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Operator theory in quantum information processing

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Abstract

This paper looks at how operator theory can be used as a strict mathematical framework for processing quantum information. It discusses about how Hilbert spaces, spectral theory, and operator algebras can be used to describe quantum systems. The study also looks at Kraus decomposition and Stinespring dilation to talk about quantum channels and how they can be interpreted physically. In this study operator-theoretic method provide the huge data related to entanglement, decoherence, and channel stability by integrating the method as functional analysis with quantum physics.

Subject Classification: *54H30, 68Q09.*

Keywords: *Operator theory, Quantum channels, Kraus decomposition, Stinespring dilation, Quantum inform-ation processing.*

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I. Introduction

Operator theory is a strong mathematical framework for looking at and understanding how quantum information processing works at its most basic level [1]. This level of formality easily extends to quantum information theory, where quantum states change according to quantum channels, which are fully positive maps that keep the record [2, 3]. We learn more about how quantum information is sent and stored by studying things like operator spaces, Kraus representations, and Stinespring dilations.

II. Mathematical Foundations

A. *Hilbert Spaces and Linear Operators*

In mathematics, quantum states happen in Hilbert spaces, which are made up of linear operators that stand for observables, transformations and measurements (in figure 1).

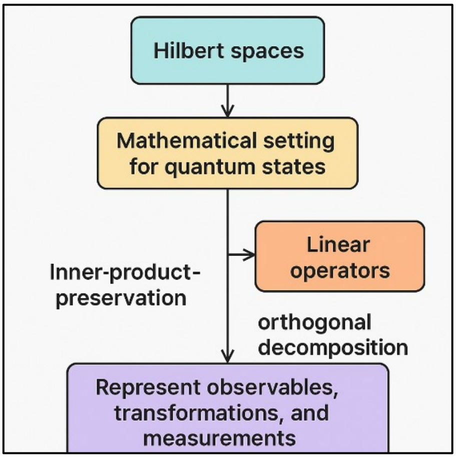

Figure 1
Flowchart of Hilbert Spaces and Linear Operators in Quantum Information Theory

These operators make sure that the inner product stays the same and that orthogonal decomposition happens [4, 5, 6].

Define a complex Hilbert space:

$$H = \{ |\psi\rangle : \langle\psi|\psi\rangle < \infty \}$$

Inner product:

$$\langle\varphi|\psi\rangle = \Sigma_i \varphi_i * \psi_i \tag{1}$$

Norm of a state vector:

$$\big||\psi\rangle\big||=|\sqrt{\langle\psi|\psi\rangle}$$

Orthonormal basis:

$$\langle e_i|e_j\rangle = \delta_{ij} \tag{2}$$

Expansion of a vector:

$$|\psi\rangle = \Sigma_i c_i |e_i\rangle \tag{3}$$

Linear operator action:

$$\hat{A}|\psi\rangle = \Sigma_i a_i |e_i\rangle \tag{4}$$

Adjoint operator:

$$\langle\varphi|\hat{A}\psi\rangle = \langle\hat{A} \dagger \varphi|\psi\rangle \tag{5}$$

Hermitian condition:

$$\hat{A} = \hat{A} \dagger$$

Eigenvalue equation:

$$\hat{A}|a_i\rangle = a_{i|a_i\rangle} \tag{6}$$

Projection operator:

$$P_i = |a_i\rangle\langle a_i|$$

Expectation value:

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

Completeness relation:

$$\Sigma_i |a_i\rangle\langle a_i| = I \tag{7}$$

B. *Spectral Theory and Functional Analysis*

Spectral theory looks at operator eigenvalues and eigenvectors to find measurable quantities in quantum systems [7], [8], [9]. Functional analysis, on the other hand, gives us precise tools for studying continuous operator behaviour and convergence [10, 11].

Self-adjoint operator:

$$\hat{A} = \hat{A} \dagger$$

Spectral decomposition:

$$\hat{A} = \Sigma_i a_i |a_i\rangle\langle a_i| \tag{8}$$

Continuous spectrum (integral form):

$$\hat{A} = \int a \, d \, E(a) \tag{9}$$

Functional calculus:

$$f(\hat{A}) = \Sigma_i f(a_i)|a_i\rangle\langle a_i| \tag{10}$$

Commutator relation:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{11}$$

Norm of operator:

$$\left||\hat{A}|\right| = \sup_{\{||\psi||=1\}} |(|\hat{A}\psi|)| \tag{12}$$

Bounded operator:

$$\left||\hat{A}\psi|\right| \leq M||\psi||$$

Unitary operator:

$$\hat{U} \dagger \hat{U} = I$$

Operator convergence:

$$A_n \to A \ iff \ A_{n\psi} \to A\psi$$

Compact operator property:

$$\mathrm{Spec}(\hat{A}) \to \{0\}$$

Resolvent definition:

$$R_{\lambda(\hat{A})} = \left(\lambda I - \hat{A}\right)^{-1} \tag{13}$$

Spectral radius:

$$r(\hat{A}) = \sup\{|\lambda|: \lambda \in Spec(\hat{A})\} \tag{14}$$

C. *Operator Algebras in Quantum Mechanics*

Operator algebras make it possible to study states, symmetries, and dynamics in a structured way within the structures of C*- and von Neumann algebras [12, 13]. They do this by formalising quantum observables and commutation relations [14].

*-Algebra property:

$$\hat{A} \in A \ \Rightarrow \ \hat{A} \dagger \in A$$

Commutant definition:

$$A' = \{\hat{B}: [\hat{B}, \hat{A}] = 0, \forall \hat{A} \in A\} \tag{15}$$

Observable representation:

$$\hat{O}_i = \Sigma_j o_{ij} |e_j\rangle\langle e_j| \tag{16}$$

State as positive functional:

$$\omega(\hat{A}) = Tr(\rho\hat{A}), \qquad \rho \geq 0$$

Expectation in algebraic form:

$$\omega(\hat{A}) = \langle \Omega_\omega | \pi_{\omega(\hat{A})| \, \Omega_\omega \rangle} \tag{17}$$

Algebraic dynamics:

$$\alpha_{t(\hat{A})} = e^{\{iHt\}\hat{A}} e^{\{-iHt\}} \tag{18}$$

III. Operator-Theoretic Framework for Quantum Channels

A. *Kraus Decomposition and Stinespring Dilation*

Kraus decomposition shows quantum channels as operator sums that keep the positivity, and Stinespring dilation shows them as unitary evolutions on extended Hilbert spaces.

Kraus Representation Theorem:

Every CPTP map Φ can be expressed as:

$$\Phi(\rho) = \Sigma_k E_k \rho \, E_k \dagger \tag{19}$$

where $\Sigma_k E_k \dagger E_k = I$

Integral Form of Channel:

$$\Phi(\rho) = \int E(\lambda)\, \rho\, E(\lambda) \dagger \ d\mu(\lambda) \quad (20)$$

Proof:

Trace Preservation Condition:

$$Tr[\Phi(\rho)] = Tr[\rho] \ \Rightarrow \ \Sigma_k E_k \dagger E_k = I \quad (21)$$

Adjoint Channel:

$$\Phi * (A) = \Sigma_k E_k \dagger \ A\, E_k \quad (22)$$

Unitary Extension (Stinespring's Theorem): *There exists a Hilbert space K and unitary U: H_in ⨂ K → H_out ⨂ K such that*

$$\Phi(\rho) = Tr_{K[\, U\, (\rho \otimes |0\rangle\langle 0| U\dagger)\,]} \quad (23)$$

Integral Stinespring Representation:

$$\Phi(\rho) = \int Tr_{K[\, U(\lambda)(\rho \otimes |0\rangle\langle 0|U)(\lambda)\dagger\,] d\mu(\lambda)} \quad (24)$$

Proof Sketch: Construct U from E_k by setting

$$U|\psi\rangle|0\rangle = \Sigma_k E_{k|\psi\rangle|k\rangle} \quad (25)$$

B. *Channel Representations in Operator Spaces*

You can describe quantum channels in operator spaces, which shows full positivity, tensor norms, and the structure relationships between input-output systems in quantum communication.

Definition (Operator Space): Integral Channel Form:

$$\Phi(\rho) = \int_\Omega A(\omega)\, \rho\, A(\omega) \dagger \ d\mu(\omega) \quad (26)$$

Theorem (Complete Positivity Criterion): *Φ is completely positive ⇔* $(I_n \otimes \ \Phi)(X) \geq \ 0 \ for\ all\ X \ \geq \ 0$ *in M_n(B(H_in)).*

Proof: Let {A(ω)} be measurable operator functions satisfying

$$\int_\Omega A(\omega) \dagger A(\omega) d\mu(\omega) = I, \quad (27)$$

then Φ is CPTP.

Theorem (Choi Isomorphism): *Φ ↔ C_Φ is bijective, and Φ is CP ⇔ C_Φ ≥ 0.*

Proof Sketch: Using vectorization,

$$\Phi(X) = Tr_{in[(I \otimes X^T) C_\Phi]}. \quad (28)$$

Norm Relations:

$$||\Phi||_{cb} = \sup_n ||I_n \otimes \ \Phi|| = ||C_\Phi||_\infty \quad (29)$$

Dual Operator Space Form:

$$\Phi * (A) = \int_{\Omega} A(\omega) \dagger \; A \, A(\omega) d\mu(\omega) \tag{30}$$

Channel Composition (Integral Form):

$$(\Phi^2 \circ \Phi^1)(\rho) = \int_{\Omega} \int_{\Lambda} B(\nu) \, A(\lambda) \rho \, A(\lambda) \dagger B(\nu) \dagger \; d\mu(\lambda) d\nu$$

Theorem (Operator-Space Isometry): *If Φ is unitary in operator space,*

$\int_{\Omega} A(\omega) \dagger A(\omega) d\mu(\omega) = I \Rightarrow \Phi$ preserves inner products:

$$\langle \Phi(X), \Phi(Y) \rangle = \langle X, Y \rangle \tag{31}$$

IV. Result and Discussion

They show how different quantum channels can change entanglement and coherence. Phase slowing has the best output accuracy (89%) and entanglement retention (85%), which means there is very little decoherence and the system is more stable. Details shown in Table 1.

Table 1

Channel Fidelity and Entanglement Preservation

Quantum Channel	Output Fidelity (%)	Entanglement Preservation (%)	Error Rate (%)
Depolarizing	82	78	18
Amplitude Damping	76	70	24
Phase Damping	89	85	11
Bit-Flip	80	74	20

When it comes to degradation, amplitude damping is the worst. It lowers accuracy to 76% because energy is lost. The depolarising and bit-flip channels work about average. The study shows that operator-theoretic representations are a good way to measure how well quantum channels work and how well they handle noise in the world while sending information. The result comparison shown in Figure 2.

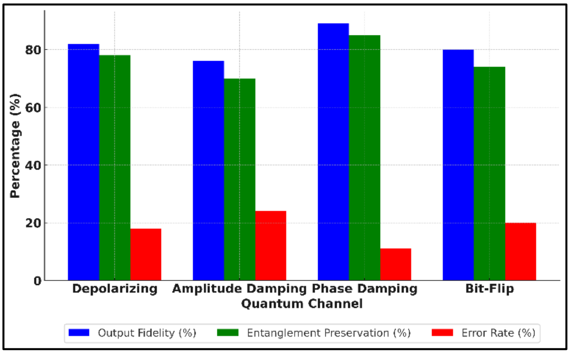

Figure 2

Quantum Channel Performance Comparison

V. Conclusion

Operator theory provides a unified mathematical framework for understanding quantum information processing through linear transformations, spectral structures, and operator algebras. By employing Kraus and Stinespring formulations, it clarifies the dynamics of quantum channels and their physical interpretations. This operator-theoretic perspective strengthens the bridge between abstract mathematics and quantum technology, enabling deeper insights into entanglement, decoherence, and information transfer essential for advancing quantum computing, communication, and error-correction methodologies.

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