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Heat transfer modeling using fractional differential equations

Prashant Khobragade †
Department of Computer Science and Engineering
G H Raisoni College of Engineering
Nagpur 440016
Maharashtra
India

Ganesh Gandal *
Department of Mathematics
Smt. Kashibai Navale College of Engineering
Pune 411041
Maharashtra
India

Avinash Somatkar §
Department of Mechanical Engineering
Vishwakarma Institute of Technology
Pune 411037
Maharashtra
India

Hrushikesh Madhukar Panchabudhe ‡
Department of Computer Technology
Yeshwantrao Chavan College of Engineering
Nagpur 441110
Maharashtra
India

† *E-mail:* prashant.khobragade@raisoni.net
* *E-mail:* ganesh.gandal_skncoe@sinhgad.edu (Corresponding Author)
§ *E-mail:* avinash.somatkar@vit.edu
‡ *E-mail:* hrushikeshpanchabudhe@gmail.com

Phatangare Prashant Ashok [@]
Department of Mechanical Engineering
Vidya Niketan College of Engineering
Sangamner 422602
Maharashtra
India

Bekzod. Madaminov [#]
Department of General Professional Sciences
Mamun University
Khiva 220900
Uzbekistan

Abstract

This study looks into how fractional differential equations (FDEs) can be used to model heat transfer. FDEs are an extension of standard Fourier-based models that add memory and nonlocal effects. Fractional derivatives enable precise modelling of the anomalous diffusion observed in mixed, porous, or microscale materials. Various analytical methods, including Laplace transforms and variable separation, are employed to resolve fractional heat transfer equations. Numerical techniques, such as finite difference and spectral techniques, are also used. The results show that lowering the fractional order slows thermal diffusion. This captures the real non-Fourier behaviour in complex materials and improves the ability to predict what will happen in advanced thermal system modelling.

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Keywords: *Fractional calculus, Heat conduction, Anomalous diffusion, Nonlocal transport, Numerical modeling, Thermal analysis.*

I. Introduction

Heat transfer is one of the most basic ideas in science and engineering. It is thought that thermal reactions happen quickly and locally in classical models of heat movement, which use Fourier's law and the standard diffusion equation [1, 2]. When tests are done on materials that are mixed, porous, or microstructured, they often behave in a way that isn't like how standard models would predict. As shown in Figure 1, fractional differential equations (FDEs) are powerful math tools that can help you understand these kinds of things better. Some of them are memory effects and effects that aren't localised through fractional-order derivatives [3, 4, 5].

Fractional time and space are two other factors that are introduced in the law of fourier to render it more generic. These models have been useful in many

[@] *E-mail:* **prashantph444@gmail.com**
[#] *E-mail:* **madaminov_bekzod@mamunedu.uz**

aspects such as transporting heat in plastics, organic tissues and in micro and nanoscale systems [6, 7].

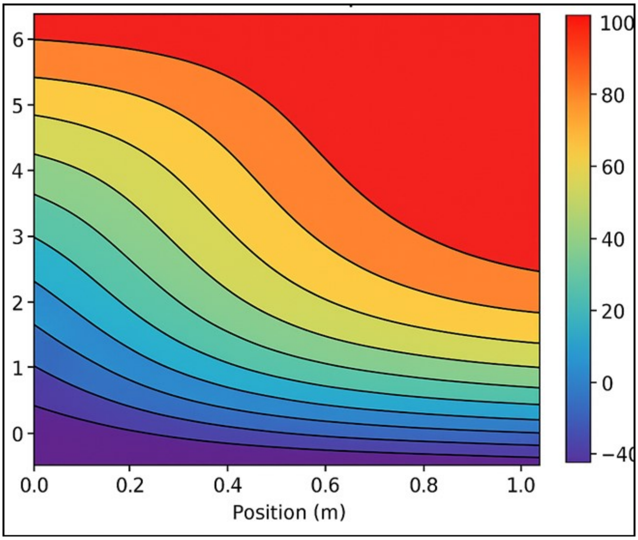

Figure 1
Temperature Distribution in Fractional Heat Transfer Model

II. Solution Methods

A. *Analytical solution approaches*

The mathematical approach involves LT and separating the variables to obtain credible and valid results to assist in determining the influence of barriers on the working flow and the partial heat transfer [8, 9, 10].

Start with the one dimensional equation of the fractional heat equation:

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = a\frac{\partial^2 T(x,t)}{\partial x^2}, \qquad 0 < \alpha \leq 1$$

Assume initial condition T(x,0) = f(x) and boundary conditions $T(0,t) = T(L,t) = 0$.

In terms of time, use the Laplace transform:

$$f(x) = a\frac{d^2\bar{T}(x,s)}{dx^2} \tag{1}$$

Solve the ODE:

$$\bar{T}(x,s) = A\sinh\left(\sqrt{\frac{s^\alpha}{a}}x\right) + B\cosh\left(\sqrt{\frac{s^\alpha}{a}}x\right) \tag{2}$$

Derive solution using Mittag-Leffler function:

$$T(x,t) = \Sigma(from\ n = 1\ to\ \infty)f_n \sin\left(\frac{n\pi x}{L}\right)E_{\alpha\left[-a\left(\frac{n\pi}{L}\right)^2 t^\alpha\right]} \tag{3}$$

Alternatively, apply separation of variables:

$$T(x,t) = X(x)\Theta(t)$$

Obtain:

$$\frac{\left(\frac{1}{\Theta(t)}\right)d^{\alpha\Theta(t)}}{dt^{\alpha}} = a\frac{\left(\frac{1}{X(x)}\right)d^{2}X(x)}{dx^{2}} = -\lambda \tag{4}$$

Solve spatial part:

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \tag{5}$$

Apply boundary conditions → $X(x) = A\sin\left(\frac{n\pi x}{L}\right)$

$$\frac{d^{\alpha\Theta(t)}}{dt^{\alpha}} = -a\lambda\,\Theta(t) \tag{6}$$

Solution:

$$\Theta(t) = E_{\alpha[-a\lambda\,t^{\alpha}]} \tag{7}$$

Combine both parts:

$$T(x,t) = \Sigma A_n \sin\left(\frac{n\pi x}{L}\right)E_{\alpha\left[-a\left(\frac{n\pi}{L}\right)^{2}t^{\alpha}\right]} \tag{8}$$

III. Mathematical Formulation

A. *Fractional Heat Conduction*

The law of Fourier is generalized in the fractional heat transfer equation where the equations involve the time-space fractional derivatives. This equation is a correct description of the effects and processes of memory that transport energy about in a nonlocal sense [11, 12].

We may start with the law of heat conduction with Fourier:

$$q(x,t) = -k\frac{\partial T(x,t)}{\partial x} \tag{9}$$

Substitute Fourier's law:

$$\rho\,c\frac{\partial T(x,t)}{\partial t} = k\frac{\partial^{2}T(x,t)}{\partial x^{2}} + Q(x,t) \tag{10}$$

Energy analysis using:

$$\rho\,c\left(\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}}\right) = k\frac{\partial^{2}T(x,t)}{\partial x^{2}} + Q(x,t) \tag{11}$$

Extend to 3D vector form:

$$\frac{\partial^{\alpha}T(r,t)}{\partial t^{\alpha}} = a\,\nabla^{2}T(r,t) + \left(\frac{1}{\rho c}\right)Q(r,t) \tag{12}$$

Suppose that the things are not identical with one another, then think of as being spatially based:

$$\frac{\partial^{\alpha}T(r,t)}{\partial t^{\alpha}} = \nabla\cdot\left[a(r)\nabla T(r,t)\right] + \left(\frac{1}{\rho(r)c(r)}\right)Q(r,t) \tag{13}$$

$$\frac{q(x,t)= -k\int^{0}{}_{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)\partial T(x,\tau)}}{\partial x}d\tau \tag{14}$$

Apply divergence theorem:

$$\int_V \rho\ c\ \left(\frac{\partial^\alpha T}{\partial t^\alpha}\right) dV\ =\ \int_V \nabla\ \cdot\ (k\ \nabla T) dV\ +\ \int_V Q\ \ dV \tag{15}$$

As equality is true of arbitrary control volume:

$$\rho\ c\ \left(\frac{\partial^\alpha T}{\partial t^\alpha}\right) =\ \nabla\ \cdot\ (k\ \nabla T) +\ Q \tag{16}$$

Include temperature-dependent properties:

$$\rho(T) c(T) \left(\frac{\partial^\alpha T}{\partial t^\alpha}\right) =\ \nabla\ \cdot\ [k(T) \nabla T] +\ Q \tag{17}$$

For zero source term, homogeneous domain:

$$\left(\frac{\partial^\alpha T}{\partial t^\alpha}\right) =\ a \frac{\partial^2 T}{\partial x^2} \tag{18}$$

Apply Laplace transform:

$$s^\alpha \bar{T}(x, s) -\ s^{\alpha-1} f(x) =\ a \frac{d^2 \bar{T}(x,s)}{dx^2} \tag{19}$$

General solution:

$$\bar{T}(x, s) =\ A \sinh\left(\sqrt{\frac{s^\alpha}{a}} x\right) +\ B \cosh\left(\sqrt{\frac{s^\alpha}{a}} x\right) \tag{20}$$

Inverse transform via Mittag-Leffler kernel:

$$T(x, t) =\ \Sigma f_n \sin\left(\frac{n\pi x}{L}\right) E_{\alpha\left[-a\left(\frac{n\pi}{L}\right)^2 t^\alpha\right]} \tag{21}$$

B. *Dimensional analysis*

It achieves this by relating derivative orders to such aspects as material memory, bizarre diffusion behaviour and scale transfers heat flow [13].

Consider fractional heat equation:

$$\frac{\partial^\alpha T}{\partial t^\alpha} =\ a \frac{\partial^2 T}{\partial x^2} \tag{22}$$

Assign fundamental dimensions:

$$[T] =\ \Theta, [x] =\ L, [t] =\ T, [a] = \frac{L^2}{T^\alpha} \tag{23}$$

Dimensional consistency check:

$$\left[\frac{L^2}{T^\alpha}\right] * \left[\frac{\Theta}{L^2}\right] = \left[\frac{\Theta}{T^\alpha}\right] \rightarrow\ consistent \tag{24}$$

Define fractional thermal diffusivity:

$$a\ = \frac{k}{\rho c} with\ dimensions \frac{L^2}{T^\alpha}$$

Rewrite in integral form:

$$\left(\frac{1}{\Gamma(1-\alpha)}\right) \int^{0\ \mathrm{t}} \left(\frac{\partial T(x,\tau)}{\partial \tau}\right) (t-\tau)^{-\alpha} d\tau\ =\ a \frac{\partial^2 T(x,t)}{\partial x^2} \tag{25}$$

Introduce dimensionless parameters:

$$X = \frac{x}{L}, \ \tau = \frac{a\, t^{\alpha}}{L^2}, \ \theta = \frac{T - T^0}{T^1 - T^0}$$

Substitution yields:

$$\frac{\partial^{\alpha}\theta}{\partial\tau^{\alpha}} = \frac{\partial^2\theta}{\partial X^2} \tag{26}$$

Take alpha to be an index of memory, α classical diffusion, α = 1 → subdiffusion (memory-dominated).

Elasticity physical flux relation with kernel memory

$$q(x,t) = -\int^{0\ \mathrm{t}} k \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)\left(\frac{\partial T}{\partial x}\right)(x,\tau)d\tau} \tag{27}$$

Introduce energy conservation integral:

$$\int_V \rho\ c\ \left(\frac{\partial^{\alpha}T}{\partial t^{\alpha}}\right) dV \ = -\int_S q \cdot n\ dS\ +\ \int_V Q\ dV \tag{28}$$

Apply divergence theorem and localize:

$$\rho\, c\ \left(\frac{\partial^{\alpha}T}{\partial t^{\alpha}}\right) = \nabla \cdot \left[\int^{0\ \mathrm{t}} k \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)\nabla T(x,\tau)d\tau}\right] + Q \tag{29}$$

Express constitutive equation compactly:

$$\rho\, c\ \left(\frac{\partial^{\alpha}T}{\partial t^{\alpha}}\right) = \nabla \cdot [k * (K * \nabla T)] + Q \tag{30}$$

Integrate over time to find total heat energy:

$$E(t) = \int^{0\ \mathrm{t}} \int_V \rho\, c\ \left(\frac{\partial^{\alpha}T}{\partial \tau^{\alpha}}\right) dV\ d\tau \tag{31}$$

Check dimensional check of the formulation:

$$\left[\frac{L^2}{T^{\alpha}}\right] \times \left[\frac{\Theta}{L^2}\right] = \left[\frac{\Theta}{T^{\alpha}}\right] \tag{32}$$

IV. Result and Discussion

Table 1 indicates the distribution of the temperatures in the space domain of the various orders of fractions (α). The temperature values remain bigger at all the positions as 0.7 is reached at the positions where 0.7 is greater than 1.0.

Table 1

Temperature Distribution for Different Fractional Orders (α)

Position (x, m)	α = 1.0 (Classical)	α = 0.9	α = 0.8	α = 0.7
0	100	100	100	100
0.2	84.5	88.7	91.3	93.8
0.4	69.2	78.6	83.2	87.1
0.6	54.8	68.3	75.4	80.6

It implies that the movement of heat is slower, and the memory effects are increased. The conventional model (α = 1.0) cools faster whereas the fractional models indicate how the heat circulates in various regions. Figure 2 reveals that when the value of α is lower, the rate of diffusion of heat is reduced and memory effects are enhanced.

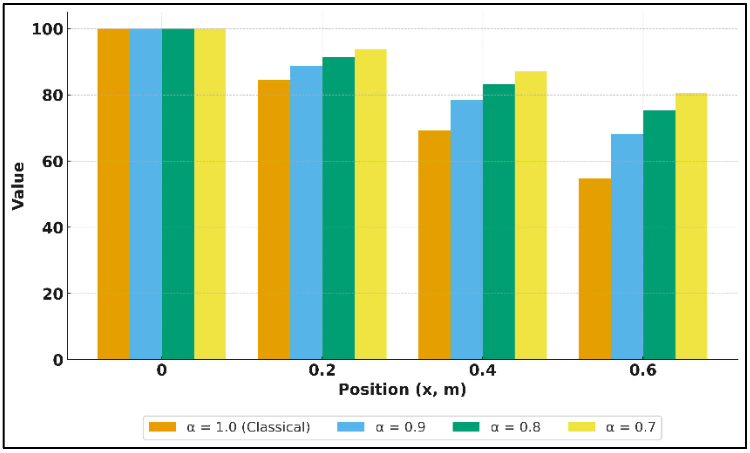

Figure 2
Comparison of Position-Dependent Output across α Parameters

This indicates that fractional derivatives are an appropriate method of modeling strange heat conduction in materials that do not respond to temperature variations across the medium in a Fourier manner.

V. Conclusion

A powerful method of modeling the behavior of a process that is not local or memory-like is through the use of fractional differential equations, which model the heat transfer. Unlike classical models, they are right in the explanation of the strange spread of microscale, complex, and diverse systems. Analytic and numerical methods meet the learning requirements and accelerate the process of computation when dealing with problems of fractional heat transfer. Incorporation of thermal modelling with fractional calculus also assists us in studying the physical world, predicting it more accurately, and in serving more sophisticated engineering applications in fields such as energy systems, materials science, and the thermal transport of heat in biological systems.

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