

On triple sequences spaces described by triple cubic maximal Foug functions

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Abstract

In this work, we offer the triple sequences spaces defined by the triple cubic maximal Foug functions. Some academics defines the concept of an Orlicz function as follows: Let $w:[0, \infty) \rightarrow [0, \infty)$ is called the Orlicz function if it is a continuous, non-decreasing, convex function that satisfies the following properties: $w(0) = 0$, $w(\vartheta) > 0$ for $\vartheta > 0$, $w(\vartheta) \rightarrow \infty$ as $\vartheta \rightarrow \infty$. Additionally, Foug function can be defined as follows: If $\psi(\mathfrak{L}) = \mathfrak{L} w(\mathfrak{L})$, where w is an Orlicz function then $\psi:[0, \infty) \rightarrow [0, \infty)$ is said to be Foug function. For the triple sequences' spaces $(\ell_{\mathfrak{N}})^3(p)$, we illustrate some of their fundamental topological and algebraic properties, $(\ell_{\mathfrak{N}})^3(p)$ is complete space, and soon. We also discuss other properties to the space.

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1. Introduction

This work presents the idea of the on triple sequences spaces identified by triple cubic maximal Foug functions. The Orlicz function is defined by Tripathy and Goswami [11] and Tripathy and Sarma [9], who also define the term modulus function as the function that results from substituting the convexity of the Orlicz function Θ for $\Theta(i + j) \leq \Theta(i) + \Theta(j)$. A space $\ell_{\Theta} = \left\{ \mathfrak{F} \in \mathbb{W} : \sum_{u=1}^{\infty} \Theta \left(\frac{|\mathfrak{F}_u|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}$. Were created by Savaş and Mursaleen [8] developed them utilizing the Orlicz function theory. ℓ_{Θ} with the norm $\|\mathfrak{F}\| = \inf \left\{ \rho > 0 : \sum_{u=1}^{\infty} \Theta \left(\frac{|\mathfrak{F}_u|}{\rho} \right) \leq 1 \right\}$ turns into a Banach space known as

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Lindenstrauss and Tazfirri's evidence states that every Orlicz sequence space ℓ_ϕ contains a subspace isomorphic to ℓ_p for any $p \geq 1$. (Refer to Tripathy [7] and Tripathy and Borgohain [10]). Battor and Niamah [12] first proposed the idea of statistically convergent double sequences in 2003. Tripathy and Sarma [13] presented the concept of sequence spaces of fuzzy real numbers provided by the Orlicz function in 2008. Tripathy and Sarma [14] used the vector valued double sequence spaces generated by the Orlicz function in 2009. Savaş and Gürdal, [5] introduced and clarified the convergent and statistically convergent for sequence spaces of fuzzy real numbers described by in 2010. Esi and Hazarika [6] were pursuing the idea of statistically convergent difference sequence spaces described by Orlicz function in 2016. Battor and Neamah [1] created statistically convergent double sequence spaces described by the Orlicz function in 2017. In 2020 saw the publication of convergent and statistically convergent double Sequences spaces of fuzzy real numbers described by double Orlicz dunctions by Battor and Dabbas [2]. Battor and Mansoor [3] discussed some research on double sequence spaces of fuzzy numbers that are double Orlicz function described in 2021. In 2022, research on certain triple sequence spaces of fuzzy numbers defined by double Orlicz functions is submitted by Battor and Al-Hussein [4].

2. Definitions and Preliminaries

A cubic maximal Fougng function $\varrho: [0, \infty) \rightarrow [0, \infty)$ is a function such that $\varrho(\varrho) = (\varrho^2)^3 \psi(\varrho)$ and ψ is a Fougng function. A triple cubic maximal Fougng function $\mathcal{H}: [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty) \times [0, \infty)$ is a function such that $\mathcal{H}(\mathfrak{G}, \mathfrak{E}, \mathfrak{C}) = (\mathcal{H}_1(\mathfrak{G}), \mathcal{H}_2(\mathfrak{E}), \mathcal{H}_3(\mathfrak{C}))$, where $\mathcal{H}_1: [0, \infty) \rightarrow [0, \infty) \ni \mathcal{H}_1(\mathfrak{G}) = (\mathfrak{G}^2)^3 \psi_1(\mathfrak{G})$, $\mathcal{H}_2: [0, \infty) \rightarrow [0, \infty) \ni \mathcal{H}_2(\mathfrak{E}) = (\mathfrak{E}^2)^3 \psi_2(\mathfrak{E})$, $\mathcal{H}_3: [0, \infty) \rightarrow [0, \infty) \ni \mathcal{H}_3(\mathfrak{C}) = (\mathfrak{C}^2)^3 \psi_3(\mathfrak{C})$. These functions are even, convex, continuous, and non-decreasing, that hold the following conditions:

- i) $\mathcal{H}_1(0) = 0, \mathcal{H}_2(0) = 0, \mathcal{H}_3(0) = 0 \Rightarrow \mathcal{H}(0, 0, 0) = (\mathcal{H}_1(0), \mathcal{H}_2(0), \mathcal{H}_3(0)) = (0, 0, 0)$
- ii) $\mathcal{H}_1(\mathfrak{G}) > 0, \mathcal{H}_2(\mathfrak{E}) > 0, \mathcal{H}_3(\mathfrak{C}) > 0 \Rightarrow \mathcal{H}(\mathfrak{G}, \mathfrak{E}, \mathfrak{C}) = (\mathcal{H}_1(\mathfrak{G}), \mathcal{H}_2(\mathfrak{E}), \mathcal{H}_3(\mathfrak{C})) > (0, 0, 0), \forall \mathfrak{G} > 0, \mathfrak{E} > 0, \mathfrak{C} > 0$ by which we say $(\mathfrak{G}, \mathfrak{E}, \mathfrak{C}) > (0, 0, 0)$ as $\mathcal{H}_1(\mathfrak{G}) > 0, \mathcal{H}_2(\mathfrak{E}) > 0, \mathcal{H}_3(\mathfrak{C}) > 0$.
- iii) $\mathcal{H}_1(\mathfrak{G}) \rightarrow \infty, \mathcal{H}_2(\mathfrak{E}) \rightarrow \infty, \mathcal{H}_3(\mathfrak{C}) \rightarrow \infty$ as $\mathfrak{G} \rightarrow \infty, \mathfrak{E} \rightarrow \infty, \mathfrak{C} \rightarrow \infty \Rightarrow \mathcal{H}(\mathfrak{G}, \mathfrak{E}, \mathfrak{C}) = (\mathcal{H}_1(\mathfrak{G}), \mathcal{H}_2(\mathfrak{E}), \mathcal{H}_3(\mathfrak{C})) \rightarrow (\infty, \infty, \infty)$ as $(\mathfrak{G}, \mathfrak{E}, \mathfrak{C}) \rightarrow (\infty, \infty, \infty)$ by which we say $\mathcal{H}(\mathfrak{G}, \mathfrak{E}, \mathfrak{C}) \rightarrow (\infty, \infty, \infty)$ as $\mathcal{H}_1(\mathfrak{G}) \rightarrow \infty, \mathcal{H}_2(\mathfrak{E}) \rightarrow \infty, \mathcal{H}_3(\mathfrak{C}) \rightarrow \infty$.

$$(\ell_{\mathcal{H}})^3(\rho) = \{\mathfrak{N} = ((\mathfrak{N}_1)_{\text{abc}}, (\mathfrak{N}_2)_{\text{abc}}, (\mathfrak{N}_3)_{\text{abc}}) \in \mathbb{W}^3: \sum_{\mathfrak{a}=1}^{\infty} \sum_{\mathfrak{b}=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|(\mathfrak{N}_1)_{\text{abc}}|}{\rho} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{N}_2)_{\text{abc}}|}{\rho} \right) \vee \mathcal{H}_3 \left(\frac{|(\mathfrak{N}_3)_{\text{abc}}|}{\rho} \right) \right]^{\rho_{\text{abc}}} < \infty, \text{ for some } \rho > 0\},$$

where $\mathbb{W}^3 = \{(\mathfrak{N} = ((\mathfrak{N}_1)_{\text{abc}}, (\mathfrak{N}_2)_{\text{abc}}, (\mathfrak{N}_3)_{\text{abc}}): \mathfrak{N} \in \mathbb{R} \text{ or } \mathbb{C}\}$ and $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ and $\mathcal{P} = (\mathcal{P}_{\text{abc}})$ is triple sequence of positive real numbers.

3. Results and Discussion

Theorem 3.1: Let $\mathbb{H} = \sup_{\text{tsr}} \mathcal{P}_{\text{tsr}}$, then over the set of complex numbers $\mathbb{C}$, the space $(\ell_{\mathcal{H}})^3(\mathcal{P})$ is linear.

Proof: Let $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3), \mathfrak{S} = (\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3) \in (\ell_{\mathcal{H}})^3(\mathcal{P})$ and $\alpha, \beta \in \mathbb{C}$. Since $\mathfrak{A}, \mathfrak{S} \in (\ell_{\mathcal{H}})^3(\mathcal{P}), \exists \rho_1, \rho_2 > 0 \exists$

$$\sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{A}_1|_{\text{tsr}}}{\rho_1} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{A}_2|_{\text{tsr}}}{\rho_1} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{A}_3|_{\text{tsr}}}{\rho_1} \right) \right]^{\mathcal{P}_{\text{tsr}}} < \infty$$

and $\sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{S}_1|_{\text{tsr}}}{\rho_2} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{S}_2|_{\text{tsr}}}{\rho_2} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{S}_3|_{\text{tsr}}}{\rho_2} \right) \right]^{\mathcal{P}_{\text{tsr}}} < \infty$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{H}$ is convex and non-decreasing,

$$\begin{aligned} & \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\alpha(\mathfrak{A}_1)_{\text{tsr}} + \hat{\alpha}(\mathfrak{S}_1)_{\text{tsr}}|}{\tilde{\mathfrak{n}}_3} \right) \vee \mathcal{H}_2 \left(\frac{|\alpha(\mathfrak{A}_2)_{\text{tsr}} + \hat{\alpha}(\mathfrak{S}_2)_{\text{tsr}}|}{\tilde{\mathfrak{n}}_3} \right) \vee \mathcal{H}_3 \left(\frac{|\alpha(\mathfrak{A}_3)_{\text{tsr}} + \hat{\alpha}(\mathfrak{S}_3)_{\text{tsr}}|}{\tilde{\mathfrak{n}}_3} \right) \right]^{\rho_{\text{tsr}}} \\ & \preceq \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\alpha(\mathfrak{A}_1)_{\text{tsr}}|}{\tilde{\mathfrak{n}}_3} + \frac{|\mathfrak{S}_1|_{\text{tsr}}}{\tilde{\mathfrak{n}}_3} \right) \vee \mathcal{H}_2 \left(\frac{|\alpha(\mathfrak{A}_2)_{\text{tsr}}|}{\tilde{\mathfrak{n}}_3} + \frac{|\mathfrak{S}_2|_{\text{tsr}}}{\tilde{\mathfrak{n}}_3} \right) \vee \mathcal{H}_3 \left(\frac{|\alpha(\mathfrak{A}_3)_{\text{tsr}}|}{\tilde{\mathfrak{n}}_3} + \frac{|\mathfrak{S}_3|_{\text{tsr}}}{\tilde{\mathfrak{n}}_3} \right) \right]^{\rho_{\text{tsr}}} \\ & \preceq \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left(\frac{1}{\hat{\mathfrak{a}}} \right)^{\rho_{\text{tsr}}} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{A}_1|_{\text{tsr}}}{1} + \frac{|\mathfrak{S}_1|_{\text{tsr}}}{2} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{A}_2|_{\text{tsr}}}{1} + \frac{|\mathfrak{S}_2|_{\text{tsr}}}{2} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{A}_3|_{\text{tsr}}}{1} + \frac{|\mathfrak{S}_3|_{\text{tsr}}}{2} \right) \right]^{\rho_{\text{tsr}}} \\ & \preceq \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{A}_1|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_1} + \frac{|\mathfrak{S}_1|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_2} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{A}_2|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_1} + \frac{|\mathfrak{S}_2|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_2} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{A}_3|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_1} + \frac{|\mathfrak{S}_3|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_2} \right) \right]^{\rho_{\text{tsr}}} \\ & \preceq \mathbb{D} \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{A}_1|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_1} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{A}_2|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_1} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{A}_3|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_1} \right) \right]^{\rho_{\text{tsr}}} \\ & + \mathbb{D} \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{S}_1|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_2} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{S}_2|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_2} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{S}_3|_{\text{tsr}}|}{\tilde{\mathfrak{n}}_2} \right) \right]^{\rho_{\text{tsr}}} \preceq (\infty, \infty, \infty) \end{aligned}$$

where $\mathbb{D} = \max(1, 2^{\mathbb{H}-1})$. Thus, $(\ell_{\mathcal{H}})^3(\mathcal{P})$ is linear space.

Theorem 3.2: If $1 \leq \mathcal{P}_{\text{abc}} < \infty$ then $(\ell_{\mathcal{H}})^3(\mathcal{P})$ is complete space with

$$\mathcal{S}(\mathfrak{G}) = \inf \left\{ \left(\rho_{\mathbb{K}}^{\frac{\mathcal{P}_{\mathfrak{n}}}{\mathbb{K}}}, \rho_{\mathbb{K}}^{\frac{\mathcal{P}_{\mathfrak{n}}}{\mathbb{K}}}, \rho_{\mathbb{K}}^{\frac{\mathcal{P}_{\mathfrak{n}}}{\mathbb{K}}} \right) : \left(\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|\mathfrak{G}_1|_{\text{abc}}|}{\rho} \right) \vee \mathcal{H}_2 \left(\frac{|\mathfrak{G}_2|_{\text{abc}}|}{\rho} \right) \vee \mathcal{H}_3 \left(\frac{|\mathfrak{G}_3|_{\text{abc}}|}{\rho} \right) \right]^{\mathcal{P}_{\text{abc}}} \right)^{\frac{1}{\mathbb{K}}} \leq (1, 1, 1), n = 1, 2, \dots \right\},$$

where $\mathbb{K} = \max(1, \sup_{\text{abc}} \mathcal{P}_{\text{abc}})$.

Proof: Let $((\mathfrak{G}_1)^{\ell_{\mathcal{H}}}), ((\mathfrak{G}_2)^{\ell_{\mathcal{H}}}), ((\mathfrak{G}_3)^{\ell_{\mathcal{H}}})$ are Cauchy triple sequence in $(\ell_{\mathcal{H}})^3(\mathcal{P})$. let r and x_0 be fixed.

Then $\forall \frac{\varepsilon}{rx_0} > 0, \exists \mathcal{N} > 0 \ni \mathcal{G}((\mathfrak{G}_1)^{\ell\ell j} - (\mathfrak{G}_1)^{\mathfrak{f}ed}) < \frac{\varepsilon}{rx_0}, \mathcal{G}((\mathfrak{G}_2)^{\ell\ell j} - (\mathfrak{G}_2)^{\mathfrak{f}ed}) < \frac{\varepsilon}{rx_0}, \mathcal{G}((\mathfrak{G}_3)^{\ell\ell j} - (\mathfrak{G}_3)^{\mathfrak{f}ed}) < \frac{\varepsilon}{rx_0}, \forall \ell, \ell, j, \mathfrak{f}, e, d \geq \mathcal{N}$.

Using $\mathcal{G}$'s definition, we can conclude that,

$$\left(\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_1)^{\ell\ell j} - (\mathfrak{G}_1)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)_{abc}^{\ell\ell j} - (\mathfrak{G}_2)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_2)^{\ell\ell j} - (\mathfrak{G}_2)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)_{abc}^{\ell\ell j} - (\mathfrak{G}_3)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_3)^{\ell\ell j} - (\mathfrak{G}_3)^{\mathfrak{f}ed})} \right) \right]^{\mathcal{P}_{abc}} \right)^{\frac{1}{\mathbb{K}}} \leq (1,1,1), \forall \ell, \ell, j, \mathfrak{f}, e, d \geq \mathcal{N}.$$

$$\text{Therefore, } \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_1)^{\ell\ell j} - (\mathfrak{G}_1)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)_{abc}^{\ell\ell j} - (\mathfrak{G}_2)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_2)^{\ell\ell j} - (\mathfrak{G}_2)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)_{abc}^{\ell\ell j} - (\mathfrak{G}_3)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_3)^{\ell\ell j} - (\mathfrak{G}_3)^{\mathfrak{f}ed})} \right) \right]^{\mathcal{P}_{abc}} \leq (1,1,1), \forall \ell, \ell, j, \mathfrak{f}, e, d \geq \mathcal{N}$$

Since $1 \leq \mathcal{P}_{abc} < \infty$, from this,

$$\left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_1)^{\ell\ell j} - (\mathfrak{G}_1)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)_{abc}^{\ell\ell j} - (\mathfrak{G}_2)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_2)^{\ell\ell j} - (\mathfrak{G}_2)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)_{abc}^{\ell\ell j} - (\mathfrak{G}_3)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_3)^{\ell\ell j} - (\mathfrak{G}_3)^{\mathfrak{f}ed})} \right) \right] \leq (1,1,1), \forall a, b, c \geq 1, \forall \ell, \ell, j, \mathfrak{f}, e, d \geq \mathcal{N}.$$

Thus, one can find $r > 0$ with $\left(\frac{x_0}{2}\right) r \mathcal{Q} \left(\frac{x_0}{2}\right) \geq 1$,

where $\mathcal{Q}$ the kernel of $\mathcal{H} \ni$

$$\left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_1)^{\ell\ell j} - (\mathfrak{G}_1)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)_{abc}^{\ell\ell j} - (\mathfrak{G}_2)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_2)^{\ell\ell j} - (\mathfrak{G}_2)^{\mathfrak{f}ed})} \right) \vee \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)_{abc}^{\ell\ell j} - (\mathfrak{G}_3)_{abc}^{\mathfrak{f}ed}|}{\mathcal{G}((\mathfrak{G}_3)^{\ell\ell j} - (\mathfrak{G}_3)^{\mathfrak{f}ed})} \right) \right] \leq \left(\left(\frac{x_0}{2}\right) r \mathcal{Q} \left(\frac{x_0}{2}\right), \left(\frac{x_0}{2}\right) r \mathcal{Q} \left(\frac{x_0}{2}\right), \left(\frac{x_0}{2}\right) r \mathcal{Q} \left(\frac{x_0}{2}\right) \right).$$

$$\text{Moreover, } (\|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}\|, \|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}\|, \|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}\|) < \left(\frac{rx_0}{2}, \frac{rx_0}{2}, \frac{rx_0}{2}\right) \cdot \left(\frac{\varepsilon}{rx_0}, \frac{\varepsilon}{rx_0}, \frac{\varepsilon}{rx_0}\right) = \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right).$$

Hence $((\mathfrak{G}_1)^{\ell\ell j}), ((\mathfrak{G}_2)^{\ell\ell j}), ((\mathfrak{G}_3)^{\ell\ell j})$ are Cauchy triple sequence in $\mathbb{R}$. $\forall \varepsilon (0 < \varepsilon < 1), \exists \mathcal{N} > 0 \ni |(\mathfrak{G}_1)^{\ell\ell j} - \mathfrak{G}_1| < \varepsilon, |(\mathfrak{G}_2)^{\ell\ell j} - \mathfrak{G}_2| < \varepsilon, |(\mathfrak{G}_3)^{\ell\ell j} - \mathfrak{G}_3| < \varepsilon, \forall \ell, \ell \geq \mathcal{N}$. Using $\mathcal{H}$'s continuity, we find that,

$$\left(\sum_{a=1}^{\mathcal{N}} \sum_{b=1}^{\mathcal{N}} \sum_{c=1}^{\mathcal{N}} \left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)_{abc}^{\ell\ell j} - \lim_{\mathfrak{f}, e, d \rightarrow \infty} (\mathfrak{G}_1)_{abc}^{\mathfrak{f}ed}|}{\rho} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)_{abc}^{\ell\ell j} - \lim_{\mathfrak{f}, e, d \rightarrow \infty} (\mathfrak{G}_2)_{abc}^{\mathfrak{f}ed}|}{\rho} \right) \vee \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)_{abc}^{\ell\ell j} - \lim_{\mathfrak{f}, e, d \rightarrow \infty} (\mathfrak{G}_3)_{abc}^{\mathfrak{f}ed}|}{\rho} \right) \right]^{\mathcal{P}_{abc}} \right)^{\frac{1}{\mathbb{K}}} \leq (1,1,1).$$

An addition,

$$\begin{aligned}
& \left(\sum_{a=1}^{\mathcal{N}} \sum_{b=1}^{\mathcal{N}} \sum_{c=1}^{\mathcal{N}} \left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)^{\ell \ell j}_{abc} - \mathfrak{G}_1|}{\rho} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)^{\ell \ell j}_{abc} - \mathfrak{G}_2|}{\rho} \right) \vee \right. \right. \\
& \left. \left. \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)^{\ell \ell j}_{abc} - \mathfrak{G}_3|}{\rho} \right) \right] \right]^{\mathcal{P}_{abc}} \Bigg)^{\frac{1}{\mathbb{K}}} \leq (1,1,1). \text{ Taking infimum of such } \rho\text{'s, we get} \\
& \inf \left\{ \rho^{\frac{\mathcal{P}_{\mathbb{K}}}{\mathbb{K}}} : \left(\sum_{a=1}^{\mathcal{N}} \sum_{b=1}^{\mathcal{N}} \sum_{c=1}^{\mathcal{N}} \left[\mathcal{H}_1 \left(\frac{|(\mathfrak{G}_1)^{\ell \ell j}_{abc} - \mathfrak{G}_1|}{\rho} \right) \vee \mathcal{H}_2 \left(\frac{|(\mathfrak{G}_2)^{\ell \ell j}_{abc} - \mathfrak{G}_2|}{\rho} \right) \vee \right. \right. \right. \\
& \left. \left. \left. \mathcal{H}_3 \left(\frac{|(\mathfrak{G}_3)^{\ell \ell j}_{abc} - \mathfrak{G}_3|}{\rho} \right) \right] \right]^{\mathcal{P}_{abc}} \right)^{\frac{1}{\mathbb{K}}} \leq (1,1,1) \Bigg\} < (\varepsilon, \varepsilon, \varepsilon), \forall \ell, \ell, j \geq \mathcal{N} \text{ and } \ell, e, d \rightarrow \infty.
\end{aligned}$$

Since $((\mathfrak{G}_1)^{\ell \ell j}), ((\mathfrak{G}_2)^{\ell \ell j}), ((\mathfrak{G}_3)^{\ell \ell j}) \in (\ell_{\mathcal{H}})^3(\mathcal{P})$ and by the continuity of $\mathcal{H}$, thus it follows $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3 \in (\ell_{\mathcal{H}})^3(\mathcal{P})$. Thus, $(\ell_{\mathcal{H}})^3(\mathcal{P})$ is complete space.

References

- [1] A.H. Battor and M.A. Neamah, "On Statistical Convergent Double Sequence Spaces Defined by Orlicz Function", Master thesis, University of Kufa, (2017).
- [2] A.H. Battor and A.F. Dabbas, "Convergent and Statistically Convergent of a Double Sequence Space of Fuzzy Real Numbers defined by Double Orlicz Functions", Master thesis, University of Kufa, (2020).
- [3] A.H. Battor and M. M. Mansoor, "Some Studies on a Double Sequence Spaces of Fuzzy Numbers Defined by Double Orlicz Functions", Master thesis, University of Kufa, (2021).
- [4] A.H. Battor and T. M. Abd Al-Hussein, "Studies on Some Triple Sequence Spaces of Fuzzy Numbers Defined by Double Orlicz Functions", Master thesis, University of Kufa, (2022).
- [5] E. Savaş and M. Gürdal, "Generalized statistically convergent sequences of functions in fuzzy 2-normed spaces," *Journal of Intelligent and Fuzzy Systems*, vol. 27, no. 4, pp. 2067-2075, Jan. 2014. <https://doi.org/10.3233/ifs-141172>.
- [6] A. Esi and B. Hazarika, "Statistically Convergent Difference Double Sequence Spaces Defined By Orlicz Functions," *Thai Journal of Mathematics*, vol. 14, no. 3, pp. 687-700 (Apr. 2015).
- [7] B.C. Tripathy, "Statistically convergent double sequences," *Tamkang Journal of Mathematics*, vol. 34, no. 3, pp. 231-237 (2003). <https://doi.org/10.5556/j.tkm.34.2003.314>.

- [8] E. Savaş and M. Mursaleen, "On statistically convergent double sequences of fuzzy numbers," *Information Sciences*, vol. 162, no. 3-4, pp. 183-192 (2004). <https://doi.org/10.1016/j.ins.2003.09.005>.
- [9] B. Tripathy and B. Sarma, "Sequence spaces of fuzzy real numbers defined by Orlicz functions," *Mathematica Slovaca*, vol. 58, no. 5, pp. 621-628 (2008). <https://doi.org/10.2478/s12175-008-0097-9>.
- [10] B.C. Tripathy and S. Borgohain, "Some Classes of Difference Sequence Spaces of Fuzzy Real Numbers Defined by Orlicz Function," *Advances in Fuzzy Systems*, vol. 2011, pp. 1-6 (2011). <https://doi.org/10.1155/2011/216414>.
- [11] B. C. Tripathy and R. Goswami, "Vector valued multiple sequence spaces defined by Orlicz function," *Boletim da Sociedade Paranaense de Matemática*, vol. 33, no. 1, pp. 67 (2014). <https://doi.org/10.5269/bspm.v33i1.21602>.
- [12] A. H. Battor and M. A. Niamah, "Vector valued double sequence spaces defined by a double Orlicz function," *International Journal of Mathematical Analysis*, vol. 11, pp. 467-477 (2017). <https://doi.org/10.12988/ijma.2017.7456>.

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