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On coefficient estimate of K-fold bi-univalent functions

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Abstract

This study examines and investigates two novel subclasses $\mathcal{MH}_{\Sigma_q}(\theta, \digamma, \eta; \delta)$ and $\tilde{\mathcal{MH}}_{\Sigma_q}(\theta, \digamma, \eta; \varphi)$ for a family Σ_q of K-fold symmetric bi-univalent function in a open unit disk. Furthermore, where function in these subclasses, bound of coefficient $|b_{q+1}|$ and $|b_{2q+1}|$ were derived. Moreover, several refined results are presented for related classes.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Analytic function, bi-univalent functions, q-fold symmetric function, Univalent functions, q-fold Symmetric Function.*

1. Introduction

If $\mathcal{S}$ family for analytic function $\mathcal{J}(\xi)$ in open unit disk $\mathcal{U} = \{\xi \in \mathbb{C}: |\xi| < 1\}$, normalize by a condition $\mathcal{J}(0) = \mathcal{J}'(0) - 1 = 0$. A Taylor's series expansions of an analytic function $\mathcal{J} \in \mathcal{S}$ takes the following form:

$$\mathcal{J}(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n \quad (\xi \in \mathcal{U}) \qquad (1).$$

Where $\mathcal{F}$ denotes functions class in $\mathcal{S}$, that is univalent in a open unit disks. Under each $\mathcal{J}(\xi) \in \mathcal{F}$, image for $\mathcal{U}$ have disk which is radius 0.25, based on Koebe's one-quarter theorem [1]. Certainly, an inverse $\mathcal{J}^{-1}$of each $\mathcal{J} \in \mathcal{F}$ satisfies the following: $\mathcal{J}^{-1}(\mathcal{J}(\xi)) = \xi, (\xi \in \mathcal{U})$ and $\mathcal{J}\left(\mathcal{J}^{-1}(z)\right) = z$, $\left(|z| < p_0(\mathcal{J}); p_0(\mathcal{J}) \geq \frac{1}{4}\right)$,

Where

$$\mathcal{G}(z) = \mathcal{J}^{-1}(z) = z - b_2 z^2 + \left(2b_2^2 - b_3\right)z^3 - (5b_2^3 - 5b_2 b_3 + b_4)z^4 + \cdots. \qquad (2)$$

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$\mathcal{J} \in \mathcal{S}$ is a bi-univalent function when both $\mathcal{J}$and $\mathcal{J}^{-1}$are univalent in open unit disk. In $\mathcal{U}$, let Σ denote to family for bi-univalent function. A coefficient bound derived in [2] is provided by$|\mathscr{b}_2| \leq 1.51$ for each $\mathcal{J} \in \Sigma$ where the author studied bi-univalent functions class Σ. At the beginning of 80s of last century, for all $\mathcal{J} \in \Sigma$, Brannan and Clunie[3] suggested $|\mathscr{b}_2|$ is less or equal to 2 (Lewin's study has inspired the work). After that, the analytical and bi-univalent function were revitalized in [4]. The studies "Bulut [5] and Adegani, Bulut, and Zireh [6]" pursued by Güney, Murugusundaramoorthy, and Sokół [7]. Author Orhan, Magesh, and Yamini in [8] introduced, defined by quasi subordinations, the bi-univalent class & determined bounds for a coefficient. The following unified subclasses, commonly referred to as Ma–Minda subclasses, were introduced by M. Haji Mohd and M. Darus [9].

$$D^*(\psi) = \left\{\mathcal{J} \in \mathcal{S} : \frac{\xi \mathcal{J}'(\xi)}{\mathcal{J}(\xi)} \prec \psi(\xi) : \xi \in \mathcal{U}\right\},$$

$$L^*(\psi) = \left\{\mathcal{J} \in \mathcal{S} : 1 + \frac{\xi \mathcal{J}''(\xi)}{\mathcal{J}(\xi)} \prec \psi(\xi) : \xi \in \mathcal{U}\right\},$$

where $\psi(\xi)$ satisfies $\psi(0) = 1$ and $\psi'(\xi) > 0$, and its analytic & univalent functions in $\mathcal{U}$ have the positive real part. In relation to **1**, the starlike region is denoted by $\psi(\mathcal{U})$, which be symmetric in a reals axis. While function in class $L^*(\psi)$ convex for Ma-Minda, the functions $D^*(\psi)$ are starlike of Ma-Minda type (see [9] for more explaination). In $(\xi) = \sqrt[q]{\mathcal{J}(\xi^{q})}$ $q \in \mathbb{N} = \{1, 2, \cdots\}$, the function q is univalent in $\mathcal{U}$ for all $\mathcal{J} \in \mathcal{F}$. Additionally, the function q maps into q-fold symmetry's region. The function, satisfying normalized condition, is formulated as follows:

$$\mathcal{J}(\xi) = \xi + \sum_{i=1}^{\infty} \mathscr{b}_{qi+1} \xi^{qi+1} \quad q \in \mathbb{N} = \{1, 2, \cdots\}, \xi \in \mathcal{U}. \tag{3}$$

In [10] [11], the form is called q-fold symmetric. In the equation (3), $\mathcal{F}_q$ denotes a family for all q-fold symmetrical univalent functions, where the last series extension normalized it. Like the symmetrical q-fold concept, in class $\mathcal{F}$, the function is one-fold symmetry where q =**1**. Conventionally, Bi-univalent symmetric κ-fold functions can be defined. In [**1**2], the antecedents of q-fold symmetric bi-univalent function have been described by Srivastava. For any positive integer q, of the class Σ, it is possible to build a q-fold symmetric bi-univalent function of the function $\mathcal{J}$ define in equation (3), $\mathcal{J}$ is a normalized form, which can be defined as $\mathcal{J}^{-1}$ as follows:

$$\mathcal{G}(z) = \mathcal{J}^{-1}(z) = z - \mathscr{b}_{q+1} z^{q+1} + \left[(q+1)\mathscr{b}_{q+1}^{2} - \mathscr{b}_{2q+1}\right] z^{2q+1} -$$
$$\left[\frac{1}{2}(q+1)(3q+2)\mathscr{b}_{q+1}^{3} - (3q+2)\mathscr{b}_{q+1}\mathscr{b}_{2q+1} + \mathscr{b}_{3q+1}\right] z^{3q+1} + \cdots, \tag{4}$$

where $\mathcal{G} = \mathcal{J}^{-1}$. Σ_q denotes the q-fold symmetric bi-univalent function class. To a family defined in equation (2), the equation (4) is corresponded when q equals one. Example for q-fold symmetric bi-univalent function shown below. $\left(\frac{\xi^{q}}{1-\xi^{q}}\right)^{\frac{1}{q}}$ and $\left[\frac{1}{2}\log\left(\frac{1+\xi^{q}}{1-\xi^{q}}\right)^{\frac{1}{q}}\right]$, based on the inverse functions, the equation below is obtained: $\left(\frac{z^{q}}{1-z^{q}}\right)^{\frac{1}{q}}$ and $\left(\frac{e^{z^{q}}-1}{e^{z^{q}}}\right)^{\frac{1}{q}}$. In the literature, with different subclasses, estimates q-fold symmetric bi-univalent functions were investigated thoroughly (see [13-17] for more details). A primary objective of this study is to introduce the new subclasses $\mathcal{MH}_{\Sigma_q}(\Theta, \text{F}, \eta; \delta)$ and $\widetilde{\mathcal{MH}}_{\Sigma_q}(\Theta, \text{F}, \eta; \varphi)$ of Σ_q & to derive estimations for initial coefficient $|b_{q+1}|$ & $|b_{2q+1}|$ of function belonging to each of these subclasses. Some important results are presented here.

Lemma 1.1: [18] *Assume* $g \in P$, so $|g_j| \leq 2$ *when* $j \in \mathbb{N}$, *and* P *be class for each analytic function* g *in* $\mathcal{U}$ *for which* $Re\{g(\xi)\} > 0$, $\xi \in \mathcal{U}$ *such that* $g(\xi) = 1 + \hbar_1 \xi + \hbar_2 \xi^2 + \hbar_3 \xi^3 + \cdots, (\xi \in \mathcal{U})$.

2. The subclass $\mathcal{MH}_{\Sigma_\kappa}(\theta, \text{F}, \eta; \delta)$

Definition 2.1: The functions $\mathcal{J} \in \Sigma_q$ given by (3) is called in a family $\mathcal{MH}_{\Sigma_q}(\Theta, \text{F}, \eta; \delta)$ if:

$$\left|arg\left(\frac{1}{\Theta}\left\{\left[\frac{\xi\mathcal{J}'(\xi)}{\mathcal{J}(\xi)}\right]^{\text{F}} + \frac{(1-\text{F})\xi^2\mathcal{J}''(\xi)}{(1-\text{F})\xi\mathcal{J}'(\xi)+\text{F}\mathcal{J}(\xi)} + \eta\xi^2\mathcal{J}'''(\xi)\right\}\right)\right| < \frac{\delta\pi}{2},\ \xi \in \mathcal{U} \tag{5}$$

and

$$\left|arg\left(\frac{1}{\Theta}\left\{\left[\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right]^{\text{F}} + \frac{(1-\text{F})z^2\mathcal{G}''(z)}{(1-\text{F})z\mathcal{G}'(z)+\text{F}\mathcal{G}(z)} + \eta z^2\mathcal{G}'''(z)\right\}\right)\right| < \frac{\delta\pi}{2},\ z \in \mathcal{U}, \tag{6}$$

where $(\Theta \in \mathbb{C}\backslash\{0\}, q \in \mathbb{N}, 0 \leq \text{F} \leq 1,\ 0 \leq \eta \leq 1\ and\ 0 < \delta \leq 1)$, $\mathcal{G} = \mathcal{J}^{-1}$ is given by (4).

Theorem 2.2: *Let* $\mathcal{J}(\xi) \in \mathcal{MH}_{\Sigma_q}(\Theta, \text{F}, \eta; \delta)(\Theta \in \mathbb{C}\backslash\{0\}, q \in \mathbb{N}, 0 \leq \text{F} \leq 1,$ $0 \leq \eta \leq 1\ and\ 0 < \delta \leq 1)$ *be given by* (3). *Then*

$$|b_{q+1}| \leq \min\{\frac{2|\Theta|\delta}{\text{F}q+q(q+1)[1-\text{F}+\eta q-\eta]},$$

$$\frac{2|\Theta|\delta}{\sqrt{\left|\Theta\left[2\text{F}q(q+1)-2\text{F}q+\text{F}(\text{F}-1)q^2 - 2q\,.\,(q+1)(1-\text{F})(q-\text{F}q+1)+2q\,.\,(q+1)(2q+1)[1-\text{F}+\eta 2q-\eta]\right]\right|}}\} \tag{7}$$

and

$$|b_{2q+1}| \leq$$

$$\min\{\frac{2\delta^2|\Theta|^2}{\Theta\left[2\text{F}q(q+1)-2\text{F}q+\text{F}(\text{F}-1)q^2-2q(q+1)(1-\text{F})(q-\text{F}q+1)+2q(q+1)(2q+1)[1-\text{F}+\eta 2q-\eta]\right]}$$

$$+\frac{|\Theta|\delta}{\text{F}q+q\,.\,(2q+1)[1-\text{F}+\eta 2q-\eta]}, \frac{2\delta^2|\Theta|^2}{(\text{F}q+q\,.\,(q+1)[1-\text{F}+\eta q-\eta])^2} + \frac{|\Theta|\delta}{\text{F}q+q\,.\,(2q+1)[1-\text{F}+\eta 2q-\eta]}\}. \tag{8}$$

Proof: Let $\mathcal{J}(\xi) \in \mathcal{MH}_{\Sigma_q}(\Theta, \text{F}, \eta; \delta)$. Then

$$\frac{1}{\theta}\left\{\left[\frac{\xi\mathcal{J}'(\xi)}{\mathcal{J}(\xi)}\right]^{F}+\frac{(1-F)\xi^{2}\mathcal{J}''(\xi)}{(1-F)\xi\mathcal{J}'(\xi)+F\mathcal{J}(\xi)}+\eta\xi^{2}\mathcal{J}'''(\xi)\right\}=[t(\xi)]^{\delta} \tag{9}$$

and

$$\frac{1}{\theta}\left\{\left[\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right]^{F}+\frac{(1-F)z^{2}\mathcal{G}''(z)}{(1-F)z\mathcal{G}'(z)+F\mathcal{G}(z)}+\eta z^{2}\mathcal{G}'''(z)\right\}=[r(z)]^{\delta}, \tag{10}$$

where $\mathcal{G}=\mathcal{J}^{-1}$, $t(\xi)$ and $r(z)$ are afunctions in P and have the form

$$t(\xi)=1+t_{q}\,\xi^{q}+t_{2q}\,\xi^{2q}+t_{3q}\,\xi^{3q}+\cdots \tag{11}$$

and

$$r(z)=1+r_{q}\,z^{q}+r_{2q}\,z^{2q}+r_{3q}\,z^{3q}+\cdots. \tag{12}$$

$$\frac{1}{\theta}(Fq+q(q+1)[1-F+\eta q-\eta])b_{q+1}=\delta t_{q}, \tag{13}$$

$$\frac{1}{\theta}\left(\begin{array}{c}2Fq+2q(2q+1)[1-F+\eta 2q-\eta]b_{2q+1}-\\ \left[Fq-\frac{F(F-1)}{2}q^{2}+(1-F)q(q+1)(q-Fq+1)\right]\end{array}\right)b_{q+1}^{2}=\delta t_{2q}+\frac{1}{2}\delta(\delta-1)t_{q}^{2}, \tag{14}$$

and

$$-\frac{1}{\theta}(Fq+q(q+1)[1-F+\eta q-\eta])b_{q+1}=\delta r_{q}, \tag{15}$$

$$\frac{1}{\theta}\left(\begin{array}{c}-(2Fq+2q(2q+1)[1-F+\eta 2q-\eta])b_{2q+1}+\\ \left[\begin{array}{c}2Fq(q+1)-Fq+\frac{F(F-1)}{2}q^{2}+2q(q+1)(2q+1)[1-F+\eta 2q-\eta]-\\ q(q+1)(1-F)(q-Fq+1)\end{array}\right]b_{q+1}^{2}\end{array}\right)$$
$$=\delta r_{2q}+\frac{1}{2}\delta(\delta-1)r_{q}^{2}. \tag{16}$$

From (13) and (15), we obtain

$$t_{q}=-r_{q} \tag{17}$$

and

$$2\left(\frac{1}{\theta}(Fq+q(q+1)[1-F+\eta q-\eta])\right)^{2}b_{q+1}^{2}=\delta^{2}\left(t_{q}^{2}+r_{q}^{2}\right). \tag{18}$$

Adding (14) and (16), we deduce that

$$\frac{1}{\theta}\left(\left[\begin{array}{c}2Fq(q+1)-2Fq+F(F-1)q^{2}-2q(q+1)(1-F)(q-Fq+1)+\\ 2q(q+1)(2q+1)[1-F+\eta 2q-\eta]\end{array}\right]b_{q+1}^{2}\right)=$$
$$\delta(t_{2q}+r_{2q})+\frac{1}{2}\delta(\delta-1)\left(t_{q}^{2}+r_{q}^{2}\right). \tag{19}$$

So,

$$\left|b_{q+1}\right|\le\frac{2|\theta|\delta}{Fq+q(q+1)[1-F+\eta(q-1)]} \tag{20}$$

$$\left|b_{q+1}\right|\le$$
$$\frac{2|\theta|\delta}{\sqrt{\left|\theta\left[2Fq(q+1)-2Fq+F(F-1)q^{2}-2q.\,(q+1)(1-F)(q-Fq+1)+2q.\,(q+1)(2q+1)[1-F+\eta 2q-\eta]\right]\right|}}, \tag{21}$$

as stated in (7), this provides the intended estimate for $\left|b_{q+1}\right|$.

Equation (16) can be subtracted from equation (14), yielding the following

$$\frac{1}{\theta}\left(\begin{array}{c}(4Fq+4q(2q+1)[1-F+\eta 2q-\eta])b_{2q+1}-\\ \left(2Fq(q+1)+2q(q+1)(1-F)(q-Fq+1)\right)b_{q+1}^{2}\end{array}\right)=$$

$$\delta(t_{2q} - r_{2q}) + \frac{1}{2}\delta(\delta - 1)\left(t_q^2 - r_q^2\right). \tag{22}$$

Putting (22) and replacing (18) with (19), The following is ascertained using Lemma 1:

$$\left|b_{2q+1}\right| \leq \frac{2\delta^2|\Theta|^2}{(Fq+q(q+1)[1-F+\eta q-\eta])^2} + \frac{|\Theta|\delta}{Fq+q(2q+1)[1-F+2\eta q-\eta]} \tag{23}$$

$$\left|b_{2q+1}\right| \leq \frac{2\delta^2|\Theta|^2}{\Theta\left[2Fq(q+1)-2Fq+F(F-1)q^2-2q(q+1)(1-F)(q-Fq+1)+2q(q+1)(2q+1)[1-F+2\eta q-\eta]\right]} + \frac{|\Theta|\delta}{Fq+q(2q+1)[1-F+2\eta q-\eta]}. \tag{24}$$

The estimate in equation (8) can be achieved from equations (23) and (24).Consequently, the proof is completed.

In case $q = 1$, the following results are determined by reducing Theorem3.

Corollary 2.3: *Let* $\mathcal{J}(\xi) \in \mathcal{MH}_{\Sigma}(\Theta, F, \eta; \delta)(\Theta \in \mathbb{C}\backslash\{0\}, 0 \leq F \leq 1,\ 0 \leq \eta \leq 1\ and\ 0 < \delta \leq 1)$ be given by (3). Then

$$|b_2| \leq min\left\{\frac{2|\Theta|\delta}{[2-F]}, \frac{2|\Theta|\delta}{\sqrt{\left|\Theta\left[5F-3F^2+12(1-F+\eta)\right]\right|}}\right\} \text{ and}$$

$$|b_3| \leq min\left\{\frac{2\delta^2|\Theta|^2}{(2-F)^2} + \frac{|\Theta|\delta}{F+3[1-F+\eta]}, \frac{2\delta^2|\Theta|^2}{\Theta\left[5F-3F^2+12(1-F+\eta)\right]} + \frac{|\Theta|\delta}{F+3[1-F+\eta]}\right\}.$$

3. The subclass $\tilde{\mathcal{MH}}_{\Sigma_\kappa}(\boldsymbol{\theta}, F, \boldsymbol{\eta}; \boldsymbol{\varrho})$

Definition 3.1: The functions $\mathcal{J} \in \Sigma_q$ given by (3) is named in a class $\tilde{\mathcal{MH}}_{\Sigma_q}(\Theta, F, \eta; \varphi)$ if:

$$Re\left(\frac{1}{\Theta}\left\{\left[\frac{\xi\mathcal{J}'(\xi)}{\mathcal{J}(\xi)}\right]^F + \frac{(1-F)\xi^2\mathcal{J}''(\xi)}{(1-F)\xi\mathcal{J}'(\xi)+F\mathcal{J}(\xi)} + \eta\xi^2\mathcal{J}'''(\xi)\right\}\right) > \varphi, \qquad \xi \in \mathcal{U} \tag{25}$$

and
$$Re\left(\frac{1}{\Theta}\left\{\left[\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right]^F + \frac{(1-F)z^2\mathcal{G}''(z)}{(1-F)z\mathcal{G}'(z)+F\mathcal{G}(z)} + \eta z^2\mathcal{G}'''(z)\right\}\right) > \varphi, \quad z \in \mathcal{U}, \tag{26}$$

where $(q \in \mathbb{N}, \Theta \in \mathbb{C}\backslash\{0\}, 0 \leq F \leq 1,\ 0 \leq \eta \leq 1\ \&\ 0 \leq \varphi < 1)$, $\mathcal{G} = \mathcal{J}^{-1}$ is given by (4).

Theorem 3.2: *Let* $\mathcal{J}(\xi) \in \tilde{\mathcal{MH}}_{\Sigma_q}(\Theta, F, \eta; \varphi)(q \in \mathbb{N}, \Theta \in \mathbb{C}\backslash\{0\}, 0 \leq F \leq 1,$ $0 \leq \eta \leq 1$ *and* $0 \leq \varphi < 1)$ *be given by* (3). Then

$$\left|b_{q+1}\right| \leq \min\left\{\frac{2|\Theta|(1-\varphi)}{Fq+q(q+1)[1-F+\eta q-\eta]}, \sqrt{\frac{2|\Theta|(1-\varphi)}{\left|\left[Fq(q+1)-Fq+\frac{F(F-1)}{2}q^2-q(q+1)(1-F)\left((q-Fq)+1\right)+q(q+1)(2q+1)[1-F+2\eta q-\eta]\right]\right|}}\right\} \tag{27}$$

and

$$\left|b_{2q+1}\right| \leq \min\{\frac{2(1-\varphi)^2|\theta|^2}{(Fq+q(q+1)[1-F+\eta q-\eta])^2}+\frac{|\theta|(1-\varphi)}{Fq+q(2q+1)[1-F+2\eta q-\eta]},$$
$$\frac{2(1-\varphi)^2|\theta|^2}{\theta\left[2Fq(q+1)-2Fq+F(F-1)q^2-2q\,.\,(q+1)(1-F)(q-Fq+1)+2q\,.\,(q+1)(2q+1)[1-F+2\eta q-\eta]\right]}+$$
$$\frac{|\theta|(1-\varphi)}{Fq+q(2q+1)[1-F+2\eta q-\eta]}\}. \tag{28}$$

Proof: From equations (25) & (26) that there exists $t, r \in P$, that

$$\frac{1}{\theta}\left\{\left[\frac{\xi J'(\xi)}{J(\xi)}\right]^F+\frac{(1-F)\xi^2 J''(\xi)}{(1-F)\xi J'(\xi)+FJ(\xi)}+\eta\xi^2 J'''(\xi)\right\}=\varphi+(1-\varphi)t(\xi) \tag{29}$$

and

$$\frac{1}{\theta}\left\{\left[\frac{zG'(z)}{G(z)}\right]^F+\frac{(1-F)z^2 G''(z)}{(1-F)zG'(z)+FG(z)}+\eta z^2 G'''(z)\right\}=\varphi+(1-\varphi)r(z), \tag{30}$$

where $t(\xi)$ and $r(z)$ have the forms (11) and (12). From the equations (29) and (30), we find that

$$\frac{1}{\theta}(Fq+q(q+1)[1-F+\eta q-\eta])b_{q+1}=(1-\varphi)t_q, \tag{31}$$

$$\frac{1}{\theta}\begin{pmatrix} 2Fq+2q(2q+1)[1-F+2\eta q-\eta]b_{2q+1}- \\ \left[Fq-\frac{F(F-1)}{2}q^2+(1-F)q(q+1)(q-Fq+1)\right]\end{pmatrix}b_{q+1}^2=(1-\gamma)t_{2q}, \tag{32}$$

$$-\frac{1}{\theta}(Fq+q(q+1)[1-F+\eta q-\eta])b_{q+1}=(1-\varphi)r_q \tag{33}$$

and

$$\frac{1}{\theta}\left(\begin{matrix} -(2Fq+2q(2q+1)[1-F+2\eta q-\eta])b_{2q+1}+ \\ \left[\begin{matrix}2Fq(q+1)-Fq+\frac{F(F-1)}{2}q^2+2q(q+1)(2q+1)[1-F+2\eta q-\eta]- \\ q(q+1)(1-F)(q-Fq+1)\end{matrix}\right]b_{q+1}^2\end{matrix}\right)$$
$$=(1-\varphi)r_{2q}. \tag{34}$$

From (31) and (33), we get $t_q=-r_q$ (35) and

$$2\left(\frac{1}{\theta}(Fq+q(q+1)[1-F+\eta(q-1)])\right)^2 b_{q+1}^2=(1-\varphi)^2(t_q^2+r_q^2). \tag{36}$$

$$\frac{1}{\theta}\left(\left[\begin{matrix}2Fq(q+1)-2Fq+F(F-1)q^2-2q(q+1)(1-F)(q-Fq+1)+ \\ 2q(q+1)(2q+1)[1-F+2\eta q-\eta]\end{matrix}\right]b_{q+1}^2\right)$$
$$=(1-\varphi)(t_{2q}+r_{2q}). \tag{37}$$

From Lemma 1, for the coefficients t_{2q}, r_{2q}, t_q^2 and r_q^2, it follows from (35) and (36), we obtain

$$\left|b_{q+1}\right| \leq \frac{2|\theta|(1-\varphi)}{Fq+q(q+1)[1-F+\eta q-\eta]} \tag{38}$$

and

$$\left|b_{q+1}\right| \leq \sqrt{\frac{2|\Theta|(1-\varphi)}{\left|\left[\digamma q(q+1)-\digamma q+\frac{\digamma(\digamma-1)}{2}q^{2}-q.\ (q+1)(1-\digamma)(q-\digamma q+1)+q\ .(q+1)(2q+1)[1-\digamma+2\eta q-\eta]\right]\right|}}. \tag{39}$$

The coefficient b_{q+1} is achieved in equation (27).Equation (34) is subtracted from (32), yielding

$$\frac{1}{\Theta}\begin{pmatrix} (4\digamma q+4q(2q+1)[1-\digamma+2\eta q-\eta])b_{2q+1}- \\ \big(2\digamma q(q+1)+2q(q+1)(1-\digamma)(q-\digamma q+1)\big)b_{q+1}^{2} \end{pmatrix}=(1-\varphi)(t_{2q}-r_{2q}). \tag{40}$$

By substituting (36) and (37) in (40). Hence from Lemma 1, we deduce that

$$\left|b_{2q+1}\right| \leq \frac{2(1-\varphi)^{2}|\Theta|^{2}}{(\digamma q+q(q+1)[1-\digamma+\eta q-\eta])^{2}}+\frac{|\Theta|(1-\varphi)}{\digamma q+q(2q+1)[1-\digamma+2\eta q-\eta]} \tag{41}$$

$$\left|b_{2q+1}\right| \leq \frac{2(1-\varphi)^{2}|\Theta|^{2}}{\Theta\left[2\digamma q(q+1)-2\digamma q+\digamma(\digamma-1)q^{2}-2q(q+1)(1-\digamma)(q-\digamma q+1)+2q(q+1)(2q+1)[1-\digamma+2\eta q-\eta]\right]}+\frac{|\Theta|(1-\varphi)}{\digamma q+q(2q+1)[1-\digamma+2\eta q-\eta]}. \tag{42}$$

The estimate in (28) can be determined from the equations (41) and (42).

In case $q=1$, the next result is achieved by reducing the Theorem 9.

Corollary 3.3: *Let* $\mathcal{J}(\xi)\in\widetilde{\mathcal{MH}}_{\Sigma_{q}}(\Theta,\digamma,\eta;\varphi)(\Theta\in\mathbb{C}\backslash\{0\},0\leq\digamma\leq 1,\ 0\leq\eta\leq 1$ *and* $0\leq\varphi<1)$ *be given by* (3). *Then*

$$|b_{2}|\leq min\left\{\frac{2|\Theta|(1-\varphi)}{(2-\digamma)},\sqrt{\frac{2|\Theta|(1-\varphi)}{\left[\frac{5\digamma}{2}-\frac{3\digamma^{2}}{2}+6(1-\digamma+\eta)\right]}}\right\}$$

and $$|b_{3}|\leq min\left\{\frac{2(1-\varphi)^{2}|\Theta|^{2}}{(2-\digamma)^{2}}+\frac{|\Theta|(1-\varphi)}{\digamma+3[1-\digamma+\eta]},\frac{2(1-\varphi)^{2}|\Theta|^{2}}{\Theta[5\digamma-3\digamma^{2}+12(1-\digamma+\eta)]}+\frac{|\Theta|(1-\varphi)}{\digamma+3[1-\digamma+\eta]}\right\}.$$

4. Conclusion

In this paper, a new subclasses $\mathcal{MH}_{\Sigma_{q}}(\Theta,\digamma,\eta;\delta)$ and $\widetilde{\mathcal{MH}}_{\Sigma_{q}}(\Theta,\digamma,\eta;\varphi)$ of q-fold symmetric bi-univalent functions defined in the open unit disk have been introduced and examined. Notably, coefficient estimates $\left|b_{q+1}\right|$ and $\left|b_{2q+1}\right|$ have been determined for functions within these specific subclasses.

References

[1] P. L. Duren, Univalent Functions, vol. 259. New York, NY, USA: Springer (2001).

[2] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proc. Amer. Math. Soc.*, vol. 18, no. 1, pp. 63–68 (1967).

[3] D. A. Brannan and J. G. Clunie, Eds., Aspects of Contemporary Complex Analysis: Proceedings of the NATO Advanced Study Institute Held at the University of Durham, Durham, July 1–20, 1979. New York, NY, USA: Academic Press (1980).

[4] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Appl. Math. Lett.*, vol. 23, no. 10, pp. 1188–1192 (2010).

[5] S. Bulut, "Coefficient estimates for general subclasses of m-fold symmetric analytic bi-univalent functions," *Turk. J. Math.*, vol. 40, no. 6, pp. 1386–1397 (2016).

[6] E. A. Adegani, S. Bulut, and A. Zireh, "Coefficient estimates for a subclass of analytic bi-univalent functions," *Bull. Korean Math. Soc.*, vol. 55, no. 2, pp. 405–413 (2018).

[7] H. Ö. Güney, G. Murugusundaramoorthy, and J. Sokół, "Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers," *Acta Univ. Sapientiae, Math.*, vol. 10, no. 1, pp. 70–84 (2018).

[8] H. Orhan, N. Magesh, and J. Yamini, "Coefficient estimates for a class of bi-univalent functions associated with quasi-subordination," *Creat. Math. Inform.*, vol. 26, no. 2, pp. 193–199 (2017).

[9] M. Haji Mohd and M. Darus, "Fekete-Szegö problems for quasi-subordination classes," *Abstr. Appl. Anal.*, vol. 2012, no. 1, pp. 1–12 (2012).

[10] H. M. Srivastava, S. Gaboury, and F. Ghanim, "Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions," Acta Univ. *Apulensis Math. Inform.*, no. 41, pp. 153–164 (2015).

[11] A. M. Delphi and K. A. Jassim, "New subclasses for estimates coefficients of m-fold symmetric bi-univalent functions and Fekete-Szegö problems," *J. Interdiscip. Math.*, vol.25 , no. 6, pp. 1857–1867 (2022).

[12] H. M. Srivastava, S. Sivasubramanian, and R. Sivakumar, "Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions," *Tbilisi Math. J.*, vol. 7, no.2 , pp. 1–10 (2014).

[13] N. H. Shehab and A. R. S. Juma, "Coefficient bounds of m-fold symmetric bi-univalent functions for certain subclasses," *Int. J. Nonlinear Anal. Appl.,* vol.12, pp. 71–82 (2021).

[14] A. R. S. Juma, A. Al-Fayadh, and N. H. Shehab, "Estimates of coefficient for certain subclasses of k-fold symmetric bi-univalent functions," *Iraqi J. Sci.,* vol. 63, no. 5, pp. 2155–2163 (2022).

[15] A. R. S. Juma, A. Al-Fayadh, S. P. Vijayalakshmi, and T. V. Sudharsan, "Upper bound on the third Hankel determinant for the class of univalent functions using an integral operator," *Afr. Mat.,* vol. 33, no.2, p. 56 (2022).

[16] Abbood, Mohaimen M., Ebrahim, Hassan H., Al-Fayadh, Ali & Obaid, Ahmed J. Measure defined on & Gamma; – algebra and some of their generalizations, *Journal of Interdisciplinary Mathematics*, 26:5, 821–827 (2023), DOI: 10.47974/JIM-1502.

[17] S. R. Swamy and L. I. Cotîrlă, "A new pseudo-type κ-fold symmetric bi-univalent function class," *Axioms*, vol. 12, no. 10, Art. no. 953 (Oct. 2023).

[18] C. Pommerenke, Univalent Functions. Göttingen, Germany: Vandenhoeck & Ruprecht (1975).

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