

A conjugate gradient framework for finite-time edge consensus in scaled multi-agent networks

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Abstract

The paper investigates the problem of achieving scaled consensus within a finite time in edge dynamics of hybrid multi-agent systems over undirected and directed networks. The main challenge lies in achieving consensus under heterogeneous edge scaling and dynamic topologies. By modeling edge interactions via the line graph and introducing edge scaling through nonzero factors $\beta_{ij} \neq 0$, we express the consensus behavior through a linear system representation. A CGM-based distributed algorithm is introduced to solve this system efficiently. According to the theoretical analysis, the algorithm reaches convergence in no more than M steps, with M being the edge count. Simulations confirm rapid convergence and improved accuracy over classical consensus protocols. The innovation lies in leveraging CGM for scalable, edge-based coordination with guaranteed finite-time performance.

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1. Introduction

Consensus and coordination in multi-agent systems (MASs) have become central themes in modern control theory and networked dynamical systems, as they underpin a wide range of distributed decision-making and computation schemes [1, 2, 3]. In a typical setting, a collection of autonomous agents repeatedly exchange information with their neighbors and adjust their local states with the aim of achieving a common objective. Classical consensus protocols generally enforce *nodal* agreement: the state of every agent converges to a common value, often interpreted as an average or weighted average of the initial conditions [4]. While this node-focused perspective has led to a rich body of theory and applications, it is not always the most natural description for systems where the key variables are inherently *relative*.

In many engineering and physical systems, the quantities of interest are associated with *interactions* between agents rather than with individual nodes. Examples include inter-vehicle distances in platooning, power flows along transmission lines, and packet flows over communication links. In such scenarios, it is more appropriate to work with dynamics defined on the *edges* of the underlying communication graph, i.e., variables that encode differences or flows between neighboring agents. This edge-centric viewpoint has stimulated a growing literature on edge consensus and coordination, illustrating its relevance in formation control, power networks, and routing problems [5, 6, 7, 8].

Another salient feature of practical MASs is that communication patterns are rarely static. Links may appear or disappear due to mobility, scheduling, failures, or environmental constraints, leading naturally to models with *switching* topologies. Such systems are often referred to as switched multi-agent systems (SMASs) and are commonly analyzed within the framework of switched and hybrid dynamical systems [9, 10]. For fixed graphs, consensus properties can be characterized using algebraic graph theory and Lyapunov methods [11]. However, once the network structure is allowed to change over time, new difficulties arise in guaranteeing convergence, especially under intermittent connectivity and potentially destabilizing switching signals [12].

Several complementary approaches have been proposed to cope with these challenges. One influential notion is that of *joint* or *uniform* connectivity, where consensus is ensured if the union of the communication graphs over sufficiently long time intervals maintains global reachability [13]. Another classical technique relies on *common quadratic Lyapunov functions (CQLFs)*, which provide convergence guarantees for arbitrary switching when all individual subsystems share a single quadratic Lyapunov function [14]. More recently, *impulsive control* strategies have attracted attention as an efficient way to reduce communication and actuation burdens. Instead of continuously updating, agents apply corrective inputs at discrete impulsive instants, which can still enforce consensus while lowering resource usage [15, 16, 17, 18]. Such protocols are particularly appealing in energy-limited platforms such as sensor networks or teams of mobile robots.

In parallel, there has been a shift from purely nodal viewpoints to formulations where the coordination objective is imposed at the level of *edges*. In edge consensus, the goal is to regulate the relative states between agents so that they satisfy prescribed relationships, rather than forcing all agents to share an identical value [5, 8, 7, 18, 19]. Mathematically, this leads to dynamics evolving on the line graph of the underlying communication graph: each edge in the original network becomes a node in the line graph, and the interactions between edges are encoded by the line graph Laplacian. The spectral properties of this operator directly influence the feasibility and performance of edge-based protocols, and consequently play a key role in the corresponding consensus analysis [20, 21].

A further generalization of the consensus concept, which has gained increasing attention in recent years, is *scaled consensus*. Instead of requiring all agents to converge to the same state, scaled consensus prescribes that the limiting configuration satisfies a set of fixed linear relationships among the states. In other words, agents asymptotically occupy values that are proportional to one another according to preassigned scaling factors. This more flexible objective is well-suited to applications where preserving ratios or distributing tasks according to given weights is more meaningful than enforcing identical values, such as in load sharing, distributed estimation with heterogeneous reliability, or resource allocation problems [22, 11].

Existing work on scaled consensus has mainly focused on nodal dynamics. For undirected graphs and constant scaling ratios, Meng and Jia [11] proposed distributed control laws guaranteeing scaled consensus and analyzed their convergence properties. For time-varying and switching topologies, Chen et al. [13] studied scaled consensus under piecewise continuous protocols using Lyapunov-based arguments. Impulsive approaches to scaled nodal consensus were further developed in [16, 17], where sufficient conditions for finite-time or fixed-time convergence were derived. These results establish important building blocks for scaled coordination, yet they predominantly remain at the node level.

Despite this progress, several important aspects remain underexplored. First, most available scaled consensus results are formulated in terms of agent states and do not explicitly incorporate edge dynamics, even though edge-level interactions are central in many practical systems. Second, many of the existing protocols are asymptotic in nature: convergence is guaranteed only as time tends to infinity, which may be inadequate for applications with stringent time constraints or limited computational resources. Third, the problem of *finite-time scaled edge consensus*, particularly for switched or hybrid multi-agent systems, has not been fully addressed. In addition, traditional consensus strategies can be computationally inefficient on large-scale or ill-conditioned networks, where slow convergence may become a serious bottleneck.

The present paper aims to address these gaps by developing a new framework for *finite-time scaled edge consensus* in multi-agent systems. Our approach builds on the conjugate gradient method (CGM), a well-established iterative scheme for solving large symmetric positive definite linear systems. By

recasting the scaled edge consensus objective as a linear algebraic problem over the edge space, we design a distributed CGM-based protocol that operates directly on edge variables. Unlike conventional diffusive consensus algorithms that rely on local averaging, the CGM exploits a sequence of carefully constructed search directions that are conjugate with respect to an appropriate inner product. This structure allows the method to recover the exact solution in a finite number of iterations, bounded by the dimension of the edge space, thereby yielding finite-time convergence in the ideal noiseless setting.

The proposed consensus protocol accommodates heterogeneous scaling factors and is implementable using only local information, without requiring global knowledge of the network. Its convergence is studied by combining spectral properties of the line graph Laplacian with the system matrices induced by the CGM update. We derive explicit conditions on the step-size parameters that guarantee stability and convergence, and we discuss both fixed and switching communication topologies, including hybrid multi-agent settings. The resulting scheme exhibits robustness with respect to changes in the network structure and provides significant benefits in terms of convergence speed and computational efficiency when compared with standard consensus protocols.

To substantiate the theoretical findings, we provide numerical simulations that benchmark the CGM-based protocol against several conventional consensus strategies. The results demonstrate faster convergence, improved scalability with respect to network size, and enhanced accuracy in achieving the desired scaled edge configuration.

The remainder of the paper is organized as follows. Section 2 introduces the necessary preliminaries and formulates the scaled edge consensus problem. In Section 3, we develop the main results on scaled consensus for edge dynamics in hybrid multi-agent systems (HMASs) under both undirected and directed communication structures, and establish conditions ensuring convergence. Section 4 presents simulation examples that illustrate and validate the effectiveness of the proposed approach. Finally, Section 5 summarizes the main contributions and outlines several directions for future research.

2. Background and Problem Statement

This section summarizes the basic tools from graph theory, matrix analysis, and iterative numerical methods that will be used in the sequel. The emphasis is on structures and properties that are particularly relevant for edge-based dynamics in multi-agent systems and for the design of a conjugate gradient method (CGM)–driven protocol with finite-time convergence.

2.1 Graph-Theoretic Foundations

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph, where $\mathcal{V} = \{v_1, \dots, v_n\}$ is the node (agent) set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix with nonnegative entries. A directed edge $(j, i) \in \mathcal{E}$ means that agent v_j can send information to agent v_i , and $a_{ij} > 0$ denotes the

corresponding communication weight. For each node v_i , its (in-)neighborhood is defined by

$$\mathcal{N}_i = \{j \in \mathcal{V} | a_{ij} > 0\},$$

that is, the set of nodes from which v_i can receive information. The in- and out-degrees of v_i are given by

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^n a_{ij}, \quad \deg_{\text{out}}(v_i) = \sum_{j=1}^n a_{ji}.$$

A directed graph is said to be *strongly connected* if, for any ordered pair of distinct nodes, there exists a directed path from the first to the second. A *directed spanning tree* of $\mathcal{G}$ is a directed subgraph containing all nodes such that one node, called the root, has directed paths to every other node.

In the undirected case, we consider a graph $G = (V, E, A)$ with vertex set $V = \{1, \dots, n\}$, edge set $E \subseteq V \times V$, and a symmetric adjacency matrix $A = [a_{ij}]$ with nonnegative entries satisfying $a_{ij} = a_{ji}$. The (combinatorial) Laplacian matrix associated with G is

$$\mathcal{L} = D - A,$$

where $D = \text{diag}(d_1, \dots, d_n)$ and $d_i = \sum_{j \neq i} a_{ij}$ is the degree of node i . It is well known that $\mathcal{L}$ is symmetric positive semidefinite and satisfies $\mathcal{L}\mathbf{1}_n = 0$, where $\mathbf{1}_n$ denotes the n -dimensional ones vector. The eigenvalues of $\mathcal{L}$ can be ordered as

$$0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \dots \leq \lambda_n(\mathcal{L}),$$

and the second smallest eigenvalue $\lambda_2(\mathcal{L})$, often called the *algebraic connectivity*, provides a quantitative measure of how well the graph is connected and directly impacts the convergence speed of many consensus algorithms.

2.2 Matrix Notation and Properties

We write $\mathbb{R}^n$ for the n -dimensional real vector space, and denote by $\mathbf{I}_n$ the $n \times n$ identity matrix. The vectors $\mathbf{1}_n$ and $\mathbf{0}_n$ respectively represent the n -dimensional all-ones and all-zeros column vectors. For a matrix A , its transpose is denoted by A^T . Given $x \in \mathbb{R}^n$, the Euclidean norm of x is

$$\|x\| = \sqrt{x^T x}.$$

A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called *nonnegative* if $a_{ij} \geq 0$ holds for all i, j , in which case we write $A \geq 0$. A row-stochastic matrix satisfies

$$\sum_{j=1}^n a_{ij} = 1, i = 1, \dots, n.$$

A stochastic matrix A is said to be *SIA* (stochastic, indecomposable, and aperiodic) if there exists a vector $v \in \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} A^k = \mathbf{1}_n v^T.$$

In this case, all rows of A^k converge to the same probability vector v^T , which is closely related to the stationary distribution of the corresponding Markov chain.

2.3 Conjugate Gradient Method (CGM) [23]

The conjugate gradient method is a classic iterative scheme for solving large-scale linear systems

$$Cx = b,$$

where $C \in \mathbb{R}^{n \times n}$ is symmetric positive definite (SPD) and $b \in \mathbb{R}^n$. The method is particularly attractive when C is sparse and high-dimensional, as often occurs in discretized partial differential equations and network-based models.

Given an initial guess $x^{(0)}$, the algorithm starts by forming the initial residual $r^{(0)} = b - Cx^{(0)}$ and setting the first search direction $p^{(0)} = r^{(0)}$. For $k = 0, 1, 2, \dots$, the iterations proceed as follows:

$$\begin{aligned} \alpha_k &= \frac{(r^{(k)})^T r^{(k)}}{(p^{(k)})^T C p^{(k)}}, & x^{(k+1)} &= x^{(k)} + \alpha_k p^{(k)}, \\ r^{(k+1)} &= r^{(k)} - \alpha_k C p^{(k)}, & \beta_k &= \frac{(r^{(k+1)})^T r^{(k+1)}}{(r^{(k)})^T r^{(k)}}, \\ p^{(k+1)} &= r^{(k+1)} + \beta_k p^{(k)}. \end{aligned}$$

The sequence $\{p^{(k)}\}$ generated in this way consists of search directions that are mutually conjugate with respect to the matrix C , in the sense that $(p^{(i)})^T C p^{(j)} = 0$ whenever $i \neq j$. Equivalently, the iterates minimize the quadratic functional

$$\varphi(x) = \frac{1}{2} x^T C x - b^T x$$

over an expanding sequence of affine subspaces. In exact arithmetic, the CGM produces the exact solution in at most n iterations.

For consensus problems, and in particular for edge-level coordination, one can construct C so that it encodes the coupling structure induced by the network Laplacian. A typical choice in the present setting is

$$C = \rho I + |\beta| Q',$$

where $\rho > 0$, the matrix Q' captures edge interactions, and the parameter β represents scaling effects in the edge dynamics. Under such a construction, the CGM can be viewed as a distributed scheme that drives edge states to a desired scaled configuration in a finite number of steps, without resorting to vanishing step sizes or purely asymptotic convergence arguments. This observation motivates the use of CGM-based protocols for fast consensus in large, sparse multi-agent networks.

We recall some standard notions that will be used later.

Definition 2.1: [21] Let $\mathcal{G}$ be an undirected graph with Laplacian matrix $\mathcal{L}$. The algebraic connectivity of $\mathcal{G}$, denoted $\lambda_2(\mathcal{L})$, is the second smallest eigenvalue of $\mathcal{L}$. It admits the variational characterization

$$\lambda_2(\mathcal{L}) = \min_{\substack{x \neq 0 \\ \mathbf{1}_n^T x = 0}} \frac{x^T \mathcal{L} x}{x^T x}.$$

Definition 2.2: Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite. A set of nonzero vectors $\{p_1, \dots, p_m\} \subset \mathbb{R}^n$ is said to be A -conjugate if

$$p_i^T A p_j = 0, \quad \text{for all } i \neq j.$$

Lemma 2.3: [24] *Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite. In exact arithmetic, the conjugate gradient method applied to the system $Ax = b$ reaches the exact solution after at most n iterations.*

2.4 Problem Formulation

Consider an undirected communication network described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. The associated adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined by

$$a_{ij} = a_{ji} = \begin{cases} 1, & \text{if } (i, j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

For directed communication, we allow asymmetric interactions. In this case, a directed edge $(i, j) \in \mathcal{E}$ indicates that node j influences node i , and the corresponding adjacency matrix is specified by

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x_{ij}(t) \in \mathbb{R}$ denote the state associated with edge $(i, j) \in \mathcal{E}$ at (discrete) time t , and let $\beta_{ij} \in \mathbb{R} \setminus \{0\}$ be a prescribed scaling coefficient for that edge. The edge dynamics are assumed to evolve according to the discrete-time update rule

$$\beta_{ij} x_{ij}(t_{k+1}) = \beta_{ij} x_{ij}(t_k) + u_{ij}(t_k), \quad (i, j) \in \mathcal{E}, \quad (2.1)$$

where $u_{ij}(t_k) \in \mathbb{R}$ denotes the control input applied to edge (i, j) at the sampling instant t_k . Let $M = |\mathcal{E}|$ be the number of edges. The initial condition of the system is encoded in the edge state vector

$$x(0) = [x_{ij}(0)]^T \in \mathbb{R}^M.$$

We now formalize the notion of scaled edge consensus adopted in this work.

Definition 2.4: The multi-agent system governed by (2.1) is said to achieve finite-time scaled edge consensus with respect to the scaling coefficients $\{\beta_{ij}\}_{(i,j) \in \mathcal{E}}$ if there exists a finite time $T > 0$ such that, for all $t \geq T$,

$$\beta_{ij} x_{ij}(t) = \beta_{ks} x_{ks}(t), \quad \forall (i, j), (k, s) \in \mathcal{E}.$$

In other words, all scaled edge states $\beta_{ij} x_{ij}(t)$ coincide after some finite time, independently of the initial edge states.

This notion generalizes classical edge consensus by allowing heterogeneous scaling factors on each edge, thereby capturing differences in interaction priorities, physical units, or desired relative weights. The main objective of this paper is to design distributed control laws $u_{ij}(t_k)$ that drive the system (2.1) to finite-time scaled edge consensus, and to establish conditions under which such convergence is guaranteed.

3. Main Results

In this section, we develop finite-time scaled edge consensus algorithms for multi-agent systems operating over both undirected and directed communication topologies. By representing interactions in terms of line graphs and incorporating edge-dependent scaling factors, the edge dynamics can be expressed as a linear discrete-time system. We then embed this system into a conjugate gradient method (CGM) framework and show that, under suitable conditions, the resulting protocol drives the scaled edge states to consensus in at most M iterations, where M is the number of edges. The admissible step size and convergence guarantees are characterized in terms of the spectral properties of the line graph Laplacian.

3.1 Undirected Communication Networks

Consider the edge-based model (2.1) for a multi-agent system whose communication structure is described by a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. At each discrete time t_k , we let $\mathcal{G}' = (\mathcal{V}, \mathcal{E}', \mathcal{A}')$ denote the interaction topology, and interpret two edges as adjacent whenever they share exactly one common endpoint. For every edge $(i, j) \in \mathcal{E}$, a nonzero scaling parameter $\beta_{ij} \in \mathbb{R} \setminus \{0\}$ is prescribed, and $h > 0$ denotes the sampling period.

We propose the following scaled edge-based consensus law:

$$u_{ij}(t_k) = h |\beta_{ij}| [\sum_{s \in \mathcal{N}_i} a_{is} (\beta_{is} x_{is}(t_k) - \beta_{ij} x_{ij}(t_k)) + \sum_{s \in \mathcal{N}_j} a_{js} (\beta_{js} x_{js}(t_k) - \beta_{ij} x_{ij}(t_k))], \quad (3.1)$$

where $\mathcal{N}_i$ denotes the neighbor set of node i , and $A(t_k) = [a_{ij}(t_k)] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix associated with $\mathcal{G}'(t_k)$.

Remark 1: If all scaling coefficients are identical, i.e., $\beta_{ij} = 1$ for every $(i, j) \in \mathcal{E}$, then (3.1) reduces to the standard edge-based diffusive protocol

$$u_{ij}(t_k) = h [\sum_{s \in \mathcal{N}_i} a_{is} (x_{is}(t_k) - x_{ij}(t_k)) + \sum_{s \in \mathcal{N}_j} a_{js} (x_{js}(t_k) - x_{ij}(t_k))], \quad (3.2)$$

which corresponds to the classical unscaled edge consensus rule.

Introduce the scaled edge variables $Y_{ij}(t_k) = \beta_{ij} x_{ij}(t_k)$ and collect them into $Y(t_k) = (Y_{ij}(t_k))^T \in \mathbb{R}^M$, where $M = |\mathcal{E}|$. Let $|\beta| = \text{diag}(|\beta_{ij}|) \in \mathbb{R}^{M \times M}$ be the diagonal matrix of absolute scaling factors. Under protocol (3.1) and a given switching rule determining $\mathcal{G}'(t_k)$, the scaled edge dynamics can be written in compact form as

$$Y(t_{k+1}) = (I_M + h |\beta| Q') Y(t_k), \quad (3.3)$$

where $Q' \in \mathbb{R}^{M \times M}$ is a symmetric matrix with zero row sums. Its off-diagonal entries are nonnegative and encode adjacency relations between edges in the line graph of $\mathcal{G}$, whereas the diagonal entries are given by

$$Q'_{ii} = - \left(\sum_{s \neq j} a_{is} + \sum_{s \neq i} a_{js} \right),$$

so that each row of Q' sums to zero.

Define

$$\beta_{\max} = \max_{(i,j) \in \mathcal{E}} |\beta_{ij}|, \quad \beta_{\min} = \min_{(i,j) \in \mathcal{E}} |\beta_{ij}|,$$

and let $\lambda_{\max}$ denote the largest eigenvalue of the Laplacian matrix associated with the line graph of $\mathcal{G}$. These parameters determine an admissible interval for the sampling period h that guarantees asymptotic scaled consensus under the diffusive dynamics (3.3).

Lemma 3.1: ([19]). *Let $\mathcal{G}$ be an undirected, connected communication graph. Consider the switched multi-agent system governed by (2.1) under the protocol (3.1). If the sampling period h satisfies*

$$0 < h < \frac{1}{2 \beta_{\max} \lambda_{\max}},$$

then the scaled edge states achieve asymptotic consensus under arbitrary switching of the interaction topology.

Theorem 3.2: *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a connected undirected graph with N nodes and M edges. For each edge $(i, j) \in \mathcal{E}$, let $x_{ij}(t) \in \mathbb{R}$ denote its associated state and define the scaled edge variable $Y_{ij}(t) = \beta_{ij} x_{ij}(t)$, where $\beta_{ij} \in \mathbb{R} \setminus \{0\}$. Stack all scaled edge states into the vector $Y(t) \in \mathbb{R}^M$, and let $|\beta| = \text{diag}(|\beta_{ij}|) \in \mathbb{R}^{M \times M}$.*

Suppose the scaled edge dynamics can be written as

$$Y(t_{k+1}) = (I_M + h |\beta| Q') Y(t_k),$$

where $h > 0$, and $Q' \in \mathbb{R}^{M \times M}$ is symmetric with zero row sums, nonnegative off-diagonal entries, and nonpositive diagonal entries. Define the matrix

$$C = \rho I_M + |\beta| Q',$$

where $\rho > 0$ is chosen such that C is symmetric positive definite.

Let

$$Y^* = \frac{1}{M} \mathbf{1}_M^T Y(0) \mathbf{1}_M \in \mathbb{R}^M$$

denote the limiting scaled consensus value, and set $b = \rho Y^$. Assume that the search directions $\{p_1, \dots, p_M\} \subset \mathbb{R}^M$ are C -conjugate,*

$$(p_i, C p_j) = 0, \quad \text{for all } i \neq j,$$

with $(\cdot, \cdot)$ denoting the standard Euclidean inner product. Then the iteration

$$Y(k) = Y(k-1) + t_k p_k, \quad t_k = \frac{(b - C Y(k-1), p_k)}{(C p_k, p_k)}, \quad k = 1, 2, \dots, M,$$

converges to Y^ in at most M steps. In particular, the scaled edge states reach finite-time scaled edge consensus.*

Proof: From (3.3), the scaled edge dynamics satisfy

$$Y(t_{k+1}) = Y(t_k) + h |\beta| Q' Y(t_k),$$

with Q' symmetric, zero-row-sum, and having nonpositive diagonal entries. Because $\mathcal{G}$ is connected, the underlying line graph is also connected, and Q' has a simple zero eigenvalue corresponding to the eigenvector $\mathbf{1}_M$, while all other eigenvalues are strictly negative.

By Lemma 3.1, suitable choices of h guarantee that the scaled edge states asymptotically converge to an average consensus value, which we denote by Y^* . At equilibrium, we have

$$Y^* = Y^* + h |\beta| Q' Y^*,$$

and thus

$$|\beta| Q' Y^* = 0. \quad (3.4)$$

Introduce a regularization term and define

$$C = \rho I_M + |\beta| Q', \quad b = \rho Y^*,$$

for some $\rho > 0$. Then (3.4) is equivalently written as

$$CY^* = (\rho I_M + |\beta| Q') Y^* = \rho Y^* = b,$$

so Y^* is the unique solution of the linear system $CY = b$.

By construction, C is symmetric, and with an appropriate choice of ρ , all its eigenvalues are positive, so C is symmetric positive definite. Hence, the conjugate gradient method can be applied to solve $CY = b$. Lemma 2.3 then ensures that, when the search directions $\{p_k\}$ are C -conjugate, the exact solution Y^* is obtained in at most M iterations. Therefore, the CGM-based update in the statement of the theorem drives $Y(k)$ to Y^* in a finite number of steps, which proves finite-time scaled edge consensus.

Corollary 3.2.1: *Suppose the assumptions of Theorem 3.2 hold, and additionally $\beta_{ij} = 1$ for all $(i, j) \in \mathcal{E}$. Then the unscaled edge state vector $X(t) = (x_{ij}(t)) \in \mathbb{R}^M$ evolves according to*

$$X(t_{k+1}) = (I_M + hQ')X(t_k),$$

and the CGM iteration

$$X(k) = X(k-1) + t_k p_k, \quad t_k = \frac{(b - CX(k-1), p_k)}{(Cp_k, p_k)},$$

with

$$C = \rho I_M + Q', \quad b = \rho X^*, \quad X^* = \frac{1}{M} \sum_{\ell=1}^M X_\ell(0) \mathbf{1}_M,$$

achieves finite-time edge consensus in at most M steps.

Proof: When $\beta_{ij} = 1$ for all edges, we have $Y(t) = X(t)$ and $|\beta| = I_M$. Equation (3.3) thus reduces to

$$X(t_{k+1}) = (I_M + hQ')X(t_k).$$

The remainder of the argument follows by applying Theorem 3.2 with $C = \rho I_M + Q'$, which is symmetric positive definite for $\rho > 0$. Consequently, CGM recovers the average edge consensus value X^* in at most M iterations.

Remark 2: Theorem 3.2 extends the classical edge consensus problem by allowing nonzero scaling parameters β_{ij} on each edge. These parameters offer an additional degree of freedom for modeling heterogeneous interaction strengths or priorities, and they can encode physical characteristics such as capacities or reliability levels associated with different edges. The diagonal scaling matrix $|\beta|$ preserves symmetry of the effective interaction matrix and thereby maintains the applicability of CGM, while enabling finite-time convergence to a non-uniform but structured edge configuration.

Remark 3: Corollary 3.2.1 shows that setting all scaling factors to unity reduces the proposed framework to the classical unscaled edge consensus scenario. In that case, the dynamics are governed directly by the (unweighted) edge interaction matrix Q' , and the limiting consensus value coincides with the average of the initial edge states. Thus, the scaled framework strictly contains the standard edge consensus problem as a special case, while retaining the finite-time convergence property guaranteed by the CGM-based design.

Remark 4: Traditional edge consensus protocols (e.g., [6]) typically rely on diffusive coupling and yield only asymptotic convergence. In contrast, the CGM-based approach exploits conjugate search directions, which are chosen to be optimal with respect to the underlying quadratic cost. This eliminates redundant directions in the update process and can dramatically accelerate convergence, especially for networks whose edge Laplacians are poorly conditioned. The finite-time result therefore highlights a clear advantage in convergence efficiency over classical asymptotic schemes, particularly in large-scale or time-critical applications.

3.2 Directed Communication Topology

We now extend the analysis to multi-agent systems operating over directed communication networks. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a strongly connected directed graph, with weighted adjacency matrix $\mathcal{A} = [a_{ij}]$. In the directed setting, it is natural to model edge-based dynamics via the *line graph* $\mathbb{L}(\mathcal{G})$, whose vertices correspond to the directed edges of $\mathcal{G}$. Each directed edge $(i, j) \in \mathcal{E}$ becomes a vertex in $\mathbb{L}(\mathcal{G})$, and there is an edge between two vertices of $\mathbb{L}(\mathcal{G})$ whenever the terminal node of one directed edge in $\mathcal{G}$ coincides with the initial node of the other. This construction preserves directionality and allows edge interactions to be encoded through the Laplacian of $\mathbb{L}(\mathcal{G})$ [5].

For each directed edge $(i, j) \in \mathcal{E}$, let $x_{ij}(t) \in \mathbb{R}$ denote its state, and define the scaled edge variable $Z_{ij}(t) = \beta_{ij}x_{ij}(t)$, where $\beta_{ij} \in \mathbb{R} \setminus \{0\}$ is a given scaling factor. Collecting all scaled edge states yields the vector $Z(t) = (Z_{ij}(t)) \in \mathbb{R}^M$, with $M = |\mathcal{E}|$, and $|\beta| = \text{diag}(|\beta_{ij}|) \in \mathbb{R}^{M \times M}$.

At discrete sampling times t_k , we consider the scaled edge protocol

$$u_{ij}(t_k) = h |\beta_{ij}| \sum_{s \in \mathcal{N}_j} a_{js} (\beta_{js} x_{js}(t_k) - \beta_{ij} x_{ij}(t_k)), \quad (3.5)$$

where $h > 0$ is the sampling period and $\mathcal{N}_j$ denotes the out-neighbor set of node j .

Remark 5: If $\beta_{ij} = 1$ for all edges, then (3.5) reduces to the familiar diffusive edge protocol

$$u_{ij}(t_k) = h \sum_{s \in \mathcal{N}_j} a_{js} (x_{js}(t_k) - x_{ij}(t_k)), \quad (3.6)$$

which governs unscaled edge consensus for directed multi-agent systems.

Let $Q' \in \mathbb{R}^{M \times M}$ denote the edge interaction matrix associated with the line graph $\mathbb{L}(\mathcal{G})$. We assume that each row of Q' sums to zero, that the off-diagonal entries are nonnegative, and that the diagonal entries satisfy

$$[Q']_{kk} = -\sum_{s \in \mathcal{N}_j \setminus \{i\}} a_{js}$$

for the edge corresponding to (i, j) . Under possibly switching directed topologies, the scaled edge dynamics admit the compact representation

$$Z(t_{k+1}) = (I_M + h |\beta| Q') Z(t_k), \quad (3.7)$$

where the matrix Q' may depend on t_k . As before, define

$$\beta_{\max} = \max_{(i,j) \in \mathcal{E}} |\beta_{ij}|, \quad \beta_{\min} = \min_{(i,j) \in \mathcal{E}} |\beta_{ij}|.$$

Let $L_e = -Q'$ be the (possibly weighted) edge Laplacian of $\mathbb{L}(\mathcal{G})$, and denote by $\lambda_{\max}$ the largest eigenvalue of L_e . These spectral quantities will again be pivotal in the analysis of stability and finite-time convergence under the CGM-based protocol.

Theorem 3.3: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a strongly connected directed graph with M edges. For each edge $(i, j) \in \mathcal{E}$, let $x_{ij}(t) \in \mathbb{R}$ be its state, and define the scaled edge state $Z_{ij}(t) = \beta_{ij} x_{ij}(t)$, where $\beta_{ij} \in \mathbb{R} \setminus \{0\}$. Let $Z(t) \in \mathbb{R}^M$ be the stacked vector of all $Z_{ij}(t)$, and let $|\beta| = \text{diag}(|\beta_{ij}|) \in \mathbb{R}^{M \times M}$.

Assume that, at each discrete time t_k ,

$$Z(t_{k+1}) = (I_M + h |\beta| Q') Z(t_k), \quad (3.8)$$

where $h > 0$ and $Q' \in \mathbb{R}^{M \times M}$ is symmetric with zero row sums, nonnegative off-diagonal entries, and diagonal entries $[Q']_{kk} = -\sum_{s \in \mathcal{N}_j \setminus \{i\}} a_{js}$ for the edge corresponding to (i, j) . Suppose that the matrix

$$C = \rho I_M + |\beta| Q'$$

is symmetric positive definite for some $\rho > 0$. Define the average scaled edge consensus value

$$Z^* = \frac{1}{M} \mathbf{1}_M^T Z(0) \mathbf{1}_M, b = \rho Z^*.$$

If the search directions $\{p_k\}_{k=1}^M \subset \mathbb{R}^M$ satisfy the C -conjugacy condition

$$(p_i, C p_j) = 0, \text{ for all } i \neq j,$$

then the CGM iteration

$$Z(k) = Z(k-1) + t_k p_k, t_k = \frac{(b - CZ(k-1), p_k)}{(Cp_k, p_k)}, k = 1, 2, \dots, M,$$

achieves finite-time scaled edge consensus in at most M steps, i.e., $Z(k) \rightarrow Z^*$.

Proof: From (3.8), the scaled edge dynamics satisfy

$$Z(t_{k+1}) = Z(t_k) + h |\beta| Q' Z(t_k).$$

Let Z^* denote the limiting consensus configuration of the scaled edge states. Under the strong connectivity assumption on $\mathcal{G}$ and the structural properties of Q' , the matrix Q' is symmetric with a simple zero eigenvalue associated with $\mathbf{1}_M$, and all remaining eigenvalues are strictly negative. Thus the equilibrium Z^* satisfies

$$Z^* = Z^* + h |\beta| Q' Z^*,$$

which implies

$$|\beta| Q' Z^* = 0.$$

Introduce the matrix

$$C = \rho I_M + |\beta| Q', b = \rho Z^*,$$

for some $\rho > 0$, so that

$$CZ^* = \rho Z^* + |\beta| Q' Z^* = \rho Z^* = b.$$

By assumption, C is symmetric positive definite, and hence the linear system $CZ = b$ has a unique solution, given by Z^* .

Applying the conjugate gradient method to $CZ = b$ and using C -conjugate search directions $\{p_k\}$, Lemma 2.3 guarantees that the exact solution is obtained in no more than M steps. Thus, the CGM iteration in the statement of the theorem drives $Z(k)$ to Z^* in at most M iterations, yielding finite-time scaled edge consensus.

Corollary 3.3.1: *Under the assumptions of Theorem 3.3, if all scaling factors are set to unity, i.e., $\beta_{ij} = 1$ for all $(i, j) \in \mathcal{E}$, then the unscaled edge state vector $X(t) = (x_{ij}(t)) \in \mathbb{R}^M$ evolves as*

$$X(t_{k+1}) = (I_M + hQ')X(t_k),$$

and finite-time edge consensus is achieved through the CGM iteration

$$X(k) = X(k-1) + t_k p_k, \quad t_k = \frac{(b - CX(k-1), p_k)}{(Cp_k, p_k)},$$

where

$$C = \rho I_M + Q', \quad b = \rho X^*, \quad X^* = \frac{1}{M} \mathbf{1}_M^T X(0) \mathbf{1}_M.$$

The iterates converge to X^* in at most M steps.

Remark 6: (Directed versus Undirected Edge Consensus via CGM). The CGM-based edge consensus protocol provides a unified mechanism for realizing finite-time edge coordination in both undirected and directed communication networks. In the undirected case, symmetry of the underlying graph naturally

leads to a symmetric positive definite update matrix. For directed networks, although the node-level adjacency structure is asymmetric, the induced edge interaction matrix Q' can be constructed to retain symmetry, for example through the structure of the line graph. Consequently, the composite matrix $C = \rho I_M + |\beta|Q'$ remains symmetric positive definite, enabling the direct application of CGM.

The main difference between the two settings lies in how Q' is generated from the underlying graph: in the directed case, the line graph explicitly encodes directionality in its adjacency relations. Nevertheless, once Q' has been symmetrized in an appropriate fashion, the same CGM framework applies, providing finite-time convergence guarantees and scalability that improve upon conventional asymptotic consensus methods.

Original Graph with 12 Agents (From Laplacian Matrix)

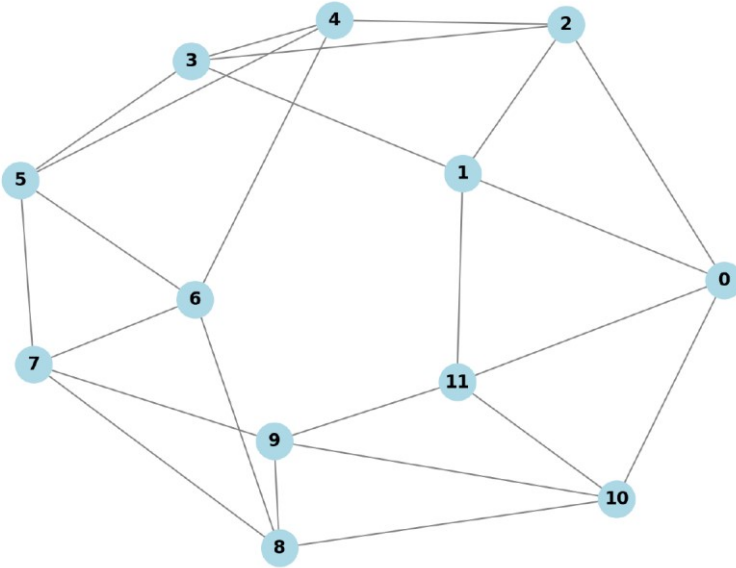


Figure 1
The communication network $\mathcal{G}$ with 12 agents.

4. Simulation Results and Interpretation

To illustrate the effectiveness of the proposed protocol based on CGM, we provide a simulation example involving a 12-agent network with edge-level interactions modeled via the line graph. The goal is to achieve finite-time scaled edge consensus under heterogeneous scaling. The example illustrates convergence properties, sensitivity to step size, and offers a performance comparison with the classical consensus protocol to highlight the advantages of the CGM approach.

Example 4.1: Let us examine a multi-agent system consisting of $N = 12$ agents, where the communication network is undirected and connected $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, as illustrated in Figure 1. The analysis is performed at the edge level using the line graph $\mathbb{L}(\mathcal{G})$ (see Figure 2), which contains $M = 24$ nodes representing edges of $\mathcal{G}$. The Laplacian matrix $L_{line} \in \mathbb{R}^{24 \times 24}$ encodes the structure of $\mathbb{L}(\mathcal{G})$, and the matrix $Q' = -L_{line}$ models edge-to-edge interaction.

Line Graph of the 12-Agent Network

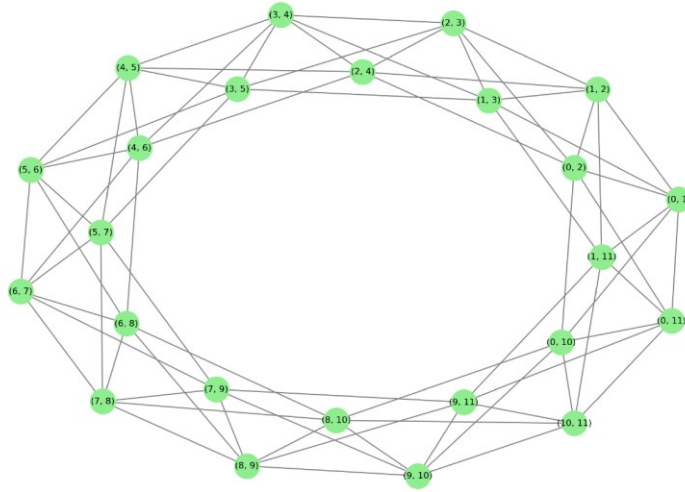


Figure 2
The line graph of $\mathcal{G}$ with 24 nodes.

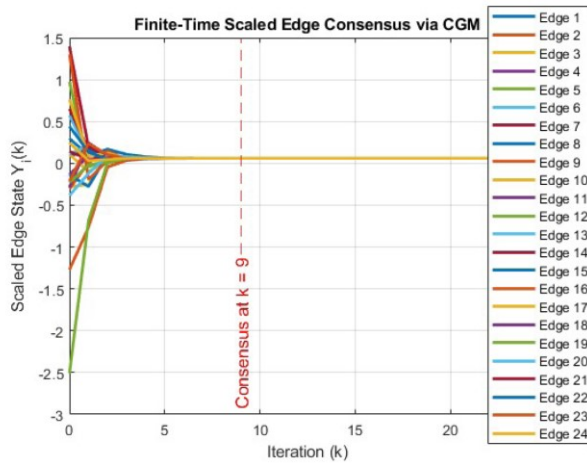


Figure 3
The scaled edge state trajectories using CGM with $h = 0.005$

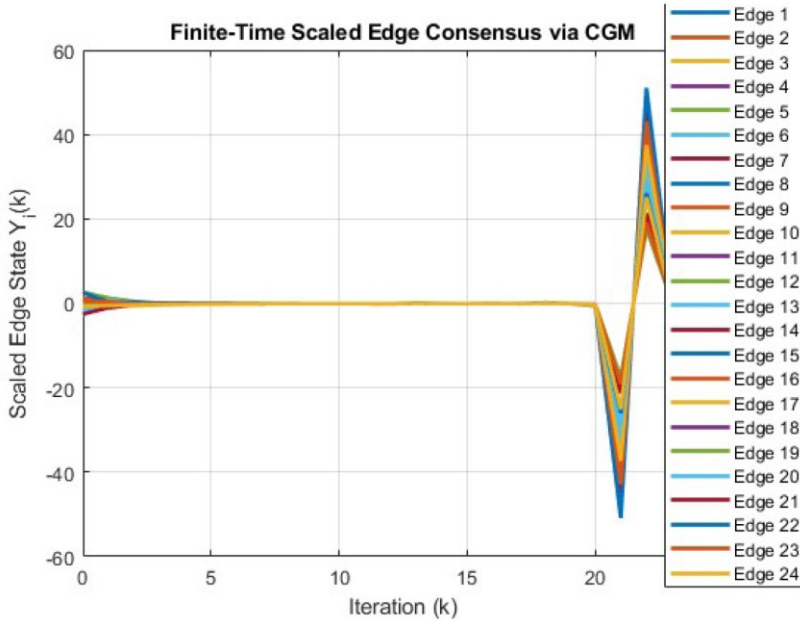


Figure 4

The scaled edge state trajectories using CGM with $h = 0.999$

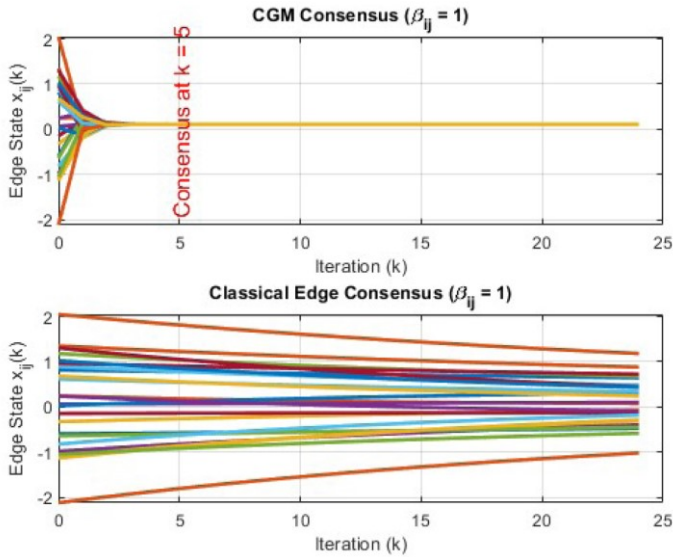


Figure 5

The scaled edge state trajectories using CGM with $h = 0.005$ and $\beta_{ij} = 1$ compared to the classical edge consensus

Every edge $(i, j) \in \mathcal{E}$ is assigned an initial state $x_{ij}(0) \in \mathbb{R}$, sampled from a standard normal distribution, i.e., $x_{ij}(0) \sim \mathcal{N}(0, 1)$. A heterogeneous positive scaling factor $\beta_{ij} \in [0.5, 2.0]$ is assigned to each edge, forming the diagonal scaling matrix $|\beta| = \text{diag}(|\beta_{ij}|) \in \mathbb{R}^{M \times M}$. The scaled edge state is then defined as:

$$Y(0) = |\beta|x(0),$$

where $x(0) \in \mathbb{R}^M$ is the vector of initial edge states. The objective is to achieve finite-time consensus over the scaled edge states, defined by the common target:

$$Y^* = \frac{1}{M} \sum_{i=1}^M Y_i(0) \cdot \mathbf{1}_M.$$

Accordingly, the conjugate gradient method (CGM) is utilized to solve the following linear equation:

$$CY = b, \text{ with } C = \rho I_M + h|\beta|Q', \text{ and } b = \rho Y^*,$$

with the regularization parameter $\rho = 0.1$, and a sampling step size $h = 0.005$. The CGM is initialized with $Y^{(0)} = Y(0)$, and updates are performed iteratively using the recurrence:

$$Y^{(k)} = Y^{(k-1)} + \alpha_k p_k, \alpha_k = \frac{(r_{k-1}^T p_{k-1})}{p_{k-1}^T C p_{k-1}}, r_k = b - CY^{(k)},$$

with p_k denoting the conjugate directions and r_k the residual vectors. Convergence is declared once the residual norm satisfies $\Pr_k P_2 < \varepsilon = 10^{-6}$.

As shown in Figure 3, exact scaled consensus is reached at iteration $k = 9$, thereby confirming the finite-time convergence guaranteed by Theorem 3.2. All scaled edge states converge to the average consensus value with high numerical accuracy.

Impact of Step Size Condition. The convergence of the proposed protocol, from a theoretical standpoint, depends on choosing the step size h such that the following stability condition holds:

$$0 < h < \frac{1}{2\beta_M \lambda_M}.$$

where $\beta_M = \max_{(i,j) \in \mathcal{E}} |\beta_{ij}|$, and λ_M denotes the largest eigenvalue of the line graph Laplacian L_{line} . This condition ensures that the matrix C remains symmetric positive definite, a prerequisite for guaranteed CGM convergence.

To demonstrate the sensitivity of the system to this condition, we simulate the protocol with an excessively large step size $h = 0.999$, which violates the bound. Figure 4 reveals the consequences: the system exhibits unstable behavior characterized by significant transient overshoots (reaching magnitudes of over 60 units) and delayed convergence, requiring 21 iterations to settle. This behavior highlights that inappropriate step-size selection may result in poor numerical performance or divergence, even under structurally sound communication topologies.

Comparison with Classical Edge Consensus. To benchmark the efficiency of the CGM approach, we consider the classical edge consensus

protocol for the special case where $\beta_{ij} = 1$ for all (i, j) . The classical discrete-time update law is:

$$x(t_{k+1}) = (I - hL_{\text{line}})x(t_k),$$

with the same step size $h = 0.005$ and initial edge state vector $x(0)$. Figure 5 illustrates that the classical protocol converges significantly slower and does not reach consensus within the 25 simulated iterations. In contrast, CGM achieves consensus in 9 steps under identical conditions, showcasing its superior convergence speed and robustness.

This simulation validates the theoretical claims made in Theorem 3.2. The CGM-based protocol not only ensures finite-time scaled consensus under heterogeneous conditions but also provides marked improvements over classical consensus approaches in terms of computational efficiency, convergence rate, and sensitivity to parameter selection.

5. Conclusion

This work studies consensus protocol for edge dynamics in multi-agent systems, founded on the conjugate gradient method (CGM), to facilitate efficient coordination under heterogeneous edge scaling. The proposed scheme is applicable to both undirected and directed communication topologies and guarantees convergence within a finite number of iterations, as rigorously demonstrated through theoretical analysis. Numerical simulations on a 12-agent network validated the theoretical results, demonstrating rapid convergence and superior performance compared to classical consensus protocols. The sensitivity of the CGM-based scheme to the choice of step size was also examined, highlighting the necessity of appropriate parameter selection to ensure numerical stability. These findings suggest the potential of CGM-based protocols for scalable and high-performance coordination in complex networked systems. Future work may explore extensions to networks with delays, switching topologies, and distributed optimization frameworks.

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Conflict of interest

The authors declare that they have no conflict of interest.

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