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Power moments of the coefficients of an L-function over a polynomial in six variables

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Abstract

Let *f* denote a normalized Hecke eigenform of an even integral weight $\kappa \geq 2$ on the group $SL_2(\mathbb{Z})$. We study power moments of the coefficients of an *L*-function associated with the eigenform *f* over square-free positive integers represented by the polynomial

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^2 + x_2^2 + \sum_{3 \leq j \leq 6} x_j(x_j + 1) \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6].$$

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1. Introduction

Let $M_\kappa(N, \chi)$ denote the space of modular forms of even integral weight $\kappa \geq 2$ for the congruence subgroup $\Gamma_0(N)$ with the nebentypus χ and $S_\kappa(N, \chi)$ denote the subspace of cusp forms in $M_\kappa(N, \chi)$. A Hecke eigenform $f \in S_\kappa(1)$ is a cusp form on the group $\Gamma_0(1) := SL_2(\mathbb{Z})$ that is an eigenfunction for all the Hecke operators T_n with the Fourier coefficients $a_f(n)$. We say that a Hecke eigenform $f \in S_\kappa(1)$ is normalized when $a_f(1) = 1$. The normalized Fourier

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coefficients $\lambda_f(n)$ are given by

$$\lambda_f(n) := \frac{a_f(n)}{n^{\frac{\kappa-1}{2}}}.$$

The normalized Fourier coefficients $\lambda_f(n)$ are real and multiplicative. These coefficients satisfy the following Hecke relation [1, Eq. (6.83)]:

$$\lambda_f(m_1)\lambda_f(m_2) = \sum_{d|\gcd(m_1,m_2)} \lambda_f\left(\frac{m_1 m_2}{d^2}\right) \tag{1}$$

for all positive integers m_1 and m_2. Let ε be any arbitrarily small positive real number. From Deligne's bound, we have

$$|\lambda_f(n)| \leq \tau(n) \ll_\varepsilon n^\varepsilon, \tag{2}$$

where $\tau(n)$ is the divisor function. Studying the average behaviour of arithmetic functions has attracted significant attention from mathematicians and is particularly interesting. In 1927, Hecke [2] established that

$$\sum_{n\leq X} \lambda_f(n) \ll X^{\frac{1}{2}}.$$

Subsequent works have established improved upper bounds for the average behaviour of $\lambda_f(n)$ (for more details, see [3], [4]). Wu [5] proved the bound

$$\sum_{n\leq X} \lambda_f(n) \ll X^{\frac{1}{3}}(\log X)^\eta$$

for some $\eta \in \mathbb{R}$ (for the value of η, see [5]). Independently, Rankin [6] and Selberg [7] established that

$$\sum_{n\leq X} \lambda_f^2(n) = c_f X + O(X^{\frac{3}{5}}),$$

where c_f is a constant. In [8], Huang refined this result with a power-saving error term

$$\sum_{n\leq X} \lambda_f^2(n) = c_f X + O(X^\alpha),$$

where $\alpha := \frac{3}{5} - \delta$ and $\delta \leq \frac{1}{560}$. Similarly, one can also study the average behaviour of these coefficients over certain polynomials. In the case of a monic quadratic polynomial $q(x)$ in one variable, Blomer [9] studied the first moment associated with Hecke eigenvalues over the sequence given by the polynomial $q(x)$. For $q(x,y) = x^2 + y^2$, Banarjee-Pandey [10] and Acharya [11] studied the distribution of $\{\lambda_f(q(k,l))\}_{k,l\in\mathbb{Z}}$, and established an asymptotic formula for

$$\sum_{\substack{k,l\in\mathbb{Z} \\ k^2+l^2\leq X}} \lambda_f(q(k,l)).$$

Let $f(z) = \sum_{n=1}^\infty \lambda_f(n) n^{\frac{\kappa-1}{2}} e^{2\pi i n z} \in S_\kappa(1)$ be a normalized Hecke eigenform. We define the L-function $L(s,f)$ associated with f as

$$L(s,f) = \sum_{n=1}^\infty \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \tag{3}$$

for $\mathrm{Re}(s) > 1$, where $\alpha_f(p),\ \beta_f(p) \in \mathbb{C}$ satisfy

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p) \quad and \quad \alpha_f(p)\beta_f(p) = 1 = |\alpha_f(p)| = |\beta_f(p)|. \tag{4}$$

Following [12], we consider the L-function associated to f defined by

$$L(s, f \otimes f \otimes \cdots \otimes_\ell f) := \sum_{n=1}^{\infty} \frac{\lambda_{f \otimes f \otimes \cdots \otimes_\ell f}(n)}{n^s} \\ = \prod_p \prod_\sigma \left(1 - \alpha_f^{\sigma(1)}(p) \alpha_f^{\sigma(2)}(p) \cdots \alpha_f^{\sigma(\ell)}(p) p^{-s}\right)^{-1} \tag{5}$$

for $\mathrm{Re}(s) > 1$, where $\sigma: \{1,2,\ldots,\ell\} \to \{1,2\}$ runs over all the functions. The coefficients $\lambda_{f \otimes f \otimes \cdots \otimes_\ell f}(n)$ satisfy

$$\lambda_{f \otimes f \otimes \cdots \otimes_\ell f}(p) = \lambda_f^\ell(p). \tag{6}$$

Recently, several researchers have investigated the asymptotic behaviour of normalized Fourier coefficients of Hecke eigenforms; see, for instance, [13, 14, 15, 16, 17, 18, 19].

Throughout the manuscript, the symbol ε represents a positive quantity, but its value may vary in different contexts. Let ℓ be a positive integer. For $r \leq \ell$, we denote the binomial coefficient by $\binom{\ell}{r}$, and define $\binom{\ell}{r} := 0$ whenever $r < 0$, and $[m]$ denotes the greatest integer $\leq m$.

Let $\mathbb{Z}[\underline{x}]$ denote the set of polynomials over $\mathbb{Z}$ in six variables. Denote $\underline{x} := (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6$. We study power moments of the Dirichlet coefficients $\lambda_{f \otimes f \otimes \cdots \otimes_\ell f}(n)$ of the L-function (defined in (5)), at the square-free positive integers represented by the polynomial

$$g(x_1, x_2, x_3, x_4, x_5, x_6) =: g(\underline{x}) = x_1^2 + x_2^2 + \sum_{3 \leq j \leq 6} x_j(x_j + 1) \in \mathbb{Z}[\underline{x}]. \tag{7}$$

For fixed $\ell, r \in \mathbb{N}$, we prove a general asymptotic result for

$$S_{\ell,r}(X) := \sideset{}{'}\sum_{\substack{n \leq X \\ n = g(\underline{x})+1, \underline{x} \in \mathbb{Z}^6}} \lambda_{f \otimes f \otimes \cdots \otimes_\ell f}^r(n) \tag{8}$$

as $X \to \infty$, where $\sum'$ indicates that the sum is taken over square-free positive integers.

In particular, we establish the following result that includes asymptotic formulae:

Theorem 1.1: *Let ℓ, r be fixed positive integers and $f \in S_\kappa(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then, for any $\varepsilon > 0$, we have*

(i) for odd ℓ and odd r,

$$S_{\ell,r}(X) = O\left(X^{3 - \frac{3}{8\gamma_{\ell,r}} + \varepsilon}\right),$$

where

$$\gamma_{\ell,r} = \frac{1}{2} + \frac{3}{16}\left(\sum_{n=0}^{[\ell r/2]} \frac{(\ell r - 2n + 1)^2}{\ell r - n + 1} \binom{\ell r}{n}\right),$$

(ii) for even ℓ or even r,

$$S_{\ell,r}(X) = X^3 P_{\ell,r}(\log X) + O\left(X^{3 - \frac{1}{2\beta_{\ell,r}} + \varepsilon}\right),$$

where $P_{\ell,r}(\log X)$ is a polynomial in $\log X$ of degree $\frac{2}{(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}}-1$ and

$$\beta_{\ell,r}=\left(\frac{1}{8}+\frac{13}{42(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}}+\frac{15}{4(\ell r+4)}\binom{\ell r}{\frac{\ell r}{2}-1}+\frac{1}{4}\left[\sum_{n=0}^{\frac{\ell r}{2}-2}\frac{(\ell r-2n+1)^2}{\ell r-n+1}\binom{\ell r}{n}\right]\right).$$

2. Preliminaries

Let $g(\underline{x})=x_1^2+x_2^2+x_3(x_3+1)+x_4(x_4+1)+x_5(x_5+1)+x_6(x_6+1)\in\mathbb{Z}[x_1,x_2,x_3,x_4,x_5,x_6]$ be a polynomial. Denote

$$R_6(g;n):=|\{\underline{x}\in\mathbb{Z}^6: n=g(\underline{x})\}|.$$

We define the generating function $\Theta_{st}(z)$ of $R_6(g;n)$ as

$$\Theta_{st}(z) \quad :=\sum_{\underline{x}\in\mathbb{Z}^6} q^{g(\underline{x})} \quad = \quad \sum_{n=0}^{\infty} R_6(g;n)q^n, \qquad q=e^{2\pi\iota z}, z\in\mathbb{H},$$

where $\mathbb{H}$ is the poincaré upper half-plane. We consider the generalized divisor function

$$\sigma_{k;\chi,\psi}(n):=\sum_{d|n}\psi(d)\chi\left(\frac{n}{d}\right)d^k,$$

where χ and ψ are Dirichlet characters modulo N. Let $E_{k;\chi,\psi}$ be the generalized Eisenstein series

$$E_{k;\chi,\psi}(z)=\sum_{n=0}^{\infty}\sigma_{k-1;\chi,\psi}(n)q^n\in M_k(\Gamma_0(N),\chi\psi).$$

It is known that $q\Theta_{st}(z)\in M_3(4,\chi_{-4})$ (see [20, Theorem 1.3]). Also, we know that $M_3(4,\chi_{-4})$ does not contain any cusp form, and it has a basis consisting $E_{3;1,\chi_{-4}}$ and $E_{3;\chi_{-4},1}$, where 1 denotes the trivial character modulo 4. Hence, Sturm's bound allows us to get $q\Theta_{st}(z)=E_{3;\chi_{-4},1}(z)$. This gives

$$R_6(g;n-1)=\sigma_{2;\chi_{-4},1}(n). \tag{9}$$

We write (8) as

$$\begin{aligned} S_{\ell,r}(X) &:= \sum_{\substack{g(\underline{x})+1\le X\\ \underline{x}\in\mathbb{Z}^6}}{}' \lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(g(\underline{x})+1)\\ &= \sum_{n\le X}{}' \lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(n)R_6(g;n-1)\\ &= \sum_{n\le X}{}' \lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(n)\sigma_{2;\chi_{-4},1}(n). \end{aligned} \tag{10}$$

To estimate $S_{\ell,r}(X)$ defined in (10), consider the Dirichlet series

$$R_{\ell,r}(s) \quad :=\sum_{n\ge1}{}' \frac{\lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(n)\sigma_{2;\chi_{-4},1}(n)}{n^s} \tag{11}$$

for $\mathrm{Re}(s)\gg 1$. To establish our main result, first we decompose $R_{\ell,r}(s)$ into smaller degrees L-functions. Then, using their analytic properties, we obtain the desired estimate.

For $m\ge 0$, we define the mth symmetric power L-function associated with the normalized Hecke eigenform $f\in S_\kappa(1)$ as

$$L(s,sym^m f):=\prod_p\prod_{j=0}^{m}\left(1-\frac{\alpha_f(p)^{m-j}\beta_f(p)^j}{p^s}\right)^{-1}=\sum_{n=1}^{\infty}\frac{\lambda_{sym^m f}(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$. The function $\lambda_{sym^m f}(n)$ is real and multiplicative. For each prime p, we have

$$\lambda_{sym^m f}(p) = \sum_{j=0}^{m} \alpha_f(p)^{m-j}\beta_f(p)^j. \tag{12}$$

It is known from Deligne's bound that

$$|\lambda_{sym^m f}(n)| \le \tau_{m+1}(n) \ll_\varepsilon n^\varepsilon$$

for any $\varepsilon > 0$, where $\tau_{m+1}(n)$ denotes the $(m+1)$ dimensional divisor function.

Next, we define the twisted mth symmetric power L-function as

$$L(s, sym^m f \times \chi) := \sum_{n\ge 1} \frac{\lambda_{sym^m f}(n)\chi(n)}{n^s} \tag{13}$$

for $\mathrm{Re}(s) > 1$. Recently, Newton and Thorne [21, 22] proved that $L(s, sym^m f)$ is an entire function for each $m \ge 1$.

We see that $L(s, sym^0 f) = \zeta(s)$, $L(s, sym^1 f) = L(s, f)$, $L(s, sym^0 f \times \chi) = L(s, \chi)$, and $L(s, sym^1 f \times \chi) = L(s, f \times \chi)$.

Throughout the paper, we denote $s := \gamma + \iota t$. Now, we state some valuable results in proving Theorem 1.1.

Lemma 2.1: [23, Lemma 2.2] *Let ℓ be a positive integer and j be a non-zero integer. Define*

$$A_{\ell,j} := \begin{cases} \binom{\ell}{\frac{\ell-j}{2}} - \binom{\ell}{\frac{\ell-j}{2}-1}, & \text{if } 0 \le j \le \ell \text{ and } j \equiv \ell \pmod 2, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, define $T_m(2x) := W_m(x)$, where $W_m(x)$ is the m th Chebyshev polynomial of the second kind. Then

$$x^\ell = \sum_{j=0}^{\ell} A_{\ell,j} T_{\ell-j}(x).$$

Lemma 2.2: *For any $\varepsilon > 0$, we have*

$$\zeta(\gamma + \iota t) \ll_\varepsilon (1 + |t|)^{\frac{13}{42}(1-\gamma)+\varepsilon}$$

$$\text{and} \qquad L(\gamma + \iota t, \chi) \ll_\varepsilon (1 + |t|)^{\frac{13}{42}(1-\gamma)+\varepsilon} \tag{14}$$

uniformly for $\gamma \in [\frac{1}{2}, 1 + \varepsilon]$ and $|t| \ge 1$.

Proof: The result follows from [24, Theorem 5] and [25, Lemma 2.7].

Lemma 2.3: [26, Corollary 1.2] *For any $\varepsilon > 0$, we have*

$$L(\gamma + \iota t, sym^2 f) \ll_{f,\varepsilon} (1 + |t|)^{\frac{5}{4}(1-\gamma)+\varepsilon} \tag{15}$$

uniformly for $\gamma \in \left[\frac{1}{2}, 1 + \varepsilon\right]$ and $|t| \ge 1$. The same subconvexity bound will work when $sym^2 f$ is twisted by a Dirichlet character χ_D modulo D and the absolute constant depends on f, ε and $|D|$.

Let $\mathfrak{L}(s)$ be a general L-function (in the sense of Perelli [27]) of degree $m \geq 2$.

Lemma 2.4: *For any* $\varepsilon > 0$, *we have*

(i) $\int_1^T |\mathfrak{L}(s)|^2 dt \ll_\varepsilon T^{m(1-\gamma)+\varepsilon}$ *uniformly for* $\gamma \in \left[\frac{1}{2}, 1+\varepsilon\right]$ *and* $T \geq 1$,

(ii) $\mathfrak{L}(s) \ll_\varepsilon (1+|t|)^{\frac{m}{2}(1-\gamma)+\varepsilon}$ *uniformly for* $\gamma \in \left[\frac{1}{2}, 1+\varepsilon\right]$ *and* $|t| \geq 1$.

Proof: It follows from [27, Theorem 4] and [28, Proposition 1].

3. Key Lemma

The next result is crucial in proving the main theorem.

Lemma 3.1: *For fixed positive integers* ℓ *and* r, *we have*

$$R_{\ell,r}(s) = L_{\ell,r}(s) U_{\ell,r}(s),$$

where for each odd ℓ *and odd* r,

$$L_{\ell,r}(s) = \prod_{n=0}^{[\ell r/2]} \left(L(s-2, sym^{\ell r-2n} f)^{\left(\binom{\ell r}{n} - \binom{\ell r}{n-1}\right)} L(s, sym^{\ell r-2n} f \times \chi_{-4})^{\left(\binom{\ell r}{n} - \binom{\ell r}{n-1}\right)} \right), \tag{16}$$

and for each even ℓ *or even* r,

$$L_{\ell,r}(s) = \zeta(s-2)^{\left(\binom{\ell r}{\ell r/2} - \binom{\ell r}{\ell r/2-1}\right)} L(s, \chi_{-4})^{\left(\binom{\ell r}{\ell r/2} - \binom{\ell r}{\ell r/2-1}\right)}$$
$$\times \prod_{n=0}^{\frac{\ell r}{2}-1} \left(L(s-2, sym^{\ell r-2n} f)^{\left(\binom{\ell r}{n} - \binom{\ell r}{n-1}\right)} L(s, sym^{\ell r-2n} f \times \chi_{-4})^{\left(\binom{\ell r}{n} - \binom{\ell r}{n-1}\right)} \right), \tag{17}$$

where $U_{\ell,r}(s)$ is a Dirichlet series converging absolutely and uniformly in the region $\gamma > \frac{5}{2}$ and $U_{\ell,r}(s)$ does not vanish at $\gamma = 3$.

Proof: By Deligne's bound, it is known that $\lambda_f(p) = 2\cos\theta_p$ and $\lambda_f(p^\ell) = T_\ell(2\cos\theta_p)$ for some $\theta_p \in [0, \pi]$, where $T_\ell(2x) = W_\ell(x)$ and $W_\ell(x)$ is the ℓth Chebyshev polynomial of the second kind. Following (6) and Lemma 2.1 , we get

$$\lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(p) = \lambda_f{}^{\ell r}(p) = \left(\sum_{0\leq m\leq[\ell r/2]} \left(\binom{\ell r}{m} - \binom{\ell r}{m-1} \right) \lambda_{sym^{\ell r-2m} f}(p) \right). \tag{18}$$

Since $\lambda_f(n)$ and $\sigma_{2;\chi_{-4},1}(n)$ are multiplicative, the function $R_{\ell,r}(s)$ has the Euler product

$$R_{\ell,r}(s):=\sum_{n\geq 1}{}' \frac{\lambda_f{}^{\ell r}(p)(n)\sigma_{2;\chi_{-4},1}(n)}{n^s}=\prod_p\left(1+\frac{(\lambda_f(p))^{\ell r}\sigma_{2;\chi_{-4},1}(p)}{p^s}\right)$$

for $\mathrm{Re}(s)>3$. Since

$$\sigma_{2;\chi_{-4},1}(p)=p^2+\chi_{-4}(p),$$

we have

$$\begin{aligned}\lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(p)\sigma_{2;\chi_{-4},1}(p) &= \lambda_f^{\ell r}(p)\sigma_{2;\chi_{-4},1}(p)\\ &=\left(\sum_{0\leq m\leq[\ell r/2]}\left(\binom{\ell r}{m}-\binom{\ell r}{m-1}\right)\lambda_{sym^{\ell r-2m}f}(p)\right)(p^2+\chi_{-4}(p))\\ &=p^2\sum_{0\leq m\leq[\ell r/2]}\left(\binom{\ell r}{m}-\binom{\ell r}{m-1}\right)\lambda_{sym^{\ell r-2m}f}(p)\\ &\quad+\sum_{0\leq m\leq[\ell r/2]}\left(\binom{\ell r}{m}-\binom{\ell r}{m-1}\right)\lambda_{sym^{\ell r-2m}f}(p)\chi_{-4}(p).\end{aligned}$$

We define

$$L_{\ell,r}(s)=\prod_{n=0}^{\left[\frac{\ell r}{2}\right]}\left(L(s-2,sym^{\ell r-2n}f)^{\left(\binom{\ell r}{n}-\binom{\ell r}{n-1}\right)}L(s,sym^{\ell r-2n}f\times\chi_{-4})^{\left(\binom{\ell r}{n}-\binom{\ell r}{n-1}\right)}\right)$$

for $\mathrm{Re}(s)>3$, and write its Euler product in the following form

$$\prod_p\left(1+\frac{A(p)}{p^s}+\frac{A(p^2)}{p^{2s}}+\cdots\right),\quad \text{where}\quad A(p)=-\lambda_f(p)^{\ell r}\sigma_{2;\chi_{-4},1}(p).$$

Define the sequence

$$B(p^r):=\begin{cases}A(p^r)+A(p^{r-1})\lambda_f^{\ell r}(p)\sigma_{2;\chi_{-4},1}(p) & \text{if}\quad r\geq 2\\ 0\quad if\quad r=1.\end{cases}$$

Since $B(n)\ll n^{2+\varepsilon}$ for any positive ε, we define the Euler product for $U_{\ell,r}(s)$ as

$$U_{\ell,r}(s)=\prod_p\left(1+\frac{B(p)}{p^s}+\frac{B(p^2)}{p^{2s}}+\cdots\right).$$

Thus, we get the result.

4. Proof of Theorem 1.1

Deligne's bound (respectively, Weil's bound) implies that

$$\lambda_{f\otimes f\otimes\cdots\otimes_\ell f}(n)\ll n^\varepsilon\quad\text{and}\quad\sigma_{2;\chi_{-4},\mathbf{1}}(n)\ll n^{2+\varepsilon}$$

for any $\varepsilon>0$. Let $1\leq Y<\frac{X}{2}$, and consider a smooth, compactly supported function $\omega(x)$ such that:

$$\omega(x)=\begin{cases}1 & \text{for } x\in[2Y,X],\\ 0 & \text{for } x<Y \text{ or } x>X+Y,\end{cases}\quad\text{and}\quad \omega^{(r)}(x)\ll_r Y^{-r}\text{ for all } r\geq 0.$$

Then, we have

$$S_{\ell,r}(X) = {\sum_{n\geq 1}}' \lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(n)\, \sigma_{2;\chi_{-4},\mathbf{1}}(n)\omega(n) + O(YX^{2+\varepsilon}).$$

For more details, see [29]. By Mellin inversion, the main term becomes

$${\sum_{n\geq 1}}' \lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(n)\, \sigma_{2;\chi_{-4},\mathbf{1}}(n)\, \omega(n) = \frac{1}{2\pi\iota}\int_{(3+\varepsilon)} \widetilde{\omega}(s)\, R_{\ell,r}(s)\, ds,$$

where $\widetilde{\omega}(s)$ is the Mellin transform of $\omega(x)$. Shifting the contour of integration from $\mathrm{Re}(s) = 3+\varepsilon$ to $\mathrm{Re}(s) = \gamma_0$, for a fixed $\frac{5}{2}+\varepsilon \leq \gamma_0 \leq 3+\varepsilon$, and applying Cauchy's residue theorem, we obtain

$$\frac{1}{2\pi\iota}\int_{(3+\varepsilon)} \widetilde{\omega}(s)\, R_{\ell,r}(s)\, ds = \operatorname*{Res}_{s=3}\big(R_{\ell,r}(s)\, \widetilde{\omega}(s)\big) + \left|V_{\ell,r}\right| + O(X^{-A})$$

for any arbitrarily large $A \in \mathbb{R}$, where

$$V_{\ell,r} = \frac{1}{2\pi\iota}\int_{(\gamma_0)} \widetilde{\omega}(s)\, R_{\ell,r}(s)\, ds. \tag{19}$$

Therefore, it suffices to find the residue at $s = 3$ and estimate the integral $V_{\ell,r}$.

Applying Lemma 3.1, we estimate

$$\left|V_{\ell,r}\right| \ll X^{\gamma_0} \int_{-T}^{T} \frac{|L_{\ell,r}(\gamma_0+\iota t)|}{|\gamma_0+\iota t|}\, dt \ll X^{\gamma_0}\left\{\int_0^1 + \int_1^T\right\}\frac{|L_{\ell,r}(\gamma_0+\iota t)|}{|\gamma_0+\iota t|}\, dt,$$

for any fixed $\gamma_0 \in \left(\frac{5}{2}, 3\right)$. Using Lemma 3.1 for the first integral and a dyadic subdivision for the second, we conclude

$$\left|V_{\ell,r}\right| \ll X^{\gamma_0} + X^{\gamma_0}\log T \cdot \max_{1\leq T_1\leq T} I_{\ell,r}(T_1), \tag{20}$$

where

$$I_{\ell,r}(T_1) = \frac{1}{T_1}\int_{T_1}^{2T_1} |L_{\ell,r}(\gamma_0+\iota t)| dt. \tag{21}$$

This implies

$$S_{\ell,r}(X) = \operatorname*{Res}_{s=3}\left(R_{\ell,r}(s)\frac{X^s}{s}\right) + |V_{\ell,r}| + O(X^{-A}) + O(YX^{2+\varepsilon}), \tag{22}$$

where $T = \frac{X^{1+\varepsilon}}{Y}$ and Y is chosen later. Thus, we are left to estimate $I_{\ell,r}(T)$ to get the desired result for $S_{\ell,r}(X)$.

(i) (Both $\boldsymbol{\ell}$ and $\boldsymbol{r}$ odd): We take $\gamma_0 = 21/8 \in (5/2, 3)$, substitute (16) in (21), and apply Cauchy-Schwarz's inequality, which implies

$$\left|I_{\ell,r}(T_1)\right| \ll \frac{1}{T_1}\int_{T_1}^{2T_1} |L_{\ell,r}(\gamma_0+\iota t)| dt$$

$$\ll \frac{1}{T_1}\log T_1 \times T_1^{\frac{1}{2}} \times \left(\int_{T_1}^{2T_1}\left|\prod_{n=0}^{[\ell r/2]} L(\gamma_0-2+\iota t, sym^{\ell r-2n}f)^{\left(\binom{\ell r}{n}-\binom{\ell r}{n-1}\right)}\right|^2 dt\right)^{\frac{1}{2}}.$$

By applying Lemma 2.4(i), we get

$$\left|I_{\ell,r}(T_1)\right| \ll \frac{1}{T_1}\log T_1 \times T_1^{\frac{1}{2}} \times T_1^{\frac{3}{16}\left(\sum_{n=0}^{\left[\frac{\ell r}{2}\right]}\frac{(\ell r-2n+1)^2}{(\ell r-n+1)}\binom{\ell r}{n}\right)+\varepsilon}.$$

Thus, we have

$$\left|I_{\ell,r}(T_1)\right| \ll T_1^{\frac{1}{2}+\frac{3}{16}\left(\sum_{n=0}^{[\ell r/2]}\frac{(\ell r-2n+1)^2}{\ell r-n+1}\binom{\ell r}{n}\right)-1+\varepsilon}. \tag{23}$$

We substitute (23) into (20) to get

$$\left|V_{\ell,r}\right| \ll X^{\frac{21}{8}} + X^{\frac{21}{8}}T^{\frac{1}{2}+\frac{3}{16}\left(\sum_{n=0}^{[\ell r/2]}\frac{(\ell r-2n+1)^2}{\ell r-n+1}\binom{\ell r}{n}\right)-1+\varepsilon}. \tag{24}$$

Since $R_{\ell,r}(s)$ is an analytic function in the region $\mathrm{Re}(s) > 5/2$, therefore, combining (22) and (24), we have

$$S_{\ell,r}(X) = {\sum_{n\le X}}' \lambda^r_{f\otimes f\otimes\cdots\otimes_\ell f}(n)\sigma_{2;\chi_{-4},1}(n) = O\left(X^{\frac{21}{8}}T^{\gamma_{\ell,r}-1+\varepsilon}\right)$$
$$+O(YX^{2+\varepsilon}) + O(X^{-A}),$$

where

$$\gamma_{\ell,r} = \frac{1}{2} + \frac{3}{16}\left(\sum_{n=0}^{[\ell r/2]}\frac{(\ell r-2n+1)^2}{\ell r-n+1}\binom{\ell r}{n}\right).$$

We put $T = \frac{X^{1+\varepsilon}}{Y}$ and then select $Y = X^{1-\frac{3}{8\gamma_{\ell,r}}+\varepsilon}$ to get

$$S_{\ell,r}(X) = O\left(X^{3-\frac{3}{8\gamma_{\ell,r}}+\varepsilon}\right).$$

(ii) (Either $\boldsymbol{\ell}$ is even or $\boldsymbol{r}$ is even): We take $\gamma_0 = \frac{5}{2}+\varepsilon$ and substitute the decomposition of the function $L_{\ell,r}(s)$ (from (17)) when ℓ or r is even in (21) to get

$$\left|I_{\ell,r}(T_1)\right| \ll \frac{1}{T_1}\int_{T_1}^{2T_1}|L_{\ell,r}(\gamma_0+\iota t)|dt$$
$$\ll \frac{1}{T_1}\times \sup_{T_1\le t\le 2T_1}\left(|\zeta(\gamma_0-2+\iota t)|^{\left(\binom{\ell r}{\ell r/2}-\binom{\ell r}{\ell r/2-1}\right)}\right.$$
$$\left.\times|L(\gamma_0-2+\iota t, sym^2 f)|^{\left(\binom{\ell r}{\ell r/2-1}-\binom{\ell r}{\ell r/2-2}-1\right)}\right)$$
$$\times\left(\int_{T_1}^{2T_1}|L(\gamma_0-2+\iota t, sym^2 f)|^2 dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_{T_1}^{2T_1}\left|\prod_{n=0}^{\frac{\ell r}{2}-2}L(\gamma_0-2+\iota t, sym^{\ell r-2n}f)^{\left(\binom{\ell r}{n}-\binom{\ell r}{n-1}\right)}\right|^2 dt\right)^{\frac{1}{2}}.$$

This implies

$$\left|I_{\ell,r}(T_1)\right| \ll T_1^{\left(\frac{13}{42(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}}+\frac{15}{4(\ell r+4)}\binom{\ell r}{\frac{\ell r}{2}-1}+\frac{1}{4}\left[\sum_{n=0}^{\frac{\ell r}{2}-2}\frac{(\ell r-2n+1)^2}{\ell r-n+1}\binom{\ell r}{n}\right]-\frac{7}{8}+\varepsilon\right)}.$$

From the above and (20), we get

$$\left|V_{\ell,r}\right| \ll X^{\frac{5}{2}+\varepsilon} + X^{\frac{5}{2}+\varepsilon} T^{\left(\frac{13}{42(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}} + \frac{15}{4(\ell r+4)}\binom{\ell r}{\frac{\ell r}{2}-1} + \frac{1}{4}\left[\sum_{n=0}^{\frac{\ell r}{2}-2} \frac{(\ell r-2n+1)^2}{\ell r-n+1}\binom{\ell r}{n}\right] - \frac{7}{8}+\varepsilon\right)}. \quad (25)$$

Note that $R_{\ell,r}(s)$ has a pole at $s = 3$ of order $\frac{2}{(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}}$. Thus, substituting the estimate of $V_{\ell,r}$ from (25) in (22), we have

$$S_{\ell,r}(X) = X^3 P_{\ell,r}(\log X) + O\left(X^{\frac{5}{2}+\varepsilon}\right) + O\left(X^{\frac{5}{2}+\varepsilon} T^{(\beta_{\ell,r}-1+\varepsilon)}\right) + O(YX^{2+\varepsilon}) + O(X^{-A}),$$

where

$$\beta_{\ell,r} = \left(\frac{1}{8} + \frac{13}{42(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}} + \frac{15}{4(\ell r+4)}\binom{\ell r}{\frac{\ell r}{2}-1} + \frac{1}{4}\left[\sum_{n=0}^{\frac{\ell r}{2}-2} \frac{(\ell r-2n+1)^2}{(\ell r-n+1)}\binom{\ell r}{n}\right]\right).$$

We put $T = \frac{X^{1+\varepsilon}}{Y}$ and then select $Y = X^{1-\frac{1}{2\beta_{\ell,r}}+\varepsilon}$ to get

$$S_{\ell,r}(X) = X^3 P_{\ell,r}(\log X) + O\left(X^{3-\frac{1}{2\beta_{\ell,r}}+\varepsilon}\right),$$

where $P_{\ell,r}(\log X)$ is a polynomial in $\log X$ of degree $\frac{2}{(\ell r+2)}\binom{\ell r}{\frac{\ell r}{2}} - 1$. This completes the proof.

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Conflict of interest The corresponding author declares, on behalf of both authors, that there are no conflicts of interest to disclose.

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