+\frac{3}{2}} - \frac{30}{137}h\phi_{\eta+2} &= \frac{6}{685}W_{\eta-1} - \frac{83}{137}W_{\eta} - \frac{12}{137}h\phi_{\eta} \end{aligned} \right\} \quad (23)$$

By substituting $W' = \tau w$ and simplifying, to obtain the stability polynomial, we evaluate the equation:

$$R(s, \bar{h}) = \det(As - B) = 0 \quad (24)$$

Which we obtain the stability polynomial as follows:

$$\begin{aligned} R(t, \bar{h}) := & s^4 + \frac{216}{1640849}\bar{h}^{-4} - \frac{11471678}{8204245}s^3 + \frac{1872}{1640849}\bar{h}^{-3} - \frac{14254385}{13126792}s^4\bar{h}^{-} \\ & - \frac{2160}{1640849}s\bar{h}^{-4} + \frac{16875}{3281698}s^4\bar{h}^{-4} - \frac{255375}{3281698}s^4\bar{h}^{-3} - \frac{13500}{1640849}s^3\bar{h}^{-4} + \frac{5749125}{13126792}s^4\bar{h}^{-2} \\ & + \frac{8100}{1640849}s^2\bar{h}^{-4} - \frac{82011}{6563396}s\bar{h}^{-} + \frac{1415433}{13126792}s^2\bar{h}^{-2} + \frac{27020943}{65633960}s^2\bar{h}^{-} - \frac{221127}{16408490}s\bar{h}^{-2} \\ & + \frac{3267433}{8204245}s^2 + \frac{25929}{16408490}\bar{h}^{-2} + \frac{4275}{56581}s^3\bar{h}^{-3} - \frac{2528355}{6563396}s^3\bar{h}^{-2} \\ & - \frac{26190}{1640849}s^2\bar{h}^{-3} - \frac{1695895}{3281698}s^3\bar{h}^{-} - \frac{5868}{1640849}s\bar{h}^{-3} \end{aligned} \quad (25)$$

To show the method is zero stable, we substitute $\bar{h} = 0$ in (25), which gives us the characteristics polynomial given as:

$$s^4 - \frac{11471678}{8204245}s^3 + \frac{3267433}{8204245}s^2 \quad (26)$$

Solving equation (26), we obtain the roots as follows:

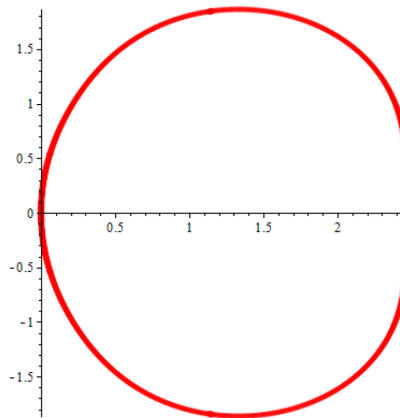
$$s = 0, s = 0, s = 1, s = \frac{3267433}{8204245}$$

The values of s imply that the proposed method is zero stable since none of the roots is above one and the root $s = 1$ is simple.

To find the boundary of the stability region of (14), we input $s = e^{-i\theta}$ into (25).

We plot the stability region with maple and is given below.

The Stability region is the area outside the circle which confirms A-stability as the stability region spans the complete left plane.

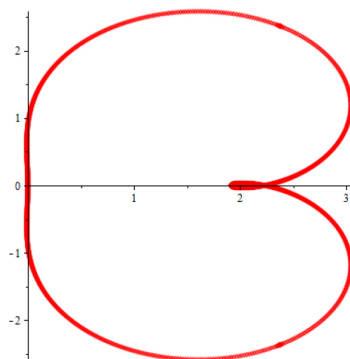


$$r = 1$$

Figure 1

Plot of stability region for $r = 1$

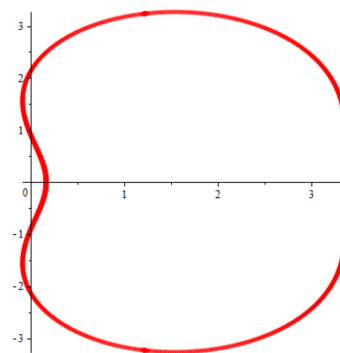
Similarly, applying the same process to the other methods gives



$$r = 2$$

Figure 2

Plot of stability region for $r = 2$



$$r = \frac{5}{8}$$

Figure 3

Plot of stability region for $r = 5/8$

The stability plots given in the Figure 1, Figure 2 and Figure 3 reveal how the method's stability region behaves as r changes. For $r = 1$ and $r = 2$, the VSBDF is A-stable, which means its stability region contains the entire left half of the complex plane. The problems with eigenvalues lying in the left half-plane are guaranteed to be solved with excellent stability. For $r = 5/8$, the method is stiffly stable. That is, the stability region does not cover the complete left half-plane in the strict A-stability sense, but it covers a large enough portion including a neighborhood of the negative real axis. This shows that the VSBDF is a robust choice for integrating stiff differential equations.

Consistency and Convergence

A Linear multistep method that satisfies $\sum_{j=0}^7 j\gamma_j = \sum_{j=0}^7 2\psi_j$ is consistent.

$$\begin{aligned} \sum_{j=0}^7 j\gamma_j &= \sum_{j=0}^7 2\psi_j \\ \gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 5\gamma_5 + 6\gamma_6 + 7\gamma_7 \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{19}{56} \\ \frac{7}{29} \\ \frac{13}{118} \\ \frac{6}{685} \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} -\frac{75}{56} \\ -\frac{4}{29} \\ -\frac{23}{59} \\ \frac{83}{137} \end{pmatrix} + 4 \begin{pmatrix} 1 \\ -\frac{32}{29} \\ \frac{61}{59} \\ -\frac{224}{137} \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ -\frac{207}{118} \\ \frac{306}{137} \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1504}{685} \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{9}{28} \\ \frac{12}{29} \\ \frac{18}{59} \\ \frac{36}{137} \end{pmatrix} \\ &2(\psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7) \\ &= \begin{pmatrix} -\frac{6}{56} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{4}{29} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{6}{59} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{12}{137} \end{pmatrix} + \begin{pmatrix} \frac{15}{56} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{10}{29} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{15}{59} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{30}{137} \end{pmatrix} = \begin{pmatrix} \frac{9}{28} \\ \frac{12}{29} \\ \frac{18}{59} \\ \frac{36}{137} \end{pmatrix} \end{aligned}$$

Therefore, the method is consistent.

A numerical method is convergent if it satisfies the conditions of consistency and zero stability, [6]. Consistency ensures that the local truncation error diminishes as the step size approaches zero, while zero stability guarantees that errors do not amplify as computations progress. We established that the proposed method has an order of 2, confirming its

consistency. In the previous sections, the method was found to be zero stable. Therefore, since both consistency and also zero stability are satisfied, the proposed method is convergent.

Conclusion

This paper introduced a method for the solution of ordinary differential equations. The method proposed enhances computational efficiency and accuracy, particularly in problems with regions of stiffness and rapid solution variations. Analysis show that the proposed method is both zero stable and A-stable. It was also found to be convergent; therefore, we conclude that it can serve as an alternative method for the solution of ordinary differential equations.

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Received November, 2025