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Soft quantum logic and Łukasiewicz fuzzy topology

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Abstract

In this paper we define Soft Quantum Logic and Łukasiewicz fuzzy topology, and study the relation between Soft Quantum Logic and Łukasiewicz fuzzy quantum logic, as well as the relation between Soft Quantum Logic and Soft Topology and other related structures, providing many illustrative examples.

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1. Preliminaries

The fuzzy set was first introduced in 1965 [1] so that for each given set $A \subseteq U$ and $x \in A$ there is a membership function

$$\mu_{\hat{A}}(x) \colon A \to [0,1]$$

such that

$$\hat{A} = \{(x, \mu_{\hat{A}}(x)) \colon x \in A, \mu_{\hat{A}}(x) \in [0,1]\}.$$

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The complement, in fuzzy sense, of the mentioned set $\hat{A}$ is given by

$$\neg\hat{A} = \{(x, 1 - \mu_{\hat{A}}(x)) : x \in A, \mu_{\hat{A}}(x) \in [0,1]\}.$$

The fuzzy union (sum) of two fuzzy sets generated by the same set A is defined by

$$\hat{E} \cup \hat{H} = \{(x, \mu_{\hat{E}\cup\hat{H}}(x)) : x \in E \text{or} x \in H, \mu_{\hat{E}\cup\hat{H}}(x) \in [0,1], \mu_{\hat{E}\cup\hat{H}}(x) = \max\{\mu_{\hat{E}}(x), \mu_{\hat{H}}(x)\}\}.$$

Accordingly, the intersection of fuzzy sets generated by the same set A is defined by

$$\hat{E} \cap \hat{H} = \{(x, \mu_{\hat{E}\cap\hat{H}}(x)) : x \in E \text{and} x \in H, \mu_{\hat{E}\cap\hat{H}}(x) \in [0,1], \mu_{\hat{E}\cap\hat{H}}(x) = \min\{\mu_{\hat{E}}(x), \mu_{\hat{H}}(x)\}\}.$$

The next two operations are not satisfied in general for fuzzy sets:

$\hat{E} \cup \neg\hat{E} \neq \hat{U}$, where $\hat{U}$ is the universal fuzzy set,

$\hat{E} \cap \neg\hat{E} \neq \hat{\emptyset}$, where $\hat{\emptyset}$ is the empty fuzzy set.

Example 1: (Example 1.1). 1 Let $U = \{a, b\}$ and

$$\hat{E} = \{(a, 0.1), (b, 0.5)\}, \neg\hat{E} = \{(a, 0.9), (b, 0.5)\}.$$

Then

$$\hat{E} \cup \neg\hat{E} = \{(a, 0.9), (b, 0.5)\} \neq \hat{U} = \{(a, 1), (b, 1)\},$$

$$\hat{E} \cap \neg\hat{E} = \{(a, 0.1), (b, 0.5)\} \neq \hat{\emptyset} = \{(a, 0), (b, 0)\}.$$

One of the most effective fuzzy quantum frameworks evolved from classical quantum models is the one introduced by Giles in 1976 [2]. The operations used in this model are often referred to as Giles operations, bounded operations, arithmetic operations, or Łukasiewicz operations. Later, in 1994, Pykacz [3] proposed a specific formula tailored for fuzzy sets within this context.

Łukasiewicz union and intersection for fuzzy sets are defined as follows:

$$\hat{E} \sqcup \hat{H} = \{(x, \mu_{\hat{E}\sqcup\hat{H}}(x)) : x \in E \text{ and } x \in H, \mu_{\hat{E}\sqcup\hat{H}}(x) \in [0,1], \mu_{\hat{E}\sqcup\hat{H}}(x) = \min\{\mu_{\hat{E}}(x) + \mu_{\hat{H}}(x), 1\}\},$$

$$\hat{E} \sqcap \hat{H} = \{(x, \mu_{\hat{E}\sqcap\hat{H}}(x)) : x \in E \text{ or } x \in H, \mu_{\hat{E}\sqcap\hat{H}}(x) \in [0,1], \mu_{\hat{E}\sqcap\hat{H}}(x) = \max\{0, \mu_{\hat{E}}(x) + \mu_{\hat{H}}(x) - 1\}\}.$$

Example 2: (Example 1.2) Let $U = \{c, d\}$ and

$$\hat{E} = \{(c, 0.2), (d, 0.7)\}, \ \neg\hat{E} = \{(c, 0.8), (d, 0.3)\}.$$

Then

$$\hat{E} \sqcup \neg\hat{E} = \{(c, 1), (d, 1)\} = \hat{U},$$

$$\hat{E} \sqcap \neg\hat{E} = \{(c, 0), (d, 0)\} = \hat{\emptyset}.$$

So, in 1994 [3], Pykacz built a fuzzy quantum logic depending on Łukasiewicz fuzzy union and fuzzy intersection operations in order to be consistent with traditional quantum logic over crisp sets. In 2007 [4], he summarized the idea as follows:

Let $LFQL(U)$ be a collection of fuzzy subsets of some universe U with Łukasiewicz operations fulfilling the following axioms:

1. $\widehat{\emptyset} \in LFQL(U)$,
2. If $\hat{E} \in LFQL(U)$, then $\neg\hat{E} \in LFQL(U)$,
3. The structure $LFQL(U)$ remains stable when applying countable Łukasiewicz unions to fuzzy sets that are weakly disjoint in pairs, meaning for every pair their Łukasiewicz intersection results in an empty fuzzy set. Symbolically,
$$\text{if } \hat{E}_i \sqcap \hat{E}_j = \widehat{\emptyset} \text{ for } i \neq j, \text{ then } \coprod_i \hat{E}_i \in LFQL(U),$$
4. If $\hat{E} \sqcap \hat{E} = \widehat{\emptyset}$, then $\hat{E} = \widehat{\emptyset}$.

Then the collection $LFQL(U)$ is known as Łukasiewicz fuzzy quantum logic. The Łukasiewicz fuzzy quantum logic with crisp sets corresponds to classical quantum logic.

Nadaban [5] introduced soft sets and the corresponding soft operations as follows:

For a universal set H, the pair (H, M), where M is a set of parameters, is said to be a soft set (simply S) on H if $F: M \to \mathcal{P}(H)$. That is, (H, M) is a parameterized collection of subsets of H, with every set $H(e)$ belonging to $\mathcal{P}(H)$ for each $e \in M$.

The intersection of two soft sets, say (F, A) and (G, B), over the same universe X, is a new soft set (H, C), i.e.
$$(F, A) \cap (G, B) = (H, C),$$
where the parameter set C is the overlap of A and B, and for every element e in C, the corresponding set $H(e)$ is formed by intersecting $F(e)$ and $G(e)$.

Notice: We can see from [6] that for an arbitrary soft set (F, E) we have
$$\neg(F, E) \ \tilde{\cup}\ (F, E) = \widetilde{U}, \quad (F, E) \ \tilde{\cap}\ \neg(F, E) = \widetilde{\emptyset},$$
where $\widetilde{\emptyset}$ and $\widetilde{U}$ are the empty soft set and the universal soft set, respectively. Therefore, there is no need to use the Giles union and intersection method to generalize intersection and union in soft sets like Łukasiewicz's method with fuzzy sets.

Given two soft sets (F, A) and (G, B) defined over the same universe X, their union results in a new soft set (H, C), where C represents the union of the parameter sets A and B, and for each element $e \in C$, the values of $H(e)$ are determined as follows:
$$(F, A) \ \tilde{\cup}\ (G, B) = (H, C), \ \ H(e) = \begin{cases} F(e), & \text{if } e \in A \backslash B, \\ G(e), & \text{if } e \in B \backslash A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

For a topology definition generated by crisp sets and its properties see [7] and [8]. Fuzzy topology was first defined by Chang in 1968 [1] depending on Zadeh's classical fuzzy sets [7]. Soft topology was defined in 2011 see [8] and [9], depending on Molodtsov's soft sets [5], as follows:

Let S be a family of soft subsets of an arbitrary universe U satisfying the following axioms:

1. $\widetilde{\emptyset} \in S$,
2. $\widetilde{U} \in S$,
3. If $\tilde{A}, \tilde{B} \in S$, then $\tilde{A} \,\tilde{\cap}\, \tilde{B} \in S$,
4. If $\{\tilde{A}_\omega\} \subseteq S$, then $\bigcup_\omega \tilde{A}_\omega \in S$.

Then the collection S is called a **soft topology**, and (U, S) is called a **soft topological space**.

Example: Let $U = \{k_1, k_2, k_3\}$, P (parameters) $= \{m_1, m_2\}$,

$$\widetilde{\emptyset} = \{(m_1, \emptyset), (m_2, \emptyset)\}, \widetilde{U} = \{(m_1, U), (m_2, U)\},$$

$$S_1 = \{\widetilde{\emptyset}, \widetilde{U}, \{(m_1, \{k_1\}), (m_2, \{k_3\})\}, \{(m_1, \{k_1, k_3\}), (m_2, \{k_2, k_3\})\}\},$$

$$S_2 = \{\widetilde{\emptyset}, \widetilde{U}, \{(m_1, \{k_1, k_2\}), (m_2, \{k_3\})\}, \{(m_1, U), (m_2, \{k_3\})\}\}.$$

Then (U, S_1) and (U, S_2) are both soft topological spaces.

In this research, we define **Łukasiewicz fuzzy topology** (simply LF-topology) according to Łukasiewicz union and intersection operations with examples. We also define **Soft Quantum Logic** depending on Molodtsovs soft sets [5] and fuzzy quantum logic definition [10]. Furthermore, we study the relations between:

- LF-topology and LF-quantum logic with Soft Topology and Soft Quantum Logic,
- LF-topology and Soft Topology with LF-quantum logic and Soft Quantum Logic.

Finally, we define **LF-concrete logic** and introduce some examples and counterexamples.

2. A Construction of Łukasiewicz Fuzzy Topological Space and Soft Quantum Logic

Definition 2.1: Let LF be a collection of fuzzy subsets of some universe U with Łukasiewicz operations fulfilling the following axioms:

1. $\widehat{\emptyset} \in LF$,
2. $\widehat{U} \in LF$,
3. If $\hat{E}, \hat{H} \in LF$, then $\hat{E} \sqcap \hat{H} \in LF$,
4. If $\{\hat{E}_\lambda\} \subseteq LF$, then $\coprod_\lambda \hat{E}_\lambda \in LF$.

Then the collection LF is said to be a **Łukasiewicz fuzzy topology** (simply LF-topology), and (U, LF) is called a **Łukasiewicz fuzzy topological space**.

Examples 2.2:

1. Let $U \neq \emptyset$, LF contains $\widehat{\emptyset}$ and $\widehat{U}$. Then (U, LF) is called the **indiscrete Łukasiewicz fuzzy topological space**.
2. Let $U = \{m, n\}$ and
$$LF = \{\{(m,0),(n,0)\},\{(m,1),(n,1)\},\{(m,0.2),(n,0.8)\}\}.$$
We have
$$\widehat{\emptyset} = \{(m,0),(n,0)\} \in LF, \quad \widehat{U} = \{(m,1),(n,1)\} \in LF,$$
$$\{(m,1),(n,1)\} \sqcup \{(m,0.2),(n,0.8)\} = \{(m,1),(n,1)\} \in LF,$$
$$\{(m,1),(n,1)\} \sqcap \{(m,0.2),(n,0.8)\} = \{(m,0.2),(n,0.8)\} \in LF.$$
Similarly for other sets. Therefore, the space (U, LF) is a Łukasiewicz fuzzy topological space.
3. Let $U = \{k\}$ and
$$LF = \{\{(k,0)\},\{(k,1)\},\{(k,0.1)\},\{(k,0.7)\}\}.$$
Then
$$\{(k,0.1)\} \sqcap \{(k,0.7)\} = \{(k, \max\{0, 0.1 + 0.7 - 1\})\} = \{(k,0)\} \in LF,$$
but
$$\{(k,0.1)\} \sqcup \{(k,0.7)\} = \{(k, \min\{0.1 + 0.7, 1\})\} = \{(k,0.8)\} \notin LF.$$
Therefore, the space (U, LF) is **not** a Łukasiewicz fuzzy topological space.
4. Let $U = \{d, e\}$ and
$$LF = \{\{(d,0),(e,0)\},\{(d,1),(e,1)\},\{(d,0.2),(e,0.8)\},\{(d,0.8),(e,0.2)\}\}.$$
Then
$$\begin{aligned}\{(d,0.2),(e,0.8)\} &\sqcap \{(d,0.8),(e,0.2)\} \\ &= \{(d, \max\{0, 0.2 + 0.8 - 1\}), (e, \max\{0, 0.8 + 0.2 - 1\})\} \\ &= \{(d,0),(e,0)\},\end{aligned}$$
$$\begin{aligned}\{(d,0.2),(e,0.8)\} &\sqcup \{(d,0.8),(e,0.2)\} \\ &= \{(d, \min\{0.2 + 0.8, 1\}), (e, \min\{0.8 + 0.2, 1\})\} \\ &= \{(d,1),(e,1)\} \in LF.\end{aligned}$$
Similarly for other sets. Therefore, the space (U, LF) is a Łukasiewicz fuzzy topological space.

Definition 2.3: Let $SQL(U)$ represent a collection of soft subsets defined over an arbitrary universe U which satisfies the following conditions:

1. The empty soft set $\widetilde{\emptyset}$ is included in $SQL(U)$.
2. If a soft set $\tilde{A} \in SQL(U)$, then its complement $\neg\tilde{A}$ also belongs to $SQL(U)$.
3. The collection is closed under countable unions of soft sets that are pairwise weakly disjoint; that is, the intersection of any two distinct sets in the collection is the empty soft set. Symbolically, if

$$\tilde{A}_i \cap \tilde{A}_j = \widetilde{\emptyset} \text{ for all } i \neq j,$$

then

$$\cup_i \tilde{A}_i \in SQL(U).$$

4. If the intersection of a soft set with itself yields the empty soft set, then that set must be the empty soft set itself; in other words,

$$\tilde{A} \cap \tilde{A} = \widetilde{\emptyset}\tilde{A} = \widetilde{\emptyset}.$$

A collection that meets all these criteria is referred to as a **Soft Quantum Logic over $\boldsymbol{U}$**.

Definition 2.4: A **soft concrete logic** $\mathcal{L}$ is defined as a collection of soft subsets drawn from a specific soft set Ω, satisfying the following conditions:

1. $\widetilde{\emptyset} \in \mathcal{L}$,
2. If $S \in \mathcal{L}$, then $\Omega - S \in \mathcal{L}$,
3. If $\{S_i : i \in \mathbb{N}\}$ is a countable collection of mutually disjoint soft sets, then their union

$$\cup_i S_i \in \mathcal{L}.$$

Examples 2.5: A soft concrete logic is a mathematical example of a Soft Quantum Logic.

Let $U = \{p_1, p_2\}$ be the universal set of two events, and E be the set of parameters corresponding to time t_1. Suppose the probabilities of the event Ψ have two possibilities p_1 and p_2 over time t_1. We express the probabilities of the event as a collection of soft sets:

$$\Psi = \{\widetilde{\emptyset}, (t_1, \{p_1\}), (t_1, \{p_2\}), (t_1, U)\}.$$

- $\widetilde{\emptyset} = (t_1, \emptyset)$ represents the empty soft set, meaning there are no events at time t_1.
- $(t_1, \{p_1\})$ means the event p_1 is happening at time t_1.
- $(t_1, \{p_2\})$ means the event p_2 is happening at time t_1.
- $(t_1, \{p_1, p_2\})$ means both events p_1 and p_2 are happening at the same time t_1.

We can see that

$$\widetilde{\emptyset} = (t_1, \emptyset) \in \Psi, \;\; \neg\widetilde{\emptyset} = (t_1, U) \in \Psi.$$

If $S_1 = (t_1, \{p_1\}) \in \Psi$, then $\neg S_1 = (t_1, \{p_2\}) \in \Psi$; if $S_2 = (t_1, \{p_2\}) \in \Psi$, then $\neg S_2 = (t_1, \{p_1\}) \in \Psi$; if $S_3 = (t_1, U) = \widetilde{U} \in \Psi$, then $\neg S_3 = (t_1, \emptyset) = \widetilde{\emptyset} \in \Psi$.

For $S_1 \neq S_2$, we have

$$S_1 \cap S_2 = (t_1, \{p_1\}) \cap (t_1, \{p_2\}) = \widetilde{\emptyset}.$$

Then

$$\bigcup_{i=1}^{2} S_i = (t_1, U) \in \Psi.$$

For any soft set S in this example, if $S \cap S = \widetilde{\emptyset}$, then $S = \widetilde{\emptyset}$, which holds only when $S = \widetilde{\emptyset}$. Therefore, the collection Ψ is a **Soft Quantum Logic** over U, i.e. $SQL(U)$.

3. A Comparison: Topology, Soft Topology and Łukasiewicz Fuzzy Topology; Classical Quantum Logic, Soft Quantum Logic and Łukasiewicz Fuzzy Quantum Logic

A topology and soft topology [11], [12] satisfy the axioms for an infinite collection of open sets (respectively, soft open sets). Consequently, these axioms are also satisfied for any countable collection of sets. Therefore, the subsequent study will be carried out over a countable set. You can see [13], [14].

Propositions 3.1: For a countable collection of sets:

Every topology is a soft topology but the converse is not necessarily true.

Proof: Since each classical set is a soft set but not necessarily the converse, therefore each open set is a soft open set but the converse is not necessarily true.

Every topology is a Łukasiewicz fuzzy topology but the converse is not necessarily true.

Proof: Since each classical set is a Łukasiewicz fuzzy set but the converse is not necessarily true, each open set is a Łukasiewicz fuzzy open set but the converse is not necessarily true.

Every Łukasiewicz fuzzy topology is a soft topology but the converse is not necessarily true.

Proof: Since each Łukasiewicz fuzzy set is a soft set but the converse is not necessarily true, each Łukasiewicz fuzzy open set is a soft open set but not necessarily the converse.

Propositions 3.2: Every classical quantum logic is a soft quantum logic but the converse is not necessarily true.

Proof: Since each classical set is a soft set but the converse is not necessarily true.

Every classical quantum logic is a Łukasiewicz fuzzy quantum logic but the converse is not neessarily true. Every classical quantum logic is a Łukasiewicz fuzzy quantum logic but the converse is not necessarily true.

Proof: Since each classical set is a Łukasiewicz fuzzy set but the converse is not necessarily true.

Every Łukasiewicz fuzzy quantum logic is a soft quantum logic but the converse is not necessarily true.

Proof: Since each Łukasiewicz fuzzy set is a soft set but the converse is not necessarily true.

Every classical quantum logic is a topology but the converse is not necessarily true.

Proof: Directly from the definitions.

Theorem 3.3: *For a countable collection of fuzzy sets: Every Łukasiewicz fuzzy quantum logic is a Łukasiewicz fuzzy topology, but the converse is not always true.*

Proof: Let U be a universal set and let $\mathcal{F}$ be a family of fuzzy sets on U. Suppose $\mathrm{LFQL}(U)$ is a Łukasiewicz fuzzy quantum logic generated by U and satisfying the axioms of Łukasiewicz fuzzy quantum logic. We need to show that this structure forms a Łukasiewicz fuzzy topology.

From the definition of $\mathrm{LFQL}(U)$, we have:

(a) $\hat{\varphi} \in \mathrm{LFQL}(U)$,

(b) $\neg\hat{\varphi} \in \mathrm{LFQL}(U)$.

But note that

$$\neg\hat{\varphi} = \widehat{U} \in \mathrm{LFQL}(U),$$

since $\widehat{U}$ is the universal fuzzy set.

Let $\hat{A}_1$ and $\hat{A}_2$ be two arbitrary fuzzy sets in $\mathrm{LFQL}(U)$. To prove that their fuzzy intersection also belongs to $\mathrm{LFQL}(U)$, we consider the Łukasiewicz intersection defined by

$$(\hat{A}_1 \cap \hat{A}_2)(x) = \max\{0, \hat{A}_1(x) + \hat{A}_2(x) - 1\}, \forall x \in U.$$

Since $\mathrm{LFQL}(U)$ is closed under the Łukasiewicz conjunction (meet operation), it follows that

$$\hat{A}_1 \cap \hat{A}_2 \in \mathrm{LFQL}(U).$$

Thus, the family $\mathrm{LFQL}(U)$ satisfies the axioms of Łukasiewicz fuzzy topology: it contains the universal fuzzy set, is closed under complements, and is closed under finite intersections and countable unions. Therefore, every Łukasiewicz fuzzy quantum logic is a Łukasiewicz fuzzy topology. The converse, however, does not necessarily hold.

Case 1: If $\hat{A}_1, \hat{A}_2 \in \mathrm{LFQL}(U)$ and $\hat{A}_1 \neq \hat{A}_2$, and

$$\neg\hat{A}_2 = \hat{A}_1, \neg\hat{A}_1 = \hat{A}_2,$$

then by Definition (c) of LFQL, for $\hat{A}_1 \neq \hat{A}_2$ we have

$$\hat{A}_1 \sqcap \hat{A}_2 = \hat{\varphi}.$$

Hence,

$$\hat{A}_1 \sqcup \hat{A}_2 \in \mathrm{LFQL}(U).$$

Therefore,

$$\neg(\hat{A}_1 \sqcup \hat{A}_2) = \widehat{\neg\hat{A}_1} \sqcap \widehat{\neg\hat{A}_2} = \hat{A}_2 ó \hat{A}_1 \in \mathrm{LFQL}(U).$$

Case 2: If $\hat{A}_1, \hat{A}_2 \in \mathrm{LFQL}(U)$ and $\hat{A}_1 \neq \hat{A}_2$ but

$$\neg\hat{A}_2 \neq \hat{A}_1 \text{ and } \neg\hat{A}_1 \neq \hat{A}_2,$$

then by Definition (2.4)(b), let

$$\neg\hat{A}_1 = \hat{E}_1, \ \ \neg\hat{A}_2 = \hat{E}_2,$$

where $\hat{E}_1, \hat{E}_2 \in \mathrm{LFQL}(U)$ and $\hat{E}_1 \neq \hat{E}_2$. (If $\hat{E}_1 = \hat{E}_2$, then $\neg\hat{A}_1 = \neg\hat{A}_2$ and $\hat{A}_1 = \hat{A}_2$, contradiction.)

By Definition (c) for $\hat{E}_1 \neq \hat{E}_2$, we have

$$\hat{E}_1 \sqcup \hat{E}_2 = \widehat{\neg\hat{A}_1} \sqcup \widehat{\neg\hat{A}_2}.$$

Then,

$$\neg(\hat{E}_1 \sqcup \hat{E}_2) = \widehat{\neg\hat{E}_1} \sqcap \widehat{\neg\hat{E}_2} = \hat{A}_1 \sqcap \hat{A}_2 = \hat{K}.$$

Hence,

$$\hat{K} \sqcap \neg\hat{K} = \hat{\varphi}, \hat{K} \sqcup \neg\hat{K} \in \mathrm{LFQL}(U).$$

Thus,

$$\neg(\hat{E}_1 \sqcup \hat{E}_2) = \hat{K} \in \mathrm{LFQL}(U),$$

and therefore,

$$\hat{A}_1 \sqcap \hat{A}_2 \in \mathrm{LFQL}(U).$$

Let $\{\hat{A}_\lambda\}$ be a collection of countable fuzzy sets. For arbitrary λ, assume that $\hat{A}_\lambda \in \mathrm{LFQL}(U)$. We want to prove that

$$\coprod_\lambda \hat{A}_\lambda \in \mathrm{LFQL}(U).$$

By LFQL definition (b), we have

$$\neg\hat{A}_\lambda \in \mathrm{LFQL}(U) \ \forall\lambda,$$

and by (c),

$$\neg\hat{A}_\lambda \sqcap \hat{A}_\lambda = \hat{\varphi} \ \forall\lambda.$$

Therefore,

$$\coprod_\lambda \hat{A}_\lambda \in \mathrm{LFQL}(U).$$

Thus, from items (1)-(3), $\mathrm{LFQL}(U)$ is a Łukasiewicz fuzzy topology over a countable family of fuzzy sets.

In the next example, we show that for a countable collection of fuzzy sets, a Łukasiewicz fuzzy topology is not necessarily a Łukasiewicz fuzzy quantum logic.

Indeed, let $\{\hat{A}_\lambda\}$ be a countable family of fuzzy sets with $\hat{A}_\lambda \in \mathrm{LFQL}(U)$ for each λ. Then, by (b),

$$\neg\hat{A}_\lambda \in \mathrm{LFQL}(U)\ \forall\lambda,$$

and by (c),

$$\neg\hat{A}_\lambda \sqcap \hat{A}_\lambda = \hat{\varphi}\ \forall\lambda.$$

Hence,

$$\coprod_\lambda \hat{A}_\lambda \in \mathrm{LFQL}(U).$$

Therefore, from (1)-(3), $\mathrm{LFQL}(U)$ is a fuzzy topology over a countable family of fuzzy sets. The following example shows that a Łukasiewicz fuzzy topology is not necessarily a Łukasiewicz fuzzy quantum logic.

Example 3.4: Let $G = \{c, d\}$ and

$$LF = \{\{(c, 0), (d, 0)\}, \{(c, 1), (d, 1)\}, \{(c, 0.25), (d, 0.75)\}\}.$$

We have

$$\hat{\varphi} = \{(c, 0), (d, 0)\} \in LF, \hat{U} = \{(c, 1), (d, 1)\} \in LF.$$

Moreover,

$$\{(c, 1), (d, 1)\} \sqcap \{(c, 0.25), (d, 0.75)\} = \{(c, 0.25), (d, 0.75)\} \in LF,$$

and

$$\{(c, 1), (d, 1)\} \sqcup \{(c, 0.25), (d, 0.75)\} = \{(c, 1), (d, 1)\} \in LF.$$

Similarly for other sets. Therefore, the space (U, LF) is a Łukasiewicz fuzzy topological space.

However, if $\hat{E} = \{(c, 0.25), (d, 0.75)\} \in LF$, then

$$\neg\hat{E} = \{(c, 0.25), (d, 0.25)\} \notin LF.$$

Thus, the collection LF is not a Łukasiewicz fuzzy quantum logic.

Theorem 3.5: For a countable collection of soft sets: Every soft quantum logic is a soft topology, but the converse is not always true.

Proof: Let U be a universal set and let S be a family of soft sets generated by U satisfying the axioms of soft quantum logic. We prove that this space is a soft topology.

From definition (a),

$$\tilde{\varphi} \in SQL(U),$$

and by (b),

$$\neg\tilde{\varphi} \in SQL(U),$$

but

$$\neg\tilde{\varphi} = \tilde{U} \in SQL(U).$$

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two different soft sets in $SQL(U)$. We prove that their soft intersection belongs to $SQL(U)$.

Case 1: If $\tilde{A}_1, \tilde{A}_2 \in SQL(U)$ and

$$\tilde{A}_1 \neq \tilde{A}_2, \neg\tilde{A}_2 = \tilde{A}_1, \neg\tilde{A}_1 = \tilde{A}_2,$$

then by definition (2.3)(c), since $\tilde{A}_1 \neq \tilde{A}_2$, we have

$$\tilde{A}_1 \cap \tilde{A}_2 = \tilde{\varphi},$$

and therefore

$$\tilde{A}_1 \cup \tilde{A}_2 \in SQL(U).$$

By definition (2.3)(b),

$$\neg(\tilde{A}_1 \cup \tilde{A}_2) = \neg\tilde{A}_1 \cap \neg\tilde{A}_2 = \tilde{A}_2 \cap \tilde{A}_1 \in SQL(U).$$

Case 2: If $\tilde{A}_1, \tilde{A}_2 \in SQL(U)$ and $\tilde{A}_1 \neq \tilde{A}_2$ but $\{\neg\tilde{A}_2 \neq \tilde{A}_1 \text{and} \neg\tilde{A}_1 \neq \tilde{A}_2\}$, then by Definition (2.3)(b), let

$$\neg\tilde{A}_1 = \widetilde{N}_1, \neg\tilde{A}_2 = \widetilde{N}_2,$$

where $\widetilde{N}_1, \widetilde{N}_2 \in SQL(U)$ and $\widetilde{N}_1 \neq \widetilde{N}_2$. (If $\widetilde{N}_1 = \widetilde{N}_2$, then $\neg\tilde{A}_1 = \neg\tilde{A}_2$ and $\tilde{A}_1 = \tilde{A}_2$, a contradiction.)

By Definition (2.3)(c), we have

$$\widetilde{N}_1 \neq \widetilde{N}_2, \widetilde{N}_1 \cup \widetilde{N}_2 = \neg\tilde{A}_1 \cup \neg\tilde{A}_2, \neg(\widetilde{N}_1 \cup \widetilde{N}_2) = \neg\widetilde{N}_1 \cap \neg\widetilde{N}_2 = \tilde{A}_1 \cap \tilde{A}_2 = \widetilde{K},$$

and

$$\widetilde{K} \cap \neg\widetilde{K} = \widetilde{\emptyset} \Rightarrow \widetilde{K} \cup \neg\widetilde{K} \in SQL(U).$$

Hence,

$$\neg\left(\widetilde{N}_1 \cup \widetilde{N}_2\right) = \widetilde{K} \in SQL(U), \text{ so } \tilde{A}_1 \cap \tilde{A}_2 \in SQL(U).$$

Countable unions: Let $\{\tilde{A}_\lambda\}$ be a countable collection of soft sets, where $\tilde{A}_\lambda \in SQL(U)$ for each λ. By Definition (2.3)(b),

$$\neg\tilde{A}_\lambda \in SQL(U)\ \forall\lambda.$$

Also, by Definition (2.3)(c),

$$\neg\tilde{A}_\lambda \cap \tilde{A}_\lambda = \widetilde{\emptyset}\ \forall\lambda.$$

Thus,

$$\cup_\lambda \tilde{A}_\lambda \in SQL(U).$$

Therefore, from 1-3, $SQL(U)$ is a soft topology over a countable collection of soft sets.

Example 3.6: Let

$$K = \{e_1, e_2, e_3\}$$

be the universal set with three events, and let E be the time parameter t. Suppose that the events e_1, e_2, e_3 occur over time t. We express the probabilities of the events through time t as a collection Ψ of soft sets:

$$\Psi = \{\tilde{\varphi}, (t, \{e_1\}), (t, \{e_1, e_2\}), (t, \{e_1, e_2, e_3\})\}.$$

Here,

$$\tilde{\varphi} = (t, \emptyset)$$

is the empty soft set meaning no events occur at time t,

$$\tilde{S}_1 = (t, \{e_1\}), \tilde{S}_2 = (t, \{e_1, e_2\}), \tilde{S}_3 = (t, \{e_1, e_2, e_3\})$$

represent the events happening at time t.

We have

$$\tilde{\varphi} \in \Psi, \neg\tilde{\varphi} = (t, \{e_1, e_2, e_3\}) = \widetilde{K} \in \Psi.$$

Moreover, the intersections and unions satisfy

$$\tilde{S}_1 \cap \tilde{S}_2 \in \Psi, \tilde{S}_1 \cap \tilde{\varphi} \in \Psi, \tilde{S}_1 \cap \widetilde{K} \in \Psi,$$
$$\tilde{S}_1 \cup \tilde{S}_2 \in \Psi, \tilde{S}$$

Conclusion

In this research, we defined Łukasiewicz fuzzy topology, Soft Quantum Logic, and LF-concrete logic, providing illustrative examples. We conclude the following:

- Soft Quantum Logic is a generalization of Łukasiewicz fuzzy quantum logic.
- Łukasiewicz fuzzy quantum logic is a generalization of classical quantum logic.
- Every topology is a Łukasiewicz fuzzy topology.
- Every Łukasiewicz fuzzy topology is a soft topology.
- Soft Quantum Logic is a soft topology.
- Łukasiewicz fuzzy quantum logic is a Łukasiewicz fuzzy topology.

All previous conclusions are summarized in the diagram presented in the paper.

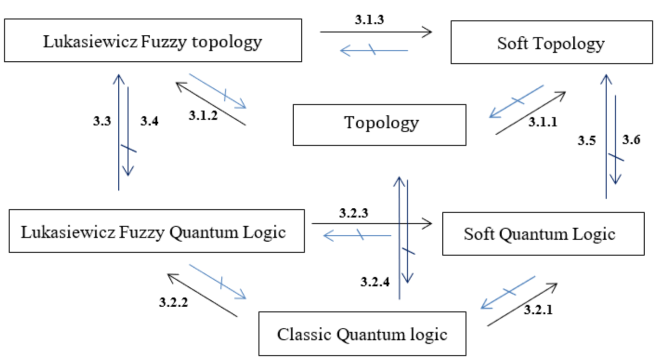

Figure 1

The diagram shows the relations between quantum and topological types over a countable collection of sets.

All previous conclusions are summarized in Figure 1.

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