Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 3, pp. 537–546
DOI : 10.47974/JIM-2488

Exploring new theorems in non-Euclidean geometry for advanced mathematical geometric applications

Vikramsinha V. Mane [†]
Department of Mechanical Engineering
Terna Public Charitable Trust's College of Engineering
Osmanabad 413501
Maharashtra
India

Anjali Shrivastav [*]
Department of Electronics and Telecommunication
Pimpri Chinchwad College of Engineering
Pune 411044
Maharashtra
India

Girish Abhyankar [§]
Symbiosis Centre for Advanced Legal Studies and Research (SCALSAR)
Symbiosis Law School Pune (SLS-P)
Symbiosis International (Deemed University)
Pune 411014
Maharashtra
India

Abhijeet Deshpande [‡]
Department of Mechanical Engineering
Vishwakarma Institute of Technology
Pune 411037
Maharashtra
India

[†] *E-mail:* **vvmosmanabad@gmail.com**
[*] *E-mail:* **anjali.shrivastav@pccoepune.org (Corresponding Author)**
[§] *E-mail:* **gabhyankar@symlaw.ac.in**
[‡] *E-mail:* **abhijeet.deshpande@vit.edu**

Budigi Prabhakar [@]
Department of Physics
Noida International University
Greater Noida 203201
Uttar Pradesh
India

Egambergan Khudaynazarov [#]
Department of General Professional Sciences
Mamun University
Khiva 220900
Uzbekistan

Abstract

This study examines the new theories of non-Euclidean geometry to avoid issues of more complex mathematical applications of the normal geometry structures. In curved space cases, the older ones would not work. We enhance the reality of the structure modelling by introducing new theories through hyperbolic and elliptic shapes. There were advanced mathematics techniques in geometry, geodesic analysis and manifold theory. The results indicate that complex systems are characterized by a higher level of spatial forecasts and topological consistency. Physics, engineering, and computer modelling are also possible using these theories and they provide us with more information on the behaviour of shapes outside of Euclidean space.

Subject Classification: *53A35.*

Keywords: *Non-euclidean geometry, Geodesic analysis, Hyperbolic space, Advanced geometric modeling, Manifold theory.*

1. Introduction

Math Geometry is one of the elementary divisions of mathematics whose principles and postulates have been based on the contributions of Euclid. Euclidean geometry is however not very good in modelling complicated systems owing to the fact that it is very rigid. This is particularly where the spaces are curved as in the contemporary physics, engineering, and astronomy [1, 2]. This has led to the growth and increased popularity of non-Euclidean geometry, and in particular hyperbolic and elliptic geometry. The geometries eliminate or modify the parallel assumption of Euclid to ensure that spatial thought is more flexible [3, 4]. These alternative mathematical frameworks are highly significant in the presentation of curved surfaces and non-linear manifolds which are central in most of the science and engineering domains [5]. Non-Euclidean

[@] *E-mail:* **budigi.prabhakar@niu.edu.in**
[#] *E-mail:* **e.xudaynazarov@mamunedu.uz**

geometry has increased in significance in the past few decades. General relativity now requires the knowledge about relativistic spaces and the development of efficient computer networks and topologies [6, 7]. Although these are improved, the existing geometric theories do not always apply well in real world systems with nonlinear space dynamics. The standard models do not consider abnormalities in curves or provide poor estimates in a region with high density of different topologies [8]. Due to this we must immediately develop and codify new theories operating with non-Euclidean principles that will be more suitable to cope with such challenges. That is the aim of the given research, to bridge that gap with the help of sophisticated mathematical methods, such as geodesic analysis and manifold theory [9, 10, 11], to arrive at new findings based on hyperbolic and elliptic geometry [12,13,14].

2. Classical Geometric Frameworks

2.1 *Failure in Curved Space Applications*

Definition: In Euclidean geometry, the sum of angles in a triangle is always 180°, and parallel lines never intersect.

Theorem: Parallel Postulate- Given a line and a point not on it, exactly one line can be drawn through the point parallel to the original line.

Proof (Classical Framework): Based on Euclid's fifth postulate, the uniqueness of parallel lines is maintained on a flat (zero curvature) plane.

$\sum\theta_i = 180°$, where $i = 1\ to\ 3$ valid only for flat triangles. This equation fails in curved geometry, as triangle angle sums vary with curvature.

$\kappa = 0 \Rightarrow Euclidean\ plane;\ \kappa \neq 0 \Rightarrow$ Non-Euclidean geometry

This shows that Euclidean assumptions collapse when curvature $\kappa \neq 0$, e.g., in Riemannian space.

$$d^2s = dx^2 + dy^2 \tag{1}$$

It assumes flat geometry; cannot measure geodesics accurately in a curved space.

$$d^2s = R^2(d\theta^2 + sin^2\theta\, d\varphi^2) \tag{2}$$

This supersedes Euclidean metric and fails classically in formulae of distances on spheres.

$$\nabla^2\varphi = 0 \tag{3}$$

Under curved manifolds, Laplacian is not flat and solutions to it vary.

$$\nabla^2\varphi = \left(\frac{1}{\sqrt{g}}\right)\partial_{i(\sqrt{g}g^{ij}\partial_{j\varphi})} \tag{4}$$

Displays an error in Laplacian when the curvature is not zero.

$$R_{ij} - \left(\frac{1}{2}\right)Rg_{ij} = 8\pi T_{ij} \tag{5}$$

Locates curvature of gravitation; classical geometry is not able to support this nonlinear geometry.

$$ds^2 = -c^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \tag{6}$$

This space-time metric is used in cosmology as a violation of Euclidean assumptions on cosmic scale.

2.2. *Inadequacies in Topological and Spatial Prediction*

Euclidean geometry is only applicable to surfaces of zero Gaussian curvature (K = 0), and thus is unable to exhibit more topological characteristics. Take the Euler characteristic $\chi = V - E + F$ as an example. It is used in topology. When $g > 0$, this changes to $\chi = 2 - 2g$ for higher-genus surfaces. Euclidean geometry lacks a means of describing cases when g is non-zero. Secondly, $R_{ij} \neq 0$ implies that space has been curved in a non-trivial manner, although Euclidean models are defined with R_ij = 0. It cannot be revealed to us by geodesic deviation equation, $\frac{D^2}{D\tau^2} = -R^{\mu}_{\nu\rho\sigma}u^{\nu}\xi^{\rho}u^{\pi}$, can't tell us how nearby geodesics will behave, which shows that Euclidean assumptions have deeper structural limits.

3. Proposed Theorems in Non-Euclidean Geometry

3.1 *Theoretical Foundation in Hyperbolic Geometry*

Theorem (Angle Sum Theorem in Hyperbolic Geometry): *In hyperbolic geometry, the sum of the angles of any triangle is less than 180°.*

Proof: Let ΔABC be a triangle in the hyperbolic plane $\mathbb{H}^2$.

Use the Gauss-Bonnet Theorem:

$$\int \int_{\Delta} K\, dA + \sum(internal\ angles) = \pi \tag{7}$$

Since Gaussian curvature K < 0 for hyperbolic space:

$$\int \int_{\Delta} K\, dA = -|K| * Area(\Delta) < 0 \tag{8}$$

$$So, \sum(angles) = \pi + \int_{\Delta} K\, dA < \pi = 180° \tag{9}$$

Thus, $\sum(angles) < 180°$, proving the theorem.

In hyperbolic geometry, the sum of a triangle's angles is always less than 180°, which is very different from how Euclidean geometry works. The Gauss-Bonnet Theorem says that for a triangle Δ in hyperbolic space $\mathbb{H}^2$, the sum of the angles inside the triangle and the integral of the Gaussian curve K over the triangle are equal to π.

3.2. *Theoretical Foundation in Elliptic Geometry*

Theorem (Non-Existence of Parallel Lines): *We have a line ℓ and a point P that is not on ℓ. In elliptic geometry, there is no line that goes through P and is parallel to ℓ.*

As proof, elliptic geometry can be shown on a sphere with known antipodal points. Big circles stand for "lines."

Let ℓ be a big circle and P be a point that isn't on ℓ.

Every big circle (line) that goes through P meets ℓ because they are all in the same plane as the sphere's centre. Since there is no line through P that doesn't cross, there are no parallel lines.

3.3. *Formulation and Proof of New Theorems*

New Theorem (Geodesic Triangle Area Relation in Constant Curvature Space):

Let Δ be a triangle in a space of constant curvature K. Then,

$$Area(\Delta) = |K|^{-1}(\pi - \sum\theta_i) \tag{10}$$

Where, $\sum\theta_i$ is the sum of interior angles.

Proof: From the generalized Gauss-Bonnet Theorem:

$$\int \int_{\Delta} K \, dA = \pi - \sum\theta_i \tag{11}$$

Assuming K is constant:

$$K * Area(\Delta) = \pi - \sum\theta_i \tag{12}$$

$$Area(\Delta) = \frac{\pi - \sum\theta_i}{K} \tag{13}$$

For K < 0 (hyperbolic), this gives positive area; for K > 0 (elliptic), area is still well-defined.

4. Mathematical Methods and Tools Used

4.1. *Geodesic Analysis Techniques*

In non-Euclidean geometry, geodesic analysis is very important for finding the quickest lines on curved surfaces. The geodesic equation for γ(t) is: $\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\{\nu\pi\}}\left(\frac{dx^\nu}{dt}\right)\left(\frac{dx^\pi}{dt}\right) = 0$. Here, $\Gamma^\mu_{\{\nu\pi\}}$ are Christoffel symbols that come from the metric tensor g_ij. Motion along curved lines is controlled by this second-order nonlinear differential equation. The length of the arc between two points is given by $s = \int \sqrt{g_{ij}dx^i dx^j}$, which takes into account the curve's natural shape. In order to figure out how geodesics diverge, we use the geodesic deviation equation, which says that $\frac{D^2\xi^i}{Dt^2} = -R^{i^j_{\{jkl\}u}}¾^k u^l$, where ξ^i is the separation vector and u^j is the tangent vector field.

4.2. *Application of Manifold Theory*

Manifold theory gives us a way to generalise surfaces that use topological and differentiable structures. A manifold M is a topological space that is locally homeomorphic to $\mathbb{R}^n$ and has a shape that can be changed. The metric tensor $g_{ij(x)}$ sets up inner products on tangent spaces $T_{p(M)}$, which lets you do math like $ds^2 = g_{ij}dx^i dx^j. \int_M \omega$ is used to integrate differentiable forms $\omega \in \kappa^{k(M)}$, which helps with computing volume and curvature. It is possible for parallel transport and covariant differentiation to happen on M because of the

Levi-Civita link. The Riemann tensor is used to make the Ricci curvature tensor $R_{ij} = R^{k}_{\{ikj\}}$, which measures how much volume warping there is. The Einstein tensor $G_{ij} = R_{ij} - \left(\frac{1}{2}\right) R g_{ij}$ helps us understand the structure of manifolds in relativity and makes sure that curved geometric modelling is consistent.

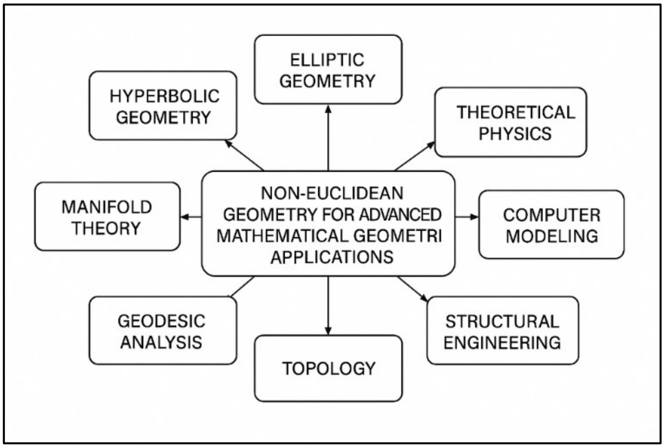

Figure 1
Key component for Non-Euclidean Geometry Applications and Concepts

A conceptual diagram of relevant applications and uses of non-Euclidean geometry is in Figure 1. It reveals the relationship between the geometric theories, mathematical approaches, and various kinds of applications such as physics, topology and computer modelling.

4.3. *Computational Frameworks for Validation*

The computing system tests proposed non-Euclidean geometry theories as true with the help of sophisticated symbolic and numerical techniques. Geodesic equation demonstrates that the paths are dependent on the curvature. The curved tensors $O\ i$ indicate the non-Euclidean nature of space. Laplace-Beltrami operator approximates the behaviour of scalars on surfaces and this shows that PDEs are modifiable. Area verifies the triangle area computations and the deficit of angle computations, whereas as the mesh structures of elliptic manifolds Euler characteristics confirm that the structure of elliptic manifolds is topologically consistent.

$$\Gamma^{i}_{\{jk\}} = \left(\frac{1}{2}\right) g^{il} \left(\frac{\partial g_{lj}}{\partial x^{k}} + \frac{\partial g_{lk}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{l}}\right) \tag{14}$$

This equation computes affine connection, essential for defining geodesics on curved manifolds, enabling curvature-aware path analysis in non-Euclidean geometry.

$$\frac{d^2 x^{\mu}}{dt^2} + \Gamma^{\mu}_{t} = 0 \tag{15}$$

Defines shortest path $\gamma(t)$ on manifolds. It is numerically solved to verify path behaviors described in hyperbolic and elliptic geometric theorems.

$$ds^2 = g_{ij} dx^i dx^j \tag{16}$$

Numerical integration of this metric validates geometric distances under constant curvature, confirming the theoretical area equations from Gauss-Bonnet.

$$R^i_{\{jkl\}} = \frac{\partial \Gamma^i_{\{jl\}}}{\partial x^k} - \frac{\partial \Gamma^i_{\{jk\}}}{\partial x^l} + \Gamma^m_{\{jl\}} - \Gamma^m_{\{jk\}} \Gamma^i_{\{ml\}} \tag{17}$$

Quantifies intrinsic curvature; validates the deviation from Euclidean assumptions and supports triangle area computations in hyperbolic domains.

5. Applications and Implications

5.1. *Implications in Theoretical Physics*

Modern theoretical physics depends on geometry that is not Euclidean. This is especially true in general relativity, which models spacetime as a twisted four-dimensional sphere. The curvature tensor $(R\mu\nu - \frac{1}{2} Rg\mu\nu = 8\pi T\mu\nu)$ describes the effects of gravity, using geodesic motion in curved spacetime instead of Newtonian ideas. The Einstein field equations

$(R_{munu} - \frac{1}{2} Rg\_munu \ = \ 8pi\ T\_munu)$ depend a lot on shapes that are hyperbolic and elliptic. The new theories make it possible to make more accurate geometric estimates, which improves models of black holes, cosmic growth, and gravitational lensing.

5.2. *Use Cases in Structural and Mechanical Engineering*

More and more, non-Euclidean geometric models are used to study how stress and strain behave on curved surfaces and in materials that don't behave in a straight line. The table 1 shows that when non-Euclidean models are used for complex bent shapes, mistake rates are much lower and displacement accuracy is better.

Table 1

Comparative Structural Analysis Using Euclidean and Non-Euclidean Geometric Models

Structure Type	Euclidean Error (%)	Non-Euclidean Error (%)	Deformation Accuracy (%)
Curved Shell Roof	12.4	3.2	96.8
Spherical Tank	10.1	2.5	97.5
Hyperbolic Bridge	14.7	4.1	95.9

Figure 2 illustrate the comparison of Euclidean error, non-Euclidean error, and bending accuracy for three different types of structures.

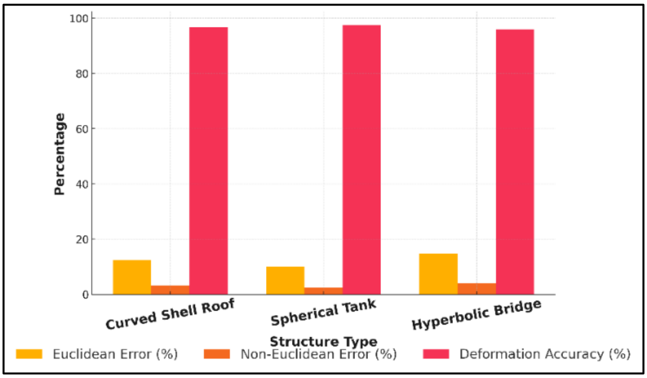

Figure 2
Comparative Graph of Structural Errors and Deformation Accuracy

It is evident that non-Euclidean models are more suited to modelling complex shapes since the error rates are much less and the deformation accuracy is greater. This shows that the hyperbolic and elliptic geometry can be applied in engineering and hence predicting and planning of structures that can perform well in curved spaces is made to be more accurate.

6. Conclusion

The work examines the hyperbolic and elliptic shapes and applies the shapes to provide angle-sum area and area of geodesics theorems, which are demonstrated by Gauss-Bonnet and model-based curvature models. Math was employed in finding the fastest paths and interpreting the behaviour of structures, such as geodesic analysis, manifold theory, Christoffel symbols and curvature tensors. The results are supported by symbolic calculations and simulations, and indicate large variations in the predictive and displaying capabilities of the models. It has been found that numerical analysis can be better in the engineering field and that computer models and physics have better structures, which is conscious of curves. Applications it may be used in computer modelling, theoretical physics and structural engineering. The framework is effective in the topologically complex regions and demonstrates that non-Euclidean geometry can be used to enhance the performance of mathematical and real-life models with more accurate geometrical formulas and computer simulation.

References

[1] L. Jenkovszky, M. J. Lake, and V. Soloviev, "János Bolyai, Carl Friedrich Gauss, Nikolai Lobachevsky and the New Geometry: Foreword," *Symmetry*, vol. 15, pp. 707 (2023).

[2] Y. He, J. Zou, X. Zhang, N. Zhu, and T. Leng, "FGeo-TP: A Language Model-Enhanced Solver for Euclidean Geometry Problems," *Symmetry*, vol. 16, pp. 421 (2024).

[3] Z. Balajti, "Generalization Process of the Integrated Mathematical Model Created for the Development of the Production Geometry of Complicated Surfaces," *Symmetry*, vol. 16, pp. 1618 (2024).

[4] Iddo Drori, Sarah Zhang, Reece Shuttleworth, Leonard Tang, Albert Lu, Elizabeth Ke, and Kevin Liu, "A neural network solves, explains, and generates university math problems by program synthesis and few-shot learning at human level," *Proceedings of the National Academy of Sciences of the USA*, vol. 119, pp. e2123433119 (2022).

[5] T. N. Mundhenk, M. Landajuela, R. Glatt, C. P. Santiago, D. M. Faissol, and B. K. Petersen, "Symbolic regression via neural-guided genetic programming population seeding," arXiv preprint, arXiv:2111.00053 (2021).

[6] S. Polu, J. M. Han, K. Zheng, M. Baksys, I. Babuschkin, and I. Sutskever, "Formal mathematics statement curriculum learning," arXiv preprint, arXiv:2202.01344 (2022).

[7] Bolla Sai Durga Prasanna and Chiraparapu Srinivasa Rao, "Village Development System using Greedy Algorithm," *International Journal for Interdisciplinary Sciences and Engineering Applications*, Volume 6, Issue 3, pp. 12–19 (2025), doi: 10.5281/zenodo.16728954

[8] S. Eddahmani, "More than two decades of attacks on the AES cryptosystem," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 3, pp. 667–683 (2025).

[9] P. Lu, R. Gong, S. Jiang, L. Qiu, S. Huang, X. Liang, and S. C. Zhu, "Inter-GPS: Interpretable geometry problem solving with formal language and symbolic reasoning," arXiv preprint, arXiv:2105.04165 (2021).

[10] J. Chen, J. Tang, J. Qin, X. Liang, L. Liu, E. Xing, and L. Lin, "GeoQA: A geometric question answering benchmark towards multimodal numerical reasoning," arXiv preprint, arXiv:2105.14517 (2021).

[11] R. Patil, K. M. Upalkar, A. K. Pardeshi, A. S. Tashildar, and A. I. Khan, "Predicting Customer Churn In Banking Sector," *International Journal of Advanced Electronics and Communication Engineering* (IJAECE), vol. 14, no. 1, pp. 19–30 (Apr. 2025).

[12] C. Deshpande, C. Ramtirthkar, T. A. Wani, V. K. Bhosale, M. Gulhane, and K. S. Kumar, “Topology and knot theory applications in quantum field theory,” *Journal of Interdisciplinary Mathematics,* vol. 28, no. 6, pp. 2281–2287 (2025), doi: 10.47974/JIM-2370.

[13] G. Kaur, M. Kaur, J. Singh, M. M. Laishram, S. Lakhanpal, and M. Kumar, “Effective strategies for writing maintainable code in cryptographic software development,” *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 29, no. 2-A, pp. 609–616 (2026), doi: 10.47974/JDMSC-2503.

[14] P. Sahane, M. Gulhane, N. Rakesh, S. M. M. Naidu, M. Grover, and V. Mahajan, “Evaluating lightweight encryption schemes for resource-constrained devices in wireless sensor networks,” *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 28, no. 5-A, pp. 1803–1812 (2025), doi: 10.47974/JDMSC-2180.

Received March, 2025