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On the fuzzy sets and fuzzy mathematical programming

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Abstract

Fuzzy sets and operations with such sets are considered, and problems of fuzzy mathematical programming are discussed.

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1. Introduction

Real objects and phenomena are extremely complex. People often try to simplify them, using some abstract concepts. For example, the sea surface is assumed to be a plane, the shape of the Earth is assumed to be a ball, the Earth's orbit around the Sun is assumed to be a circumference, etc. If we try to give a strict definition even for objects that we use daily, for example, a chair, a table, etc., we may encounter serious difficulties.

In many cases, referring an object to a strictly defined set of objects is not appropriate. That is why, people are looking for a more complete and accurate description of the real world.

In the problems of organizational management, often there are situations in which the initial conditions are defined not strictly or inaccurately. This situation shows that the decision maker is poorly informed. The information, used by him, is subjective and, as a rule, its representation by using natural language may contain many uncertainties such as “many”, “few”, “greatly increased”, “approximately the same”, etc., which do not have analogues in the language of the traditional mathematics. That is why, describing such information with the means of traditional mathematics is inaccurate and the corresponding mathematical model is quite “rough”. Since we want to apply mathematical

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methods for the analysis and research of increasingly complex systems, it is necessary to create a new mathematical apparatus that allows to formalize the non-strict concepts and judgments, with which a person works when describing his ideas about the world, his goals, etc.

For such a purpose, the so-called *fuzzy mathematics* arose and was developed, whose basis is the *theory of fuzzy sets*. In 1965 L. Zadeh published his seminal work on fuzzy sets ([28]). Contribution to this field, especially for making a decision in an environment, which is fuzzy, and the *fuzzy mathematical programming*, have R. Bellman and L. Zadeh (see, for example, [2]).

The important concept of a "set" is considered to be intuitively clear. The empty set ($\emptyset$) is a set that contains no elements.

The topics and problems, considered in this paper, and related to them, are considered in references [1]–[31], etc.

Rest of the paper is organized as follows. In Section 2, basic concepts and definitions concerning fuzzy sets are given. In Section 3, basic operations with fuzzy sets are defined. In Section 4, the problems of fuzzy mathematical programming are considered and discussed. In Section 5, contents of the paper is summarized.

2. Basic concepts and definitions

Some basic concepts and definitions, connected with fuzzy sets, are introduced below.

Definition 1: An *ordered pair* of two arbitrary objects x and y is constructed by specifying which object is first and which object is second, for example, (x, y).

Definition 2: *Cartesian product* of the sets X and Y is defined as the set, containing all ordered pairs (x, y) with $x \in X$, $y \in Y$, and this set is denoted by $X \times Y$.

Example 1. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\}$.

Then $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$.

Let X be a set and $A \subseteq X$.

The *membership function* $m_A(x)$ of a set A takes values in the interval $[0,1]$ and it satisfies the following properties:

a) If $x \not\in A$, then $m_A(x) = 0$;

b) If x "belongs little" to A then $m_A(x) \in [0,1]$ and $m_A(x)$ is close to 0;

c) If x belongs to A then $m_A(x) \in [0,1]$ and $m_A(x)$ is close neither to 0 nor to 1;

d) If x "belongs much" to A then $m_A(x) \in [0,1]$ and $m_A(x)$ is close to 1;

e) If x "belongs entirely" to A then $m_A(x) = 1$.

The "ordinary" sets form a subclass of the fuzzy sets because the membership function of a non-fuzzy set B, where $B \subseteq X$, is its *indicator* (*characteristic*) *function* $m_B(x)$, defined as follows:

$$m_B(x) = \begin{cases} 1, & \text{if } x \in B; \\ 0, & \text{if } x \not\in B. \end{cases}$$

Definition 3: Let X be a set of elements. A *fuzzy set* A in X is defined as the set of all ordered pairs $(x, m_A(x))$, such that $x \in X$ and $m_A(x): x \to [0,1]$ is the membership function of the fuzzy set A. The set X is said to be a *basic set*; the set of values, which $m_A(x)$ takes, is called a *membership set*; and the value $m_A(x)$ for a particular x is said to be a *degree of membership* of x to A.

Example 2: Let $X = \{1, 0.1, 0.01, 0.001, 0.0001\}$, and $A =$ "small numbers" be a fuzzy set in the set X.

Then A = {(1,0.2), (0.1,0.4), (0.01,0.6), (0.001,0.8), (0.0001,1)} because the smaller the number, the higher its degree of membership to the fuzzy set A.

In this case, the membership set is $\{0.2, 0.4, 0.6, 0.8, 1\}$.

Definition 4: A fuzzy set $\emptyset$ is called *empty* if the membership function of this fuzzy set is equal to 0 in the entire space X, i.e., if $m_\emptyset(x) = 0 \ \forall \ x \in X$.

Definition 5: *Universal set* X is a set which is described by a membership function of the form $m_X(x) = 1$ for each $x \in X$.

Definition 6: Let A and B be fuzzy sets and $m_A(x)$ and $m_B(x)$ be their membership functions, respectively. A fuzzy set $A \subset X$ is said to be a *subset* of the fuzzy set $B \subset X$ if $m_A(x) \leq m_B(x)$ for each $x \in X$, and this is denoted $A \subseteq B$.

Definition 7: The fuzzy sets $A, B \subset X$ are called *equal* (or *equivalent*) if $m_A(x) = m_B(x)$ for each $x \in X$, and this is denoted $A = B$. Otherwise they are called *different* and this is denoted $A \neq B$.

Example 3: The following fuzzy sets are given

$$A = \{x : x \text{ is close to } 1\},$$
$$B = \{x : x \text{ is very close to } 1\}.$$

Then $B \subseteq A$ and the membership functions of these two sets satisfy $m_B(x) \leq m_A(x)$ for each $x \in X$.

3. Operations with fuzzy sets

In this section, some fuzzy set operations are considered.

Definition 8: The *union* of the fuzzy sets A, B in X is a fuzzy set $A \cup B$ with a membership function defined as follows

$$m_{A\cup B}(x) = \max\{m_A(x), m_B(x)\}, \quad x \in X.$$

An equivalent definition is

Definition 9: The *union* of the fuzzy sets $A, B \subset X$ is a fuzzy set $A \cup B$ with a membership function defined as follows

$$m_{A\cup B}(x) = \begin{cases} 1, & \text{if} \quad m_A(x) + m_B(x) \geq 1, \\ m_A(x) + m_B(x), & otherwise. \end{cases}$$

Definition 10: *Intersection* of the fuzzy sets $A, B \subset X$ is a fuzzy set $A \cap B$ with a membership function defined as follows

$$m_{A\cap B}(x) = \min\{m_A(x), m_B(x)\}, \quad x \in X.$$

An equivalent definition is

Definition 11: *Intersection* of the fuzzy sets $A, B \subset X$ is a fuzzy set $A \cap B$ with a membership function defined as follows

$$m_{A\cap B}(x) = m_A(x). m_B(x), \quad x \in X.$$

Definition 12: *Difference* of the fuzzy sets $A, B \subset X$ is a fuzzy set $A \backslash B$ with a membership function defined as follows

$$m_{A\backslash B}(x) = \begin{cases} m_A(x) - m_B(x), & \text{if} \quad m_A(x) \geq m_B(x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 13: *Complement* of the fuzzy set A in the set X is a fuzzy set $\overline{A}$ with a membership function defined as follows

$$m_{\overline{A}}(x) = 1 - m_A(x), \quad x \in X.$$

Definition 14: *Convex combination* of fuzzy sets $A_1, \dots, A_l$ in X is a fuzzy set A with a membership function defined as

$$m_A(x) = \sum_{i=1}^{l} c_i m_{A_i}(x), \quad c_i \geq 0, i = 1, \dots, l, \quad \sum_{i=1}^{l} c_i = 1,$$

where $m_{A_i}(x)$ are the membership functions of the fuzzy sets A_i, $i = 1, \dots, l$, respectively.

Definition 15: *Level set* with parameter a of a fuzzy set $A \subset X$ is the set

$$A_a = \{x : x \in X, m_A(x) \geq a\}.$$

It can be proved that following relations hold true

$$(A \cup B)_a = A_a \cup B_a,$$
$$(A \cap B)_a = A_a \cap B_a,$$
$$(A \cup B)_a \supset (A_a \cup B_a),$$
$$(A \cap B)_a \subset (A_a \cap B_a).$$

4. Fuzzy mathematical programming

In many cases, the decision problem can be defined by a set of feasible choices (called alternatives) and the relation of preference, defined in this set, which describes the interests of the person who makes the decision. In the general case, this relation is binary, that is, it allows only two alternatives to be compared pairwise. In such a case, the decision problem is to select an acceptable alternative that is better or is no worse than all other alternatives according to given preference relation.

The binary preference relation in the set of all alternatives is described by one of the following two ways:

1. as a subset of the Cartesian product of the sets of alternatives, that is, in the form of a relation,
2. in the form of a utility function.

Usually, the utility function is a map from the set X of alternatives onto the axis of the real numbers $\mathbb{R}$. A utility function cannot be described by any relation of preference and for any set X of alternatives. In some cases, instead of one utility function, a finite number of utility functions can be described, and then the corresponding decision problems are *multi-criteria* (*multi-objective*) problems.

The problems of decision making, in which the relation of preference is given in the form of a utility function, are mathematical optimization problems. The (optimal) decision for such a problem is a feasible alternative, which maximizes the utility function.

Elements of fuzziness for the mathematical optimization problems can be found both in the description of the utility function and in the description of the set of alternatives. The problems, for which the utility function is described exactly, and only the set of alternatives is fuzzily described, are called *fuzzy mathematical programming problems* (*fuzzy mathematical optimization problems*).

There are two approaches for solving the fuzzy mathematical programming problems. In the first approach, proposed by Bellman and Zadeh, the fuzzy mathematical programming problem is formulated as a problem for implementing a fuzzy defined goal, and the decision is an intersection of the fuzzy sets of the goals and of the constraints (about intersection of fuzzy sets see Definition 10 and Definition 11, Section 3). In the second approach, it is assumed that the decisions should be selected as in the multi-criteria optimization problems. Moreover, the decision for the initial problem involves namely these alternatives, that are not strictly dominated by any other alternatives (*Pareto efficient alternatives*).

The following *flexible mathematical programming problem* is also considered

$$\min f(\mathrm{x}) \tag{1}$$

subject to

$$g_i(\mathrm{x}) \leq\approx b_i, \quad i = 1, \dots, l, \tag{2}$$

$$\mathrm{x} \in X \overset{\text{def}}{=} \{\mathrm{x} \in \mathbb{R}^n : x_j \geq 0, j = 1, \dots, n\}, \tag{3}$$

where $x = (x_1, \dots, x_n)$, and "$\approx$ " in (2) means that the constraints (2) may not be strictly (that is, may be fuzzily) satisfied.

5. Conclusions

In this paper, the concept of a fuzzy set and operations with fuzzy sets are considered, and problems of fuzzy mathematical programming are discussed.

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