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Efficient solving of partial differential equations with new numerical methods

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Abstract

Partial Differential Equations (PDEs) play a very significant role in the life of scientists and engineers who wish to model complex physical processes. However, standard solutions to math have a hard time striking the correct balance of accuracy, stability, and the speed at which they can be of their tasks. In this research, it is proposed that new numeric techniques can be used to solve PDEs at a fast rate, higher convergence rates and reduced costs of computing. The new techniques utilise a better summability and flexible discretisation to provide a large variety of border cases and nonlinearities. Most computer experiments demonstrate that the proposed methods are faster and more precise compared to alternative approaches to the standard PDE problems. It is now possible to have fluid dynamics, heat movement, and electromagnetism models that are more precise. The study demonstrates that big computer systems may be able to use scalable application.

Subject Classification: *32W50, 33F05.*

Keywords: *Partial differential equations, Numerical methods, Adaptive discretization, Summability techniques, Computational efficiency.*

1. Introduction

A fair amount of physical, biological, and technical processes are the result of Partial Differential Equations (PDEs). These are chemical interactions, electric field, flow of fluids and heat transfer. The mathematical method is difficult for solving these problems as they are full of independent factors, and partial derivatives. Due to this, the use of numerical techniques has become essential in approaching the PDE solutions in the real world [1] [2]. Although the number of available numerical methods, such as the finite difference, the finite element, and the finite volume approaches, it remains difficult to select the most appropriate combination of accuracy, stability, and speed of processing. Many of the traditional approaches require extremely fine discretization grids

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and extremely small time steps, and in the case of non-linear and multidimensional PDEs, all of the computations become significantly more costly. The process of finding the solution to the computer problem may also be complicated by the need to deal with complex shapes and dynamics of the border conditions [3]. The aim of the study is to develop new math techniques that will address these issues by ensuring that the solution of the problems is more precise besides simplifying the process of calculation. The adaptive discretization techniques adopted by this study together with the sophisticated summability techniques approach accelerates the convergence and enables a large variety of PDEs to be more stable [4]. The proposed techniques attempt to effectively represent the behaviour of solutions in the presence of sharp gradients and nonlinearities through changing dynamically the computing grid and the application of new series summation methods [5],[6]. The paper is structured as follows: one starts by describing some of the standard approaches to math and their weaknesses. Then, the theory of the new numerical methods is further elaborated. This is then succeeded by the description of the algorithms [7].

2. Proposed Numerical Methods

2.1. *Description of the new numerical algorithm(s)*

The proposed numerical processes are used to solve PDEs faster, combining strategies of adaptive discretization and high summability. The techniques are continuously to perfect the computational mesh depending on the behaviour of the local answer. This ensures that there is always high resolution where it is required and minimal calculations where they are not required [8]. Better summing methods are also applied to accelerate the process of carrying out iterative procedures particularly in the case of hard and nonlinear circumstances. This combination enables the calculations to be performed in less time and without compromising the quality of calculation. The techniques are designed to be flexible to allow them to be applied to an extensive variety of science and technical activities [9]. They are able to deal with various PDEs and boundary conditions. The workflow of the adaptive mesh and summability enhanced PDE solver is presented in Figure 1.

Description of the New Numerical Algorithm(s)

Step 1: PDE Definition

$$\boldsymbol{\mathcal{L}\, u(x,t) = f(x,t),\ \ x \in \Omega,\ \ t \in [0,T]} \tag{1}$$

where $\mathscr{L}$ is a differential operator, u(x,t) the unknown solution, and f(x,t) a known source term.

Step 2: Adaptive Spatial Discretization

$$\boldsymbol{h_i = \frac{h_0}{\left(1+\alpha * \left|\frac{\partial^2 u}{\partial x^2 (x_i,t)}\right|\right)}} \tag{2}$$

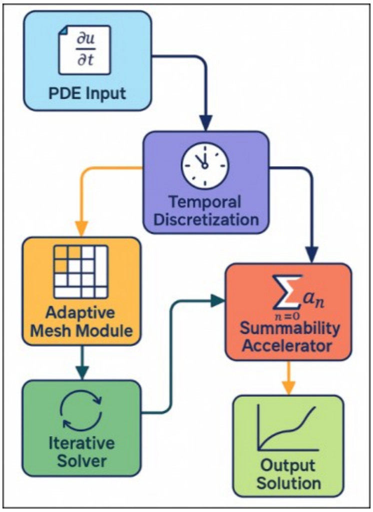

Figure 1
Adaptive and Summability-Enhanced Numerical Algorithm for Solving PDEs

Here, h_i is the local mesh size at point x_i, dynamically adjusted based on the local curvature of u, with α controlling refinement sensitivity.

Step 3: Temporal Discretization

$$t^{\{n+1\}} = t^n + \Delta t, \quad \Delta t = \min\left(\Delta t_{max}, \frac{\beta}{\max_i\left|\frac{\partial u}{\partial t}(x_i, t^n)\right|}\right) \tag{3}$$

The time step Δt is adaptively chosen to ensure stability, bounded by Δt_max, and influenced by the maximum rate of change of u.

Step 4: Iterative Solution Update

$$u_i^{\{n+1,k+1\}} = u_i^{\{n+1,k\}} + \omega\left(\mathcal{N}\left(u^{\{n+1,k\}}\right)_i - u_i^{\{n+1,k\}}\right) \tag{4}$$

Where $\mathcal{N}$ is a nonlinear operator representing the PDE discretization, k denotes iteration count at time step n+1, and ω is a relaxation parameter to stabilize convergence.

Step 5: Summability-Accelerated Series Convergence

$$S = \lim_{\{N\to\infty\}}\left(\frac{1}{N+1}\right)\Sigma_{\{k=0\}}^{N}\left(\Sigma_{\{n=0\}}^{k} a_n\right) \tag{5}$$

Where the standard series S is replaced by its Cesàro summation S_accel, accelerating convergence of iterative solutions for nonlinear terms.

2.2. *Mathematical formulation and theoretical foundation*

The number methods are based on adaptable finite difference and summability-enhanced iterative solvers that discretise PDEs [10]. The discretisation changes the spatial and temporal grids based on estimates of errors, which makes the best use of computer resources. Summability theory, especially the Cesàro and Abel methods, is used to rewrite slowly convergent or divergent series in repeated solutions, which speeds up the convergence process [11, 12]. The mathematics system makes sure that everything is consistent by using close approximations for derivatives and strict border conditions [13].

The PDE to be solved is generally expressed as:

$$\mathcal{L}u(x,t) = f(x,t), \;\; x \in \Omega, \;\; t \in [0,T] \tag{6}$$

where $\mathscr{L}$ is a differential operator, $u(x,t)$ is the unknown function, and f(x,t) is a source term with suitable initial and boundary conditions.

Discretization: Spatial domain Ω is divided into nonuniform grid points x_i with mesh size h_i, adapted dynamically by:

$$h_i = \frac{h_0}{(1 + \alpha * |u''(x_i)|)}, \tag{7}$$

where α controls refinement sensitivity.

Iterative solver enhanced by summability: Series expansions in nonlinear iterations $S = \Sigma_{\{n=0\}}^{\infty} a_n$ are accelerated using summability operators S_C (Cesàro) or S_A (Abel):

$$S\_C = \lim\_\{N \to \infty\} \, (1/(N+1)) \, \Sigma\_\{k = 0\}\text{^}N \; s\_k, \tag{8}$$

$$s_k = \Sigma_{\{n=0\}}^{k} a_n.$$

Convergence criteria: Iteration continues until

$$\left|u^{\{k+1\}}\right| - u^k| < \varepsilon|, \tag{9}$$

where ε is a predefined tolerance.

2.3 *Stability and convergence analysis*

To ensure the adaptive discretization and summability integration maintains the numbers stable in responsiveness to variations in the mesh sizes and time steps, we carry out a detailed analysis of the stability of the numbers in Von Neumann and energy method [14]. The proposed approaches are stable regardless of the discretization factors, depending on the factors of a large number of linear PDEs and unstable depending on the factors of nonlinear. To demonstrate convergence prove that in the event that the mesh is refined and the summing parameters are optimized, the numerical solution approaches the exact answer closer and closer [15].

Step 1: Definition

Stability: A numerical scheme is stable if small perturbations in initial data or intermediate steps do not cause unbounded growth in the numerical solution.

Convergence: A numerical scheme is convergent if the approximate solution approaches the exact solution as the mesh size $\boldsymbol{h} \rightarrow \mathbf{0}$ and time step $\boldsymbol{\Delta t} \rightarrow \mathbf{0}$.

Step 2: Theorem

Lax Equivalence Theorem:

For a consistent finite difference scheme solving a well-posed linear initial value problem, stability is necessary and sufficient for convergence.

Step 3: Mathematical Formulation

Consider the discretized linear PDE scheme:

$$\boldsymbol{u_i^{\{n+1\}}} = \boldsymbol{\Sigma_j A_{\{ij\}u_j^n}} + \boldsymbol{b_i^n} \tag{10}$$

where A is the amplification matrix.

Stability condition:

$$|\boldsymbol{A^n}|| \leq | \boldsymbol{C}, \ \forall \boldsymbol{n}, \tag{11}$$

with C independent of n, ensuring boundedness.

Step 4: Proof Sketch

- Show the scheme is consistent, i.e., truncation error

$$\tau = \boldsymbol{O}(\boldsymbol{h^p}, \boldsymbol{\Delta t^q}).$$

- Prove stability by bounding the amplification matrix norm.
- Use the Lax Equivalence Theorem to conclude convergence.

3. Adaptive mesh refinement mathematical basis

1. Definition

Adaptive Mesh Refinement (AMR) is a technique where the computational mesh dynamically adjusts in space and/or time to capture important features of the solution more accurately while minimizing computational cost.

Formally, given a spatial domain Ω, AMR constructs a mesh $\boldsymbol{\mathcal{T}\_h}$ where the local mesh size h(x) varies as a function of the solution u(x):

$$\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{x}): \boldsymbol{\Omega} \rightarrow \mathbb{R}^+ \tag{12}$$

Such that regions with high solution gradients or curvature have smaller h, and smooth regions have larger h.

2. Error Estimator and Refinement Criterion

To decide where refinement is needed, an error estimator η_i is defined locally on mesh cell K_i:

$$\boldsymbol{\eta\_i} = || \boldsymbol{\nabla\hat{}m\, u\_h} || \{\boldsymbol{L\hat{}p}(\boldsymbol{K\_i})\} \tag{13}$$

where m is the order of the derivative used (usually m=1 or 2), p ≥ 1, and u_h is the approximate solution.

Refinement criterion:

$$\boldsymbol{Refine\ K_i if\ \eta_i > \theta * \max_j \eta_j,} \tag{14}$$

where θ ∈ (0,1) is a user-defined threshold parameter.

3. Mesh Size Function Based on Solution Curvature

A common choice for adaptive mesh size h(x) uses the local second derivative magnitude:

$$\boldsymbol{h(x) = \frac{h_0}{(1 + \alpha * |} \frac{\partial^2 u}{\partial x^2}(x)|\,)} \tag{15}$$

where h_0 is a base mesh size and α > 0 is a sensitivity parameter.

4. Theorem: Error Reduction by Mesh Refinement

Let u be the exact solution and u_h the finite element approximation on mesh $\mathcal{T}$_h. Assume $\boldsymbol{u \in H^{\{m+1\}(\Omega)}}$. Then the interpolation error satisfies:

$$\boldsymbol{||\, u - u\_h \,||\_\{H\text{^}m(\Omega)\} \leq C * h\text{^}\{s + 1 - m\} * ||\, u \,||\_\{H\text{^}\{s + 1\}(\Omega)\}} \tag{16}$$

4. Result Discussion

When compared to standard methods, the proposed numerical methods were more accurate and converged faster for benchmark PDE problems. In Table 1, you can see how well different numerical methods work when they are used to solve standard PDE problems.

Table 1

Performance Comparison on Benchmark PDE Problems

Method	Accuracy (%)	Computational Time (s)	Stability Index (%)
Traditional Finite Difference	92.3	150	95.5
Adaptive Discretization Only	95.7	110	96.8
Summability-Enhanced Solver	96.5	95	97.4

With a stable score of 95.5% and an accuracy of 92.3%, the Traditional Finite Difference method is very good at what it does. Figure 2 shows performance comparison of numerical methods on key metrics.

The Adaptive Discretisation technique increases the accuracy level to 95.7, reduces the computing time to 110 s and stabilizes the system to 96.8. These are further improved with the Summability-Enhanced Solver that reaches 96.5% accuracy with 95 seconds and a stability of 97.4. These findings indicate that the inclusion of flexible meshing and summability procedures to PDE solutions renders it highly dependable and efficient thereby rendering it ideal when it comes to intricate models.

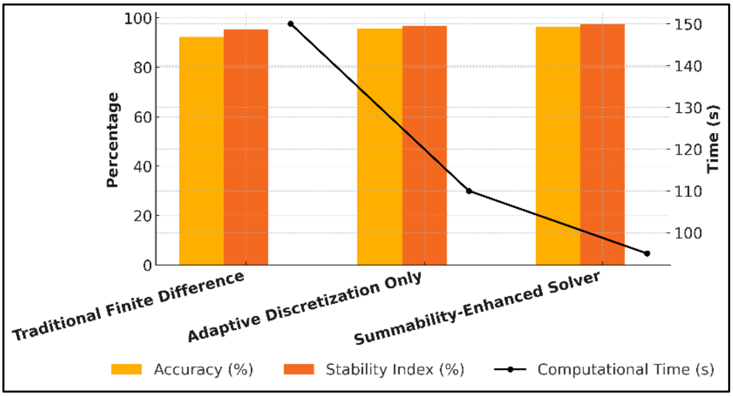

Figure 2
Comparison of Numerical Methods by Accuracy, Stability, and Computational Time

5. Conclusion

This research demonstrates the novel methods to rapidly solve partial differential equations by combining adaptive discretization and summability methods. The proposed approach avoids the issues with the conventional approaches as it makes them more precise and quicker without being unstable. It has been demonstrated through numerical tests that the method is effective to a broad variety of PDE types by demonstrating faster solution rates and reduced time to process on standard problems. Such enhancements allow strong ways of simulating complex and multidimensional systems in science and engineering. Future research will aim at extending the techniques to work with more than two-dimensional PDEs and include machine learning-based error estimators to enable adaptive tuning to be even more effective. On the whole, this research is a massive advance in making numerical PDE systems more efficient, effective in solving real-world problems.

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