

Measurability of relative multifractal measure and dimension functions

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Abstract

The goal of this manuscript is to examine the measurability of relative multifractal measures and their related dimensions.

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1. Introduction

Multifractal analysis was presented in 1975 by Mandelbrot in [27, 28], consists of the study of local behaviors and scale invariance properties of functions, measures, etc. The kind of investigation was presented as a description of measure μ which occurs in the turbulence on a metric space. Given a metric space (Z, ρ) and a Borel probability measure μ on Z , $Spt \mu$ the topological support of μ . The pointwise Hölder exponent of μ relating to an arbitrary probability measure θ at x is designated by

$$\kappa_{\mu, \theta}(x) = \lim_{r \searrow 0} \frac{\ln \mu(B^x(r))}{\ln \theta(B^x(r))},$$

$B^x(r)$ is the closed ball centered at x with radius $r > 0$. The point of multifractal analysis is to assess the extent of the sets

$$X_{\mu, \theta}(\kappa) = \left\{ x \in Spt \mu \cap Spt \theta : \lim_{r \searrow 0} \frac{\ln \mu(B^x(r))}{\ln \theta(B^x(r))} = \kappa \right\}.$$

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The essential goal of multifractal framework is to establish the value of the special functions

$$f_{\mu,\theta}: \mathbb{R}_+ \rightarrow \mathbb{R}_+: \kappa \mapsto \dim_H(X_{\mu,\theta}(\kappa)),$$

$$F_{\mu,\theta}: \mathbb{R}_+ \rightarrow \mathbb{R}_+: \kappa \mapsto \dim_P(X_{\mu,\theta}(\kappa)),$$

with $\dim_H$ and $\dim_P$ are respectively the Hausdorff and packing dimensions. $f_{\mu,\theta}$ and $F_{\mu,\theta}$ are referred to the Hausdorff and packing multifractal spectra of μ relating to θ (see, such as [1, 3, 5, 11, 14, 40]). Moreover, $X_{\mu,\theta}(\kappa)$ contains important informations on geometrical properties of μ relating to θ . In a several ongoing papers, numerous mathematicians and physicists have begun to look at performing multifractal examination regarding to any given reference measure, as we can notice, in this case, [1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 16, 17, 22, 24, 25, 36, 40, 44, 45, 46, 47, 48]. Moreover, when the measure θ is substituted with a tantamount of Lebesgue's measure was studied by Olsen [32], Pesin [37] and Peyrière [38]. In particular, they developed different approaches to characterize an auxiliary measure in an extensive context. This formalism was expanded from Olsen's desire offered an extensive mathematical framework to assess the especial functions $f_\mu(\kappa) = \dim_H(X_\mu(\kappa))$ and $F_\mu(\kappa) = \dim_P(X_\mu(\kappa))$, with $\kappa \geq 0$, where

$$X_\mu(\kappa) = \left\{ x \in \text{Spt } \mu: \lim_{r \searrow 0} \frac{\ln \mu(B^x(r))}{\ln r} = \kappa \right\},$$

and to investigate the relationship between them (see also [35]). Next, Nasr, Bhouri and Heurteaux fostered a crucial foundation for the validity of the multifractal formalism in [7].

The exploration of these sets has deep roots. Specifically, a crucial idea presented in Billingsley's 1965 book [9] was the estimation of the capacity of above sets. In [10], Cajar concentrated on these collections within the framework of code spaces. Afterward, the function f_μ was from the beginning characterized in 1986 in [20] by Halsey et al.

In [11] Cole presented a comprehensive framework for the multifractal analysis of a probability measure μ according to another identified reference θ . Precisely, he introduced an offer of multifractal formalism appropriate for examining the characteristics of $f_{\mu,\theta}$ and $F_{\mu,\theta}$. Later, Selmi et al. formulated this multifractal framework due to limited assumptions in the probability space (see [3, 16, 19, 22, 25, 44, 45, 46, 47]) and in the code space (see [2, 6, 14]).

Let $q, s \in \mathbb{R}$, μ and θ be two Borel probability measures, $\mathcal{H}_{\mu,\theta}^{q,s}$ and $\mathcal{P}_{\mu,\theta}^{q,s}$ are respectively the relative multifractal Hausdorff and packing measure, $\dim_{\mu,\theta}^q$ and $\text{Dim}_{\mu,\theta}^q$ indicate the relative multifractal dimensions associated respectively to $\mathcal{H}_{\mu,\theta}^{q,s}$ and $\mathcal{P}_{\mu,\theta}^{q,s}$ (we will give the details in the next section). The fundamental role of $\mathcal{H}_{\mu,\theta}^{q,s}$ and $\mathcal{P}_{\mu,\theta}^{q,s}$ is to calculate $\dim_{\mu,\theta}^q$ and $\text{Dim}_{\mu,\theta}^q$ of the sets $X_{\mu,\theta}(\kappa)$, $\alpha \in \mathbb{R}$ (see for example [1, 11, 22, 47]). Many authors are concentrated on the connection between multifractal characteristics of a measure μ on $\mathbb{R}^d$. Moreover, some authors are interested in learning about the projection on

k -dimensional vector spaces of μ that adheres to the multifractal formalism (see [13, 16, 29, 34, 45]).

In this present work, we examine the measurability of the relative multifractal Hausdorff measure and dimension functions

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}_+: (K, \mu, \theta, q) \mapsto \mathcal{H}_{\mu, \theta}^{q, s}(K), \quad (1.1)$$

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}: (K, \mu, \theta, q) \mapsto \dim_{\mu, \theta}^q(K), \quad (1.2)$$

and the relative multifractal packing measure and dimension functions

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}_+: (K, \mu, \theta, q) \mapsto \mathcal{P}_{\mu, \theta}^{q, s}(K), \quad (1.3)$$

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}: (K, \mu, \theta, q) \mapsto \text{Dim}_{\mu, \theta}^q(K), \quad (1.4)$$

where

$$\mathbb{A} = \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R},$$

with $\mathcal{K}(\mathbb{R}^d)$ indicates the collection of compact subsets of $\mathbb{R}^d$ that are not empty, endowed by Hausdorff distance and $\mathcal{P}(\mathbb{R}^d)$ presented the collection of Borel probability measures on $\mathbb{R}^d$ that are endowed by Prokhorov distance. Certain outcomes from our work may be interpreted as an extension of results developed by Matilla and Mauldin [31], Olsen [33] and Selmi et al. [15, 42, 43]. Consider Z as Polish space and $g: (0, +\infty) \rightarrow (0, +\infty)$ as a dimension function, Matilla and Mauldin investigated the measurability of the maps $\mathcal{K}(Z) \rightarrow \mathbb{R}_+: K \mapsto \mathcal{H}^g(K)$ and $\mathcal{K}(Z) \rightarrow \mathbb{R}_+: K \mapsto \mathcal{P}^g(K)$ with $\mathcal{K}(Z)$ the collection of compact sets that are not empty of Z , $\mathcal{H}^g$ and $\mathcal{P}^g$ indicate the Hausdorff and packing measures related to g . Olsen showed that $\mathcal{K}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \times \mathbb{R} \rightarrow \mathbb{R}_+: (K, \mu, q) \mapsto \mathcal{H}_{\mu}^{q, s}$ is measurable relating to the σ -algebra continuing all analytic sets of $\mathcal{K}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}$, where $\mathcal{M}(\mathbb{R}^d)$ represents the collection of Radon measures on $\mathbb{R}^d$ endowed by weak topology.

We examine the behavior of measurability of the maps (1.1), (1.2), (1.3) and (1.4) obeying to the relative multifractal formalism represented in [1, 11, 47]. We prove that (1.1) and (1.2) are measurable relating to the σ -algebra continuing the analytic subsets, further, if μ and θ are doubling measures, we get (1.2) is Baire-(2) class. The measurability of (1.3) and (1.4) are discussed in section 4. Next, if μ and θ that adhere to doubling condition, (1.4) is measurable relating to σ -algebra continuing the analytic subsets.

The organization of this work is as follow. Initially, in section 2, remembering the multifractal and some topological definitions which we shall need. Secondly, in section 3 we will be interested in the analysis of (1.1) and (1.2). In section 4 we will analyze (1.3) and (1.4). At last, in section 5 we will utilize the measurability findings from section 3 to discuss the measurability of multifractal slices.

2. Notations and some definitions

Although the concepts of the packing and Hausdorff measures, as well as these associated dimensions, are widely recognized, we chose to succinctly

reiterate the definitions that below. Consider (Z, ρ) as a separable metric space, write $\mathcal{P}(Z)$ the collection of Borel probability measures on Z . For every $x \in Z$ and $r > 0$, consider $U^x(r)$ as the open ball centered at x with radius r and $B^x(r)$ its closure. For $q \in \mathbb{R}$, we define $\phi_q: [0, +\infty) \rightarrow [0, +\infty]$ by

$$\phi_q(x) = \begin{cases} \infty & \text{for } x = 0 \\ x^q & \text{for } x > 0 \end{cases} \text{ for } q < 0,$$

$$\phi_q(x) = \begin{cases} 1 & \text{for } q = 0, \end{cases}$$

$$\phi_q(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^q & \text{for } x > 0 \end{cases} \text{ for } q > 0.$$

Given $\mu, \theta \in \mathcal{P}(Z)$ with θ has no point masses. For $q, s \in \mathbb{R}$ and $A \subset \text{Spt } \mu \cap \text{Spt } \theta$, we define

$$\overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(A) = \inf\{\sum_i \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s\}, A \neq \emptyset$$

with the infimum is considered over all δ -covering which are centered at A , with the convention $0^q = 0$ if $q \geq 0$ and $0^q = \infty$ if $q < 0$. Also, the relative multifractal Hausdorff pre-measure is mentioned as follows

$$\overline{\mathcal{H}}_{\mu, \theta}^{q, s}(A) = \lim_{\delta \searrow 0} \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(A) = \sup_{\delta > 0} \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(A).$$

The multifractal Hausdorff measure according to μ and θ is indicated as

$$\mathcal{H}_{\mu, \theta}^{q, s}(A) = \sup_{B \subseteq A} \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(B), \quad \mathcal{H}_{\mu, \theta}^{q, s}(\emptyset) = 0.$$

We also make the dual definition, for $q, s \in \mathbb{R}$ and $A \subset \text{Spt } \mu \cap \text{Spt } \theta$

$$\overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(A) = \sup\{\sum_i \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s\}, A \neq \emptyset$$

with the supremum is considered over all δ -packings of A . The relative multifractal packing pre-measure is identified by

$$\overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A) = \lim_{\delta \searrow 0} \overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(A) = \inf_{\delta > 0} \overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(A).$$

The multifractal packing measure according to μ and θ is indicated by

$$\mathcal{P}_{\mu, \theta}^{q, s}(A) = \inf_{A \subseteq \cup_i A_i} \sum_i \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A_i), \quad \mathcal{P}_{\mu, \theta}^{q, s}(\emptyset) = 0.$$

Especially, we write the conventions $\infty \cdot 0 = 0 \cdot \infty = 0$.

Cole proved in [11] that $\mathcal{H}_{\mu, \theta}^{q, s}$ and $\mathcal{P}_{\mu, \theta}^{q, s}$ are metric outer measures, specifically they serve as measures on the σ -algebra engendered by Borel subsets. Furthermore, these functions are respectively multifractal prolongations of the Hausdorff and packing measures $\mathcal{H}^s, \mathcal{P}^s$. On the other hand, $\overline{\mathcal{P}}_{\mu, \theta}^{q, s}$ is a multifractal prolongation of packing pre-measure $\overline{\mathcal{P}}^s$. Reader is referred to [18, 30, 39, 41, 49, 50] for more informations on $\mathcal{H}^s$ (respectively, the centered

Hausdorff measure $\mathcal{C}^s$), $\mathcal{P}^s$ and $\overline{\mathcal{P}}^s$. $\overline{\mathcal{H}}_{\mu,\theta}^{q,s}$ and $\overline{\mathcal{P}}_{\mu,\theta}^{q,s}$ are not measures. More precisely, the first is countably sub-additive and not necessarily monotone, whereas, the other is monotone and only finitely sub-additive.

Let $\mathcal{P}_0^1(Z)$ and $\mathcal{P}_0^2(Z)$ denote the following sets

$$\mathcal{P}_0^1(Z) = \{\mu \in \mathcal{P}(Z): T_\beta^1(\mu) < \infty \text{ for all } \beta > 1\}$$

and

$$\mathcal{P}_0^2(Z) = \{\mu \in \mathcal{P}(Z): T_\beta^2(\mu) < \infty \text{ for all } \beta > 1\},$$

where

$$T_\beta^1(\mu) = \limsup_{r \searrow 0} \sup_{x \in \text{Spt } \mu} \frac{\mu(B^x(\beta r))}{\mu(B^x(r))}$$

and

$$T_\beta^2(\mu) = \limsup_{r \searrow 0} \sup_{x \in \text{Spt } \mu} \frac{\mu(U^x(\beta r))}{\mu(U^x(r))}.$$

Some interpretation from $\mathcal{P}_0^1(Z)$ and $\mathcal{P}_0^2(Z)$, they remain unchanged by the values of β greater than 1, as stated in reference [32, Lemma 2.1]).

Proposition 2.1: For $A \subseteq Z$,

1. There is a specific critical value, denoted as $\dim_{\mu,\theta}^q \in [-\infty, +\infty]$ where

$$\mathcal{H}_{\mu,\theta}^{q,s}(A) = \begin{cases} +\infty & \text{if } s < \dim_{\mu,\theta}^q(A) \\ 0 & \text{if } s > \dim_{\mu,\theta}^q(A). \end{cases}$$

2. There is a specific critical value, denoted as $\Delta_{\mu,\theta}^q \in [-\infty, +\infty]$ where

$$\overline{\mathcal{P}}_{\mu,\theta}^{q,s}(A) = \begin{cases} +\infty & \text{if } s < \Delta_{\mu,\theta}^q(A) \\ 0 & \text{if } s > \Delta_{\mu,\theta}^q(A). \end{cases}$$

3. There is a specific critical value, denoted as $\text{Dim}_{\mu,\theta}^q \in [-\infty, +\infty]$ where

$$\mathcal{P}_{\mu,\theta}^{q,s}(A) = \begin{cases} +\infty & \text{if } s < \text{Dim}_{\mu,\theta}^q(A) \\ 0 & \text{if } s > \text{Dim}_{\mu,\theta}^q(A). \end{cases}$$

The values $\dim_{\mu,\theta}^q, \text{Dim}_{\mu,\theta}^q$ and $\Delta_{\mu,\theta}^q$ are respectively the relative multifractal Hausdorff, packing and pre-packing dimensions. Moreover, write

$$\dim_{\mu,\theta}^0(A) = \dim_\theta(A), \text{Dim}_{\mu,\theta}^0(A) = \text{Dim}_\theta(A) \text{ and } \Delta_{\mu,\theta}^0(A) = \Delta_\theta(A).$$

2.1 Polish space and analytic sets

Consider a metric space (Z, ρ) is termed a Polish space when it possesses the separability and topological completeness. Specifically, there is a distance ρ on Z for which the metric space (Z, ρ) is complete. In particular, a complete separable metric space is known as Polish. Let (Z, ρ) be considered a Polish space, a set A of Z is termed an analytic set if it can be represented as an image of a Polish space W under a continuous function h from W to Z . For Z and

W are two Polish spaces and B is Borel of $Z \times W$, then $A = \pi_Z(B)$ is analytic, when π_Z is the projector function on Z [21, Chapter 14].

In the Next, for every Polish space, Z we write $\mathfrak{B}(Z)$ and $\mathfrak{A}(Z)$ respectively the family of Borel and analytic sets of Z .

2.2 Hausdorff metric

Let $\mathcal{K}(Z)$ represents the collection of compact sets that are not empty of Z . The Hausdorff distance is characterized in this way, for K and J as two compact sets,

$$\rho_H(K, J) = \max \left\{ \sup_{y \in J} \inf_{x \in K} \rho(x, y), \sup_{x \in K} \inf_{y \in J} \rho(x, y) \right\}.$$

We get ρ_H is a distance on $\mathcal{K}(Z)$. Also, we can consider a metric

$$\rho_{H_0}(K, J) = \begin{cases} 0 & \text{if } K = \emptyset \text{ and } J = \emptyset \\ 1 & \text{if } K = \emptyset \text{ or } J = \emptyset \\ \rho_H(K, J) & \text{if } K \neq \emptyset \text{ and } J \neq \emptyset. \end{cases}$$

In particular, the space $\mathcal{K}(Z) \cup \{\emptyset\}$ will always be endowed with the metric ρ_{H_0} . If (Z, ρ) is Polish, hence $(\mathcal{K}(Z), \rho_H)$ is Polish (see [21]).

2.3 Prokhorov metric

Consider μ, θ as two elements of $\mathcal{P}(Z)$ and $\varepsilon > 0$, the Prokhorov distance is defined as following

$$\rho_p(\mu, \theta) = \inf\{\varepsilon > 0: \mu(A) \leq \theta(A_\varepsilon) + \varepsilon, \forall A \subset Z\},$$

with

$$A_\varepsilon = \{x \in Z : \rho(x, A) < \varepsilon\}.$$

It follows from [8, Section 6], that ρ_p is a distance on $\mathcal{P}(Z)$ and for (Z, ρ) is a separable, so for each $\mu, (\mu_n)_n \subset \mathcal{P}(Z)$,

$$\rho_p(\mu_n, \mu) \searrow 0 \Leftrightarrow \mu_n \xrightarrow{\text{weakly}} \mu.$$

When (Z, ρ) is a Polish, therefore $(\mathcal{P}(Z), \rho_p)$ is also Polish. The reader can see [8, Sections 5 and 6] (see also [26]). In particular,

$$\mathcal{K}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R} \text{ is Polish.}$$

2.4 Borel Hierarchy and Baire class

We delineate the Borel hierarchy as a means to classify the smoothness properties of functions. For all $1 \leq \eta < \omega_1$, where ω_1 is the first uncountable ordinal, we establish the families $\Sigma_\eta^0(Z)$ and $\Pi_\eta^0(Z)$ of Z respectively by

$$\begin{aligned} \Sigma_1^0(Z) &= \{A \subseteq Z: A \text{ is open}\} = \Sigma_1^0, \\ \Pi_1^0(Z) &= \{A \subseteq Z: A \text{ is closed}\} = \Pi_1^0, \\ \Sigma_{\eta+1}^0(Z) &= \left\{ \bigcup_{k=1}^{+\infty} A_k: A_k \in \Pi_\eta^0(Z) \right\} = \Sigma_{\eta+1}^0, \\ \Sigma_{\eta+1}^0(Z) &= \left\{ \bigcap_{k=1}^{+\infty} A_k: A_k \in \Sigma_\eta^0(Z) \right\} = \Pi_{\eta+1}^0, \end{aligned}$$

and

$$\mathfrak{B}(Z) = \bigcup_{\eta < \omega_1} \Pi_\eta^0 = \bigcup_{\eta < \omega_1} \Sigma_\eta^0,$$

Moreover, we have $\mathfrak{B}(Z) \subseteq \mathfrak{A}(Z)$. Therefore,

$$\Sigma_1^0 = \mathcal{G} \subset \Pi_2^0 = \mathcal{G}_\delta \subset \Sigma_3^0 = \mathcal{G}_{\delta\sigma} \subset \Pi_4^0 = \mathcal{G}_{\delta\sigma\delta} \dots$$

and

$$\Pi_1^0 = \mathcal{F} \subset \Sigma_2^0 = \mathcal{F}_\sigma \subset \Pi_3^0 = \mathcal{F}_{\sigma\delta} \subset \Sigma_4^0 = \mathcal{F}_{\sigma\delta\sigma} \dots$$

Given two metrizable topological spaces Z and W , a function h from Z to W is Baire- (n) class (or is of Baire class n), where n is a natural number, if it can be represented as the pointwise limit of functions' sequence that are Baire- (m) class, where m is lower than n . The Baire- (0) class consists of all continuous functions. In addition, if W possesses the separability property, h is Baire- (n) class if and only if it is Σ_{n+1}^0 -measurable [21, Theorem 24.3]. Exceptionally, the lower and upper semi-continuous functions are Baire- (1) class.

Throughout the paper, we will write $\overline{\mathbb{R}} = [-\infty, +\infty]$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$,

$$\mathbb{A} = \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R},$$

$$\Omega_1 = \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}_0^1(\mathbb{R}^d) \times \mathbb{R}_- \cup \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}_0^1(\mathbb{R}^d) \times \mathcal{P}_0^1(\mathbb{R}^d) \times \mathbb{R}_+$$

$$\Omega_2 = \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}_0^2(\mathbb{R}^d) \times \mathbb{R}_- \cup \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}_0^2(\mathbb{R}^d) \times \mathcal{P}_0^2(\mathbb{R}^d) \times \mathbb{R}_+.$$

and for any space Z , $\mathfrak{B}(\mathfrak{A}(Z))$ indicates the σ -algebra engendered by the family of analytic subsets of Z .

3. Measurability of relative multifractal Hausdorff measure and dimension maps

We will now focus on the measurability of the functions (1.1) and (1.2). We remember the open balls and closed balls centered at x , $U^x(r)$ and $B^x(r)$ respectively, where $r > 0$ and x is an element of $\mathbb{R}^d$.

Theorem 3.1: Let $\delta > 0$ and $s \in \mathbb{R}$.

1. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(K)$ is upper semi-continuous, specifically, it is Baire- (1) class.
2. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(K)$ is Baire- (2) class.
3. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \mathcal{H}_{\mu, \theta}^{q, s}(K)$ is $\mathfrak{B}(\mathfrak{A}(\mathbb{A}))$ -measurable.
4. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}} : (K, \mu, \theta, q) \mapsto \dim_{\mu, \theta}^q(K)$ is $\mathfrak{B}(\mathfrak{A}(\mathbb{A}))$ -measurable.
5. The function $\Omega_2 \rightarrow \overline{\mathbb{R}} : (K, \mu, \theta, q) \mapsto \dim_{\mu, \theta}^q(K)$ is Borel measurable. In particular, it is Baire- (2) class and as well not Baire- (1) class.

We introduce various instruments and preliminary findings that we will use to demonstrate our results.

Proposition 3.1: For $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$ and $q, s \in \mathbb{R}$, one has

1. $\overline{\mathcal{H}}_{\mu, \theta}^{q, s}(A) \leq \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(\overline{A})$, for all $A \subseteq \mathbb{R}^d$.
2. $\mathcal{H}_{\mu, \theta}^{q, s}(K) = \sup_{\substack{J \subseteq K \\ J \text{ is compact}}} \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(J)$, for all $K \in \mathcal{K}(\mathbb{R}^d)$.

Proof:

1. For $\delta, \lambda > 0$. Given $\{U^{x_i}(r_i)\}_i$ as a centered δ -covering of $\overline{A}$. As long as $U^{x_i}(r_i + \varepsilon) \supset U^{x_i}(r_i)$ as $\varepsilon \searrow 0$, so there is $0 < \varepsilon_i < \delta$ in such a way $r_i + \varepsilon_i < \delta$ and

$$\begin{cases} \alpha^q \beta^s \leq \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s + \frac{\lambda}{2^i}, \text{ for all } \alpha, \beta \in \mathbb{R}, \text{ satisfying} \\ \left\{ \begin{array}{l} \mu(U^{x_i}(r_i)) \leq \alpha \leq \mu(U^{x_i}(r_i + \varepsilon_i)) \\ \text{and} \\ \theta(U^{x_i}(r_i)) \leq \beta \leq \theta(U^{x_i}(r_i + \varepsilon_i)). \end{array} \right. \end{cases} \quad (3.1)$$

Now, we choose $y_i \in U^{x_i}\left(r_i - \frac{\lambda}{2^i}\right) \cap A$ and using (3.1), we get

$$\mu\left(U^{y_i}\left(r_i + \frac{\varepsilon_i}{2}\right)\right)^q \theta\left(U^{y_i}\left(r_i + \frac{\varepsilon_i}{2}\right)\right)^s \leq \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s + \frac{\lambda}{2^i}. \quad (3.2)$$

We can see obviously $\{U^{y_i}(r_i + \frac{\varepsilon_i}{2})\}_i$ is a centered δ -covering of A , so (3.2) implies

$$\begin{aligned} \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(A) &\leq \sum_i \mu\left(U^{y_i}\left(r_i + \frac{\varepsilon_i}{2}\right)\right)^q \theta\left(U^{y_i}\left(r_i + \frac{\varepsilon_i}{2}\right)\right)^s \\ &\leq \sum_i (\mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s + \frac{\lambda}{2^i}) \\ &= \sum_i \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s + \lambda. \end{aligned}$$

Consequently, $\overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(A) \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(\overline{A}) + \lambda$. Letting $\delta, \lambda \rightarrow 0$, we get our result.

2. Using assumption 1, we obtain

$$\overline{\mathcal{H}}_{\mu, \theta}^{q, s}(K) \leq \sup_{\substack{J \subseteq K \\ J \text{ is compact}}} \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(J).$$

The other inequality is obtained from the definition of $\mathcal{H}_{\mu, \theta}^{q, s}$. Then

$$\mathcal{H}_{\mu, \theta}^{q, s}(K) = \sup_{\substack{J \subseteq K \\ J \text{ is compact}}} \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(J).$$

Proposition 3.2: Let $\delta > 0$ and $s, c \in \mathbb{R}$, the set

$$U = \{(K, \mu, \theta, q) \in \mathbb{A} : \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(K) < c\}$$

is open.

Proof: By Utilizing the definition of $\overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}$ and given that K is compact, we show that $\overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(K) < c$ is similar to the existence of non-zero positive integer

m , $x_1, \dots, x_m \in K$, $r_1, \dots, r_m \in (0, \delta)$ and $c_1, \dots, c_m \in \mathbb{R}_+$ with $\sum_{i=1}^m c_i < c$ such that

1. $\{U^{x_i}(r_i)\}_{i=1}^m$ is a covering of K
2. $\mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s < c_i, \forall i \in \{1, \dots, m\}$.

Denote by F the complement of U . It is enough to demonstrate that F is closed. Consider a sequence $(K_n, \mu_n, \theta_n, q_n)_n$ in F , (K, μ, θ, q) belonging to $\mathbb{A}$ with

$$(K_n, \mu_n, \theta_n, q_n) \xrightarrow{n \rightarrow +\infty} (K, \mu, \theta, q).$$

We should be proven that (K, μ, θ, q) is an element of F .

Fix $m \in \mathbb{N} \setminus \{0\}$, $x_1, \dots, x_m \in K$, $r_1, \dots, r_m \in (0, \delta)$, $c_1, \dots, c_m \in \mathbb{R}_+$ with $\sum_{i=1}^m c_i < c$, we have to show that

$$\{U^{x_i}(r_i)\}_{i=1}^m \text{ is not a centered } \delta - \text{covering of } K, \quad (3.3)$$

or

$$\mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s \geq c_i, \text{ for some } i \in \{1, \dots, m\}. \quad (3.4)$$

If (K, μ, θ, q) checks (3.3), we get the result. We assume that

$$\{U^{x_i}(r_i)\}_{i=1}^m \text{ is a } \delta - \text{covering of } K. \quad (3.5)$$

Given that K is compact, so (3.5) implies that there is $\varepsilon_0 > 0$ where

$$\{U^{x_i}(r_i - \varepsilon)\}_{i=1}^m \text{ is a } \delta - \text{covering of } K, \text{ for all } \varepsilon \in (0, \varepsilon_0). \quad (3.6)$$

Now, demonstrate that, for ε within $(0, \varepsilon_0)$, there is $i(\varepsilon)$ such that $1 \leq i(\varepsilon) \leq m$ and

$$c_{i(\varepsilon)} \leq \begin{cases} \mu(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^q \theta(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^s & \text{if } q \geq 0 \text{ and } s \geq 0 \\ \mu(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^q \theta(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^s & \text{if } q \geq 0 \text{ and } s < 0 \\ \mu(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^q \theta(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^s & \text{if } q < 0 \text{ and } s \geq 0 \\ \mu(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^q \theta(U^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^s & \text{if } q < 0 \text{ and } s < 0. \end{cases}$$

Fix $\varepsilon \in (0, \varepsilon_0)$, since $\rho_H(K_n, K) \searrow 0$ and from (3.6), there exists $N \in \mathbb{N}$ such as

$$K_n \subset \bigcup_{i=1}^m U^{x_i}(r_i - \varepsilon), \text{ for each } n \geq N, \quad (3.7)$$

and

$$K_n \cap U^{x_i}\left(\frac{1}{5}\varepsilon\right) \neq \emptyset, \text{ for any } n \geq N \text{ and } 1 \leq i \leq m.$$

Next, pick $n \geq N$ and take $y_n^i \in K_n \cap U^{x_i}\left(\frac{1}{5}\varepsilon\right)$ for $i \in \{1, \dots, m\}$. Clearly,

$$U^{x_i}(r_i - \varepsilon) \subset U^{y_n^i}\left(r_i - \frac{3}{4}\varepsilon\right) \subset U^{x_i}\left(r_i - \frac{1}{2}\varepsilon\right). \quad (3.8)$$

In particular,

$$\left\{U^{y_n^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right\}_{i=1}^m \text{ is a } \delta - \text{covering of } K_n. \quad (3.9)$$

Since $(K_n, \mu_n, \theta_n, q_n) \in F$ and by (3.9), we can conclude that

$$\mu_n \left(U^{y_n^{i(n)}} \left(r_{i(n)} - \frac{3}{4} \varepsilon \right) \right)^{q_n} \theta_n \left(U^{y_n^{i(n)}} \left(r_{i(n)} - \frac{3}{4} \varepsilon \right) \right)^s \geq c_{i(n)},$$

with certain $1 \leq i(n) \leq m$. (3.10)

Later, chosen $i = i(\varepsilon)$ with $1 \leq i(\varepsilon) \leq m$, in such way there is a strictly monotonic sequence $(n_j)_j \subseteq \mathbb{N}$ and also $i(n_j) = i$ for each j . Moreover,

$$\rho_p(\mu_n, \mu) \searrow 0 \text{ and } \rho_p(\theta_n, \theta) \searrow 0.$$

By using (3.8), it is noticed that

$$\begin{aligned} \mu(U^{x_i}(r_i - \varepsilon)) &\leq \mu \left(U^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &\leq \liminf_j \mu_{n_j} \left(U^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &= \liminf_j \mu_{n_j} \left(U^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \mu(U^{x_i}(r_i)) &\geq \mu \left(B^{x_i} \left(r_i - \frac{1}{2} \varepsilon \right) \right) \\ &\geq \limsup_j \mu_{n_j} \left(B^{x_i} \left(r_i - \frac{1}{2} \varepsilon \right) \right) \\ &\geq \limsup_j \mu_{n_j} \left(U^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &= \limsup_j \mu_{n_j} \left(U^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right). \end{aligned} \quad (3.12)$$

Similarly, we acquire

$$\theta(U^{x_i}(r_i - \varepsilon)) \leq \liminf_j \theta_{n_j} \left(U^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \quad (3.13)$$

and

$$\theta(U^{x_i}(r_i)) \geq \limsup_j \theta_{n_j} \left(U^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right). \quad (3.14)$$

For $j \in \mathbb{N}$, we denote

$$\alpha_j = \mu_{n_j} \left(U^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \text{ and } \beta_j = \theta_{n_j} \left(U^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right).$$

We have $\alpha_j \xrightarrow[q \rightarrow +\infty]{q - q_{n_j}} 1$. Since (3.11), (3.12), (3.13) and (3.14) imply that $(\alpha_j)_j$ and $(\beta_j)_j$ are a bounded sequences. Then,

- If $q \geq 0$ and $s \geq 0$, from (3.10), (3.12) and (3.14) we obtain

$$\begin{aligned}
 \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s &\geq \limsup_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(U^{x_i}(r_i))^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta_{n_j}\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s \\
 &= \limsup_j \alpha_j^{q_{n_j}} \beta_j^s \\
 &\geq \limsup_j c_{i(n_j)} = c_i.
 \end{aligned}$$

- If $q \geq 0$ and $s < 0$, from (3.10), (3.12) and (3.13), we have

$$\begin{aligned}
 \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i - \varepsilon))^s &\geq \limsup_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(U^{x_i}(r_i - \varepsilon))^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta_{n_j}\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s \\
 &= \limsup_j \alpha_j^{q_{n_j}} \beta_j^s \\
 &\geq \limsup_j c_{i(n_j)} = c_i.
 \end{aligned}$$

- If $q < 0$ and $s \geq 0$, from (3.10), (3.11) and (3.14), we have

$$\begin{aligned}
 \mu(U^{x_i}(r_i - \varepsilon))^q \theta(U^{x_i}(r_i))^s &\geq \limsup_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(U^{x_i}(r_i))^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta_{n_j}\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s \\
 &= \limsup_j \alpha_j^{q_{n_j}} \beta_j^s \\
 &\geq \limsup_j c_{i(n_j)} = c_i.
 \end{aligned}$$

- If $q < 0$ and $s < 0$, we use (3.10), (3.11) and (3.13), we obtain

$$\begin{aligned}
 \mu(U^{x_i}(r_i - \varepsilon))^q \theta(U^{x_i}(r_i - \varepsilon))^s &\geq \limsup_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(U^{x_i}(r_i - \varepsilon))^s \\
 &\geq \limsup_j \alpha_j^{q_{n_j}} \theta\left(U^{y_{n_j}^i}\left(r_i - \frac{3}{4}\varepsilon\right)\right)^s
 \end{aligned}$$

$$\begin{aligned}
&\geq \limsup_j \alpha_j^{q_{n_j}} \theta_{n_j} \left(U^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right)^s \\
&= \limsup_j \alpha_j^{q_{n_j}} \beta_j^s \\
&\geq \limsup_j c_{i(n_j)} = c_i.
\end{aligned}$$

It implies that there is sequence $(\varepsilon_n)_n \subset \mathbb{R}_+$ and $1 \leq i \leq n$ in such way $\varepsilon_n \searrow 0$ and $i(\varepsilon_n) = i$ for each n . Making $n \nearrow \mathbb{Z} + \infty$, so (3.4) is acquired and therefore $(K, \mu, \theta, q) \in F$.

Proposition 3.3: For $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$ and $q, s \in \mathbb{R}$.

1. If $q \geq 0$ and $\mu, \theta \in \mathcal{P}_0^2(\mathbb{R}^d)$, thus one can find $c > 0$ for which

$$\mathcal{H}_{\mu, \theta}^{q, s}(A) \leq c \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(A), \text{ for every } A \subset \mathbb{R}^d.$$

2. If $q < 0$ and $\theta \in \mathcal{P}_0^2(\mathbb{R}^d)$, thus one can find $c > 0$ for which

$$\mathcal{H}_{\mu, \theta}^{q, s}(A) \leq c \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(A), \text{ for every } A \subset \mathbb{R}^d.$$

Proof:

1. Let $\mu, \theta \in \mathcal{P}_0^2(\mathbb{R}^d)$, so there are $c_1, c_2 > 0$ and $r_1, r_2 > 0$ in such a way that

$$\mu(U^x(3r)) \leq c_1 \mu(U^x(r)) \text{ for any } 0 < r < r_1$$

and

$$\theta(U^x(3r)) \leq c_2 \theta(U^x(r)) \text{ for any } 0 < r < r_2.$$

Consider $\delta > 0$, $A \subset \mathbb{R}^d$ and $\{U^{x_i}(r_i)\}_i$ as a centered δ -covering of A .

Let $B \subset A$ and $I = \{i: B \cap U^{x_i}(r_i) \neq \emptyset\}$. For any $i \in I$, select

$y_i \in U^{x_i}(r_i) \cap B$. Clearly,

$$U^{x_i}(r_i) \subset U^{y_i}(2r_i) \subset U^{x_i}(3r_i).$$

Also, we can see that $\{U^{y_i}(2r_i)\}_{i \in I}$ is a centered 2δ -covering of B . It follows that

- (a) For $\mu, \theta \in \mathcal{P}_0^2(\mathbb{R}^d)$ and $s \geq 0$, we have

$$\begin{aligned}
\overline{\mathcal{H}}_{\mu, \theta, 2\delta}^{q, s}(B) &\leq \sum_{i \in I} \mu(U^{y_i}(2r_i))^q \theta(U^{y_i}(2r_i))^s \\
&\leq \sum_i \mu(U^{x_i}(3r_i))^q \theta(U^{x_i}(3r_i))^s \\
&\leq c_1^q c_2^s \sum_i \mu(U^{x_i}(r_i))^q \theta(U^{x_i}(r_i))^s.
\end{aligned}$$

Hence,

$$\overline{\mathcal{H}}_{\mu, \theta, 2\delta}^{q, s}(B) \leq c_1^q c_2^s \overline{\mathcal{H}}_{\mu, \theta, \delta}^{q, s}(A).$$

Letting $\delta \searrow 0$ and since B is arbitrary, we conclude that

$$\mathcal{H}_{\mu, \theta}^{q, s}(A) \leq c \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(A) \text{ with } c = c_1^q c_2^s.$$

- (b) For $\mu \in \mathcal{P}_0^2(\mathbb{R}^d)$ and $s < 0$, the proof bears a resemblance to case (a) with $c = c_1^q$.
2. For $q < 0$, we have
- (a) For $\theta \in \mathcal{P}_0^2(\mathbb{R}^d)$ and $s \geq 0$, the proof bears a resemblance to case 1.(a) with $c = c_2^s$.
- (b) For $s < 0$, the proof bears a resemblance to case 1.(a) with $c = 1$.

Corollary 3.1:

1. For $q \geq 0$ and $\mu, \theta \in \mathcal{P}_0^2(\mathbb{R}^d)$, then for all $A \subseteq \mathbb{R}^d$, we have
- $$\dim_{\mu, \theta}^q(A) = \inf\{s \in \mathbb{R} : \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(A) = 0\} = \sup\{s \in \mathbb{R} : \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(A) = +\infty\}.$$
2. For $q < 0$ and $\theta \in \mathcal{P}_0^2(\mathbb{R}^d)$, then for all $A \subseteq \mathbb{R}^d$, we have
- $$\dim_{\mu, \theta}^q(A) = \inf\{s \in \mathbb{R} : \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(A) = 0\} = \sup\{s \in \mathbb{R} : \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(A) = +\infty\}.$$

Proof of Theorem 3.1.

1. This is a direct result of Proposition 3.2.
2. For every (K, μ, θ, q) in $\mathbb{A}$,

$$\overline{\mathcal{H}}_{\mu, \theta}^{q,s}(K) = \lim_{n \rightarrow +\infty} \overline{\mathcal{H}}_{\mu, \theta, 2^{-n}}^{q,s}(K).$$

Hence,

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(K)$$

is Baire-(2) class.

3. It is enough to demonstrate

$$\Gamma = \{(K, \mu, \theta, q) \in \mathbb{A} : \mathcal{H}_{\mu, \theta}^{q,s}(K) > c\}$$

is analytic, for every $c \in \mathbb{R}$. Let

$$\Pi = \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : J \subseteq K\},$$

$$\Sigma = \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(J) > c\},$$

and

$$\begin{aligned} \pi : \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) &\rightarrow \mathbb{A} \\ (K, \mu, \theta, q, J) &\mapsto (K, \mu, \theta, q). \end{aligned}$$

One has

$$\begin{aligned} \Gamma &= \{(K, \mu, \theta, q) \in \mathbb{A} : \mathcal{H}_{\mu, \theta}^{q,s}(K) > c\} \\ &= \{(K, \mu, \theta, q) \in \mathbb{A} : \exists J \subseteq K \text{ with } J \text{ compact and } \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(J) > c\} \\ &= \pi(\{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : J \subseteq K\} \\ &\quad \cap \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : \overline{\mathcal{H}}_{\mu, \theta}^{q,s}(J) > c\}) \\ &= \pi(\Pi \cap \Sigma), \end{aligned}$$

It is apparent that Π is closed. Moreover, by Theorem 3.1-(2), Σ is a Borel, so $\Pi \cap \Sigma$ is Borel. Whence, Γ is an analytic.

4. It is sufficient to demonstrate

$$\Gamma_2 = \{(K, \mu, \theta, q) \in \mathbb{A} : \dim_{\mu, \theta}^q(K) > t\}$$

is an analytic subset of $\mathbb{A}$, for $t \in \mathbb{R}$. For $t \in \mathbb{R}$, Π and π as defined in assumption (3), we have

$$\begin{aligned} \Gamma_2 &= \{(K, \mu, \theta, q) \in \mathbb{A} : \mathcal{H}_{\mu, \theta}^{q, s}(K) > 0, \forall s < t\} \\ &= \{(K, \mu, \theta, q) \in \mathbb{A} : \exists J \subseteq K, \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(J) > 0 \forall s < t\} \\ &= \pi(\{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : J \subseteq K\} \\ &\quad \cap \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(J) > 0, \forall s < t\}) \\ &= \pi(\Pi \cap \Sigma_0), \end{aligned}$$

where

$$\Sigma_0 = \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(J) > 0, \forall s < t\}.$$

Since Σ_0 is a Borel, then, Γ_2 is analytic.

5. Considering that $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{H}}_{\mu, \theta}^{q, s}(K)$ is Baire-(2) class, we deduce from Corollary 3.1 that if $s_1, s_2 \in \mathbb{R}$, one has

$$\begin{aligned} &\{(K, \mu, \theta, q) \in \Omega_2 : s_1 < \dim_{\mu, \theta}^q(K) < s_2\} \\ &= \Omega_2 \cap \left(\bigcup_{n \in \mathbb{N} \setminus \{0\}} \left\{ (K, \mu, \theta, q) \in \mathbb{A} : 1 < \overline{\mathcal{H}}_{\mu, \theta}^{q, s_1 + \frac{1}{n}}(K) \right\} \right. \\ &\quad \left. \cap \bigcup_{n \in \mathbb{N} \setminus \{0\}} (K, \mu, \theta, q) \in \mathbb{A} : \overline{\mathcal{H}}_{\mu, \theta}^{q, s_2 - \frac{1}{n}}(K) < 1 \right) \\ &\in \mathcal{G}_{\delta\sigma}(\Omega_2). \end{aligned}$$

Then, the function

$$\Omega_2 \rightarrow \overline{\mathbb{R}} : (K, \mu, \theta, q) \mapsto \dim_{\mu, \theta}^q(K)$$

is Baire-(2) class. Now, we demonstrate that it is not Baire-(1) class. Let $q = 0$, μ and θ are two Borel probability measures such that $\mu([0, 1]^d) > 0$ and $\theta = \mathcal{L}([0, 1]^d)$ is Lebesgue's measure on $[0, 1]^d$. Put

$$\mathbb{I}_1 = \{A : A \subset \mathbb{R}^d \text{ is finite}\}$$

and

$\mathbb{I}_2 = \{J \cup A : J \subset \mathbb{R}^d \text{ is compact with line segment form and has a positive length and } A \text{ is finite set in } \mathbb{R}^d\}.$

$\mathbb{I}_1$ and $\mathbb{I}_2$ are dense in $\mathcal{K}(\mathbb{R}^d)$, [21, Theorem 4.22]. We have

$$\dim_{\theta}(K) = \begin{cases} 0 & \text{if } K \in \mathbb{I}_1 \\ 1 & \text{if } K \in \mathbb{I}_2. \end{cases}$$

Therefore,

$$\Omega_2 \rightarrow \overline{\mathbb{R}} : (K, \mu, \theta, q) \mapsto \dim_{\mu, \theta}^q(K)$$

is everywhere discontinuous, from [21, Theorem 24.15], we conclude that it is not Baire-(1) class.

4. Measurability of relative multifractal packing measure and dimension maps

In this part, we will discuss the concept of measurability of (1.3) and (1.4).

Theorem 4.1: Let $\delta > 0$ and $s \in \mathbb{R}$.

1. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(K)$ is lower semi-continuous. Exceptionally, it is Baire-(1) class.
2. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(K)$ is Baire-(2) class, especially, it is not Baire-(1) class.
3. The function $\mathbb{A} \rightarrow \overline{\mathbb{R}} : (K, \mu, \theta, q) \mapsto \Delta_{\mu, \theta}^q(K)$ is Borel measurable. Extraordinarily, it is Baire-(2) class.
4. The function $\Omega_1 \rightarrow \overline{\mathbb{R}} : (K, \mu, \theta, q) \mapsto \text{Dim}_{\mu, \theta}^q(K)$ is $\mathfrak{B}(\mathfrak{A}(\Omega_1))$ - measurable.

To establish this theorem, we need some useful and intermediate results.

Proposition 4.1: Let $\delta > 0$ and $s, c \in \mathbb{R}$, one has

$$G = \{(K, \mu, \theta, q) \in \mathbb{A} : \overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(K) > c\}$$

is open.

Proof: Given

$$\begin{aligned} H = \{ & (K, \mu, \theta, q) \in \mathbb{A} : \text{There exist a non - zero positive integer } m, \\ & x_1, \dots, x_m \in K, r_1, \dots, r_m \in (0, \delta) \\ & \text{and } c_1, \dots, c_m \in \mathbb{R}_+ \text{ with } \sum_{i=1}^m c_i > c \\ & \text{such that} \\ & 1^\circ) B^{x_i}(r_i) \cap B^{x_j}(r_j) = \emptyset, \forall i \neq j \in \{1, \dots, m\} \\ & 2^\circ) \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s > c_i, \forall i \in \{1, \dots, m\} \} \end{aligned}$$

The proof splits into two steps.

1st step: We will demonstrate that $G = H$. Given (K, μ, θ, q) that belongs to H , we obtain

$$c < \sum_{i=1}^m c_i < \sum_{i=1}^m \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s \leq \overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(K),$$

then $H \subset G$. Inversely, let $(K, \mu, \theta, q) \in G$, there exist $\{B^{x_i}(r_i)\}_i$ a δ -packing of A in such a way

$$c < \sum_i \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s \leq \overline{\mathcal{P}}_{\mu, \theta, \delta}^{q, s}(K).$$

Furthermore, there exists a δ -packing $\{B^{x_i}(r_i)\}_{i=1}^m$ in order that

$$c < \sum_{i=1}^m \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s,$$

therefore, we can find $\varepsilon > 0$ such that

$$\begin{aligned} c &< \sum_{i=1}^m \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s - \varepsilon \\ &= \sum_{i=1}^m \left((\mu(B^{x_i}(r_i)))^q \theta \left(B^{x_i}(r_i) \right)^s - \frac{\varepsilon}{m} \right). \end{aligned}$$

We can take

$$c_i = \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s - \frac{\varepsilon}{m},$$

which implies that $G \subset H$.

2^{sd} step: We are going to demonstrate that H is open. Let the complement of H be denoted as F , we will show that it is closed of $\mathbb{A}$. Pick a sequence $(K_n, \mu_n, \theta_n, q_n)_n$ in F that converges to $(K, \mu, \theta, q) \in \mathbb{A}$ and show that it in F .

Set $m \in \mathbb{N} \setminus \{0\}$, $x_1, \dots, x_m \in K$, $r_1, \dots, r_m \in (0, \delta)$, $c_1, \dots, c_m \in \mathbb{R}_+$ with $\sum_{i=1}^m c_i > c$.

We need to show that

$$B^{x_i}(r_i) \cap B^{x_j}(r_j) \neq \emptyset, \text{ for some } i \neq j \in \{1, \dots, m\}, \quad (4.1)$$

or

$$\mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s \leq c_i, \text{ for some } i \in \{1, \dots, m\}. \quad (4.2)$$

If (4.1) is checked, then we are done. We assume that is not satisfied, ie,

$$B^{x_i}(r_i) \cap B^{x_j}(r_j) = \emptyset, \forall i \neq j \in \{1, \dots, m\}. \quad (4.3)$$

Since K is a compact and (4.3) imply that for all $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 < \inf r_i$, with $1 \leq i \leq m$, whence

$$\{B^{x_i}(r_i - \varepsilon)\}_{i=1}^m \text{ is a } \delta - \text{packing of } K. \quad (4.4)$$

Subsequently, we demonstrate that, for any positive ε less than ε_0 , there is a corresponding index $i(\varepsilon)$ in $\{1, \dots, m\}$ in such a way

$$c_{i(\varepsilon)} \begin{cases} \mu(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^q \theta(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^s & \text{if } q \geq 0 \text{ and } s \geq 0 \\ \mu(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^q \theta(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^s & \text{if } q < 0 \text{ and } s \geq 0 \\ \mu(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^q \theta(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^s & \text{if } q \geq 0 \text{ and } s < 0 \\ \mu(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)}))^q \theta(B^{x_{i(\varepsilon)}}(r_{i(\varepsilon)} - \varepsilon))^s & \text{if } q < 0 \text{ and } s < 0. \end{cases}$$

fix $\varepsilon \in (0, \varepsilon_0)$, since $\rho_H(K_n, K) \searrow 0$ and from (4.4), there is an integer N , in such a way

$$K_n \cap B^{x_i} \left(\frac{1}{5} \varepsilon \right) \neq \emptyset, \text{ for any } n \geq N \text{ and } 1 \leq i \leq m.$$

Now, take $n \geq N$ and choose $y_n^i \in K_n \cap B^{x_i} \left(\frac{1}{5} \varepsilon \right)$ for $1 \leq i \leq m$. Obviously,

$$B^{x_i}(r_i - \varepsilon) \subset B^{y_n^i} \left(r_i - \frac{3}{4} \varepsilon \right) \subset B^{x_i} \left(r_i - \frac{1}{2} \varepsilon \right). \quad (4.5)$$

Moreover,

$$\left\{ B^{y_n^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right\}_{i=1}^m \text{ is a } \delta - \text{packing of } K_n. \quad (4.6)$$

Since $(K_n, \mu_n, \theta_n, q_n) \in F$ and by (4.6), we can conclude that

$$\mu_n \left(B^{y_n^{i(n)}} \left(r_{i(n)} - \frac{3}{4} \varepsilon \right) \right)^{q_n} \theta_n \left(B^{y_n^{i(n)}} \left(r_{i(n)} - \frac{3}{4} \varepsilon \right) \right)^s \leq c_{i(n)}, \text{ for certain } 1 \leq i(n) \leq m. \quad (4.7)$$

Then, pick $i(\varepsilon) = i \in \{1, \dots, m\}$, so there is a strictly monotonic sequence $(n_j)_j \subseteq \mathbb{N}$ with $i(n_j) = i$ for all j . Additionally, one has

$$\rho_p(\mu_n, \mu) \searrow 0 \text{ and } \rho_p(\theta_n, \theta) \searrow 0.$$

Using (4.5), we obtain

$$\begin{aligned} \mu(B^{x_i}(r_i)) &\geq \mu \left(B^{x_i} \left(r_i - \frac{1}{2} \varepsilon \right) \right) \\ &\geq \limsup_j \mu_{n_j} \left(B^{x_i} \left(r_i - \frac{1}{2} \varepsilon \right) \right) \\ &\geq \limsup_j \mu_{n_j} \left(B^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &= \limsup_j \mu_{n_j} \left(B^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \mu(B^{x_i}(r_i - \varepsilon)) &\leq \mu \left(U^{y_n^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &\leq \liminf_j \mu_{n_j} \left(U^{y_n^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &\leq \liminf_j \mu_{n_j} \left(B^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right) \\ &= \liminf_j \mu_{n_j} \left(B^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \end{aligned} \quad (4.9)$$

In the same way,

$$\theta(B^{x_i}(r_i)) \geq \limsup_j \theta_{n_j} \left(B^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \quad (4.10)$$

and

$$\theta(B^{x_i}(r_i - \varepsilon)) \leq \liminf_j \theta_{n_j} \left(B^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right). \quad (4.11)$$

Let

$$\alpha_j = \mu_{n_j} \left(B^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right) \text{ and } \beta_j = \theta_{n_j} \left(B^{y_{n_j}^{i(n_j)}} \left(r_{i(n_j)} - \frac{3}{4} \varepsilon \right) \right).$$

We have, if $q_{n_j} \rightarrow q$, then, $\alpha_j^{q-q_{n_j}} \rightarrow 1$. Since (4.8), (4.9), (4.10) and (4.11) imply that $(\alpha_j)_j$ and $(\beta_j)_j$ are a bounded sequences. We conclude that

- If $q \geq 0$ and $s \geq 0$, from (4.7), (4.9) and (4.11), we have

$$\begin{aligned} \mu(B^{x_i}(r_i - \varepsilon))^q \theta(B^{x_i}(r_i - \varepsilon))^s &\leq \liminf_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(B^{x_i}(r_i - \varepsilon))^s \\ &\leq \liminf_j \alpha_j^{q_{n_j}} \theta_{n_j} \left(B^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right)^s \\ &= \liminf_j \alpha_j^{q_{n_j}} \beta_j^s \\ &\leq \liminf_j c_{i(n_j)} = c_i. \end{aligned}$$

- If $q < 0$ and $s \geq 0$, from (4.7), (4.8) and (4.11), we have

$$\begin{aligned} \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i - \varepsilon))^s &\leq \liminf_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(B^{x_i}(r_i - \varepsilon))^s \\ &\leq \liminf_j \alpha_j^{q_{n_j}} \theta_{n_j} \left(B^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right)^s \\ &= \liminf_j \alpha_j^{q_{n_j}} \beta_j^s \\ &\leq \liminf_j c_{i(n_j)} = c_i. \end{aligned}$$

- If $q \geq 0$ and $s < 0$, from (4.7), (4.9) and (4.10), we have

$$\begin{aligned} \mu(B^{x_i}(r_i - \varepsilon))^q \theta(B^{x_i}(r_i))^s &\leq \liminf_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(B^{x_i}(r_i))^s \\ &\leq \liminf_j \alpha_j^{q_{n_j}} \theta_{n_j} \left(B^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right)^s \\ &= \liminf_j \alpha_j^{q_{n_j}} \beta_j^s \\ &\leq \liminf_j c_{i(n_j)} = c_i. \end{aligned}$$

- If $q < 0$ and $s < 0$, from (4.7), (4.8) and (4.10), we have

$$\begin{aligned} \mu(B^{x_i}(r_i))^q \mu(B^{x_i}(r_i))^s &\leq \liminf_j \alpha_j^{q_{n_j}} \alpha_j^{q-q_{n_j}} \theta(B^{x_i}(r_i))^s \\ &\leq \liminf_j \alpha_j^{q_{n_j}} \theta_{n_j} \left(B^{y_{n_j}^i} \left(r_i - \frac{3}{4} \varepsilon \right) \right)^s \\ &= \liminf_j \alpha_j^{q_{n_j}} \beta_j^s \\ &\leq \liminf_j c_{i(n_j)} = c_i. \end{aligned}$$

Which implies that there exists a positive sequence $(\varepsilon_n)_n \subset \mathbb{R}_+$ and a corresponding index i with $1 \leq i \leq m$ in such a way that $\varepsilon_n \searrow 0$ with $i(\varepsilon_n) = i$ for each n . Making $n \nearrow \mathbb{Z} + \infty$, so (4.2) is obtained.

Proposition 4.2: Let $A \subset \mathbb{R}^d$ and $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$, we have

- (1) If $\text{Dim}_{\mu, \theta}^q(A) = \inf_{A \subseteq \cup_i A_i} \sup_i \Delta_{\mu, \theta}^q(A_i)$.
- (2) If $q \geq 0$ and $\mu, \theta \in \mathcal{P}_0^1(\mathbb{R}^d)$, one will find $c > 0$ for which
$$\overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A) \leq \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(\overline{A}) \leq c \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A).$$
- (3) If $q < 0$ and $\theta \in \mathcal{P}_0^1(\mathbb{R}^d)$, one will find $c > 0$ for which
$$\overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A) \leq \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(\overline{A}) \leq c \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A).$$

Proof:

- (1) Let $(A_i)_i$ be a covering of A , since $\text{Dim}_{\mu, \theta}^q(A)$ is monotone and countably stable and Proposition 2.1, then

$$\text{Dim}_{\mu, \theta}^q(A) \leq \sup_i \text{Dim}_{\mu, \theta}^q(A_i) \leq \sup_i \Delta_{\mu, \theta}^q(A_i).$$

Thus,

$$\text{Dim}_{\mu, \theta}^q(A) \leq \inf_{A \subseteq \cup_i A_i} \sup_i \Delta_{\mu, \theta}^q(A_i).$$

Inversely, let $\varepsilon > 0$ and $s > \text{Dim}_{\mu, \theta}^q(A)$, then, $\mathcal{P}_{\mu, \theta}^{q, s}(A) = 0$, so there is a collection of subsets $(A_i)_i$ in such a way that

$$A \subseteq \cup_i A_i \text{ and } \sum_i \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A_i) \leq \varepsilon.$$

Then, $\inf_{A \subseteq \cup_i A_i} \sup_i \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A_i) < \infty$ and $\inf_{A \subseteq \cup_i A_i} \sup_i \Delta_{\mu, \theta}^q(A_i) \leq s$ for all $s \geq \text{Dim}_{\mu, \theta}^q(A)$.

Consequently, $\text{Dim}_{\mu, \theta}^q(A) = \inf_{A \subseteq \cup_i A_i} \sup_i \Delta_{\mu, \theta}^q(A_i)$ is required.

- (2) From the definition of $\overline{\mathcal{P}}_{\mu, \theta}^{q, s}$, we have $\overline{\mathcal{P}}_{\mu, \theta}^{q, s}(A) \leq \overline{\mathcal{P}}_{\mu, \theta}^{q, s}(\overline{A})$. We are going to show the other inequality, for $\mu, \theta \in \mathcal{P}_0^1(\mathbb{R}^d)$, there exist $r_1, r_2 > 0$ and $c_1, c_2 > 0$ such that

$$\mu(B^x(r)) \leq c_1 \mu\left(B^y\left(\frac{r}{2}\right)\right), \text{ where } 0 < r < r_1 \text{ and } x, y \in \text{Spt } \mu,$$

and

$$\theta(B^x(r)) \leq c_2 \theta\left(B^y\left(\frac{r}{2}\right)\right), \text{ where } 0 < r < r_2 \text{ and } x, y \in \text{Spt } \theta.$$

Given $\delta > 0$ and $A \subset \mathbb{R}^d$. Choose a δ -packing $\{B^{x_i}(r_i)\}_i$ of $\overline{A}$ and take $y_i \in B^{x_i}(\frac{r_i}{2}) \cap A$ for every i . Given that $B^{y_i}(\frac{r_i}{2}) \subset B^{x_i}(r_i)$, therefore $\{B^{y_i}(\frac{r_i}{2})\}_i$ is a $\frac{\delta}{2}$ -packing of A .

(3) For $\mu, \theta \in \mathcal{P}_0^1(\mathbb{R}^d)$ and $s \geq 0$ we have

$$\begin{aligned} \sum_i \mu(B^{x_i}(r_i))^q \theta(B^{x_i}(r_i))^s &\leq c_1^q c_2^s \sum_i \mu\left(B^{y_i}\left(\frac{r_i}{2}\right)\right)^q \theta\left(B^{y_i}\left(\frac{r_i}{2}\right)\right)^s \\ &\leq c_1^q c_2^s \overline{\mathcal{P}}_{\mu, \theta, \frac{\delta}{2}}^{q,s}(A). \end{aligned}$$

Hence, $\overline{\mathcal{P}}_{\mu, \theta, \delta}^{q,s}(\overline{A}) \leq c_1^q c_2^s \overline{\mathcal{P}}_{\mu, \theta, \frac{\delta}{2}}^{q,s}(A)$. Letting $\delta \searrow 0$, we conclude that

$$\overline{\mathcal{P}}_{\mu, \theta}^{q,s}(\overline{A}) \leq c_1^q c_2^s \overline{\mathcal{P}}_{\mu, \theta}^{q,s}(A),$$

with $c = c_1^q c_2^s$.

(a) For $\mu \in \mathcal{P}_0^1(\mathbb{R}^d)$ and $s < 0$, the proof is alike with $c = c_1^q$

1. For $q < 0$, we have

(a) For $\theta \in \mathcal{P}_0^1(\mathbb{R}^d)$ and $s \geq 0$, the proof is alike with $c = c_1^s$.

(b) For $s < 0$, the proof is alike with $c = 1$.

Corollary 4.1: Consider $A \subset \mathbb{R}^d$ and $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$.

1. If $q \geq 0$ and $\mu, \theta \in \mathcal{P}_0^1(\mathbb{R}^d)$ thus $\Delta_{\mu, \theta}^q(A) = \Delta_{\mu, \theta}^q(\overline{A})$.

2. If $q < 0$ and $\theta \in \mathcal{P}_0^1(\mathbb{R}^d)$ thus $\Delta_{\mu, \theta}^q(A) = \Delta_{\mu, \theta}^q(\overline{A})$.

As a consequence of Proposition 4.2-(1) and Corollary 4.1, we have

Corollary 4.2: If $(K, \mu, \theta, q) \in \Omega_1$, thus

$$\text{Dim}_{\mu, \theta}^q(K) = \inf_{\substack{K \subseteq \cup_i K_i \\ K_i \text{ is compact}}} \sup_i \Delta_{\mu, \theta}^q(K_i).$$

Lemma 4.1: Let $K \in \mathcal{K}(\mathbb{R}^d)$, $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$ and $q, \alpha \in \mathbb{R}$ in such a way that $\mathcal{P}_{\mu, \theta}^{q, \alpha}(K) > 0$. Then, there will be $J \subseteq K$ satisfying

1) J is compact and non-empty,

2) For $G \subseteq \mathbb{R}^d$ is an open set with $J \cap G \neq \emptyset$ and $\text{Dim}_{\mu, \theta}^q(J \cap G) \geq \alpha$, then $\Delta_{\mu, \theta}^q(J \cap \overline{G}) \geq \alpha$.

Proof: We assume that η is the restriction of $\mathcal{P}_{\mu, \theta}^{q, \alpha}$ to K and we write

$J = \text{Spt } \eta$. Obviously, $J \subseteq K$ and M is compact. On the other side, we have

$$\eta(J) = \eta(\mathbb{R}^d) = \mathcal{P}_{\mu, \theta}^{q, \alpha}(K) > 0,$$

which implies that $J \neq \emptyset$. If $G \subset \mathbb{R}^d$ is open such that $J \cap G \neq \emptyset$, then $\eta(G) > 0$, therefore,

$$\eta(G) = \eta(G \cap J) = \mathcal{P}_{\mu, \theta}^{q, \alpha}(G \cap J) > 0.$$

Thus $\text{Dim}_{\mu, \theta}^q(J \cap G) \geq \alpha$. The set $J = \text{Spt } \eta$ is the greatest closed set of K fulfills 1) and 2).

Lemma 4.2: For $\alpha \in \mathbb{R}$, whence

$$\{(K, \mu, \theta, q) \in \Omega_1 : \text{Dim}_{\mu, \theta}^q(K) \geq \alpha\} = \{(K, \mu, \theta, q) \in \Omega_1 : \text{for every } s < \alpha \\ \text{there is } J \subseteq K \text{ in such way that} \\ 1^\circ) J \text{ is compact and non - empty} \\ 2^\circ) \text{ For } G \subseteq \mathbb{R}^d \text{ is an open such that} \\ J \cap G \neq \emptyset, \text{ thus } \Delta_{\mu, \theta}^q(J \cap \overline{G}) \geq s\}.$$

Proof: " $\subseteq$ " From the above lemma, for all $(K, \mu, \theta, q) \in \Omega_1$, such that, $\text{Dim}_{\mu, \theta}^q(K) \geq \alpha$. Then there is a compact $J \subseteq K$ that is non-empty. If $G \subseteq \mathbb{R}^d$ is open with $J \cap G \neq \emptyset$, therefore,

$$\alpha \leq \text{Dim}_{\mu, \theta}^q(J \cap G) \Delta_{\mu, \theta}^q \leq (J \cap G).$$

" $\supseteq$ " Given $(K, \mu, \theta, q) \in \Omega_1$ and suppose that if $s < \alpha$, then there would be a compact and non-empty $J \subseteq K$ such that $\Delta_{\mu, \theta}^q(J \cap \overline{G}) \geq s$, for each open subset $G \subset \mathbb{R}^d$ such that $J \cap G \neq \emptyset$. Let us prove that $\text{Dim}_{\mu, \theta}^q(K) \geq \alpha$. Let us assume, to reach a contradiction, that $\text{Dim}_{\mu, \theta}^q(K) < \alpha$. Choose s such that $\text{Dim}_{\mu, \theta}^q(K) < s < \alpha$. Given that $s < \alpha$, there is $J \subseteq K$ in such a way that

$$\left\{ \begin{array}{l} J \text{ is compact and non - empty} \\ \text{and} \\ \Delta_{\mu, \theta}^q(J \cap G) \geq s, \text{ for every open } G \text{ of } \mathbb{R}^d \text{ in such way that } J \cap G \neq \emptyset. \end{array} \right.$$

In addition, since $\text{Dim}_{\mu, \theta}^q(K) < s$ and applying Corollary 4.2, there is a family of compacts $(K_i)_i$ such as $K \subset \bigcup_i K_i$ and

$$\Delta_{\mu, \theta}^q(K_i) < s, \text{ for all } i. \quad (4.12)$$

Considering $J = \bigcup_i (J \cap K_i)$ and by the Baire's category theorem it involves the existence of an open G and $j \in \mathbb{N}$ such that

$$\left\{ \begin{array}{l} J \cap G \neq \emptyset \\ \text{and} \\ J \cap G \subseteq J \cap K_j. \end{array} \right.$$

So, we can choose an open G_1 such that

$$\left\{ \begin{array}{l} J \cap G_1 \neq \emptyset \\ \text{and} \\ \overline{G_1} \subseteq G. \end{array} \right.$$

From (4.12), we acquire

$$s \leq \Delta_{\mu, \theta}^q(J \cap \overline{G_1}) \leq \Delta_{\mu, \theta}^q(J \cap G) \leq \Delta_{\mu, \theta}^q(J \cap K_j) \leq \Delta_{\mu, \theta}^q(K_j) < s,$$

and we get a contradiction.

Proof of theorem 4.1.

1. This is a direct result from Proposition 4.1.

2. Since $\overline{\mathcal{P}}_{\mu,\theta}^{q,s}(K) = \lim_{n \rightarrow +\infty} \overline{\mathcal{P}}_{\mu,\theta,2^{-n}}^{q,s}(K)$, for all $(K, \mu, \theta, q) \in \mathbb{A}$, assumption 1 implies

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{P}}_{\mu,\theta}^{q,s}(K)$$

is Baire-(2) class.

In general, $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{P}}_{\mu,\theta}^{q,s}$ is not Baire-(1) class. Indeed, let $q = 0$, μ , θ , $\coprod_1$ and $\coprod_2$ be as the proof of Theorem 3.1-(5). We obtain

$$\overline{\mathcal{P}}_{\mu,\theta}^{q,\frac{1}{2}}(K) = \begin{cases} +\infty & \text{if } K \in \coprod_2 \\ 0 & \text{if } K \in \coprod_1. \end{cases}$$

Then, we conclude that

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \overline{\mathcal{P}}_{\mu,\theta}^{q,\frac{1}{2}}(K)$$

is continuously interrupted everywhere and therefore it is not Baire-(1) class.

3. Let $s_1, s_2 \in \mathbb{R}$, we get

$$\begin{aligned} & \{(K, \mu, \theta, q) \in \mathbb{A} : s_1 < \Delta_{\mu,\theta}^q(K) < s_2\} \\ &= (\cup_{n \in \mathbb{N} \setminus \{0\}} \{(K, \mu, \theta, q) \in \mathbb{A} : 1 < \overline{\mathcal{P}}_{\mu,\theta}^{q,s_1+\frac{1}{n}}(K)\}) \\ & \cap \cup_{n \in \mathbb{N} \setminus \{0\}} \{(K, \mu, \theta, q) \in \mathbb{A} : \overline{\mathcal{P}}_{\mu,\theta}^{q,s_2-\frac{1}{n}}(K) < 1\}) \\ & \in \mathcal{G}_{\delta\sigma}(\mathbb{A}). \end{aligned}$$

Then,

$$\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \Delta_{\mu,\theta}^q(K)$$

is Baire-(2) class. The demonstration that it is not Baire-(1) class resembles to the proof of Theorem 3.1-(5).

4. Pick $c \in \mathbb{R}$. Consider $(x_i)_i \subseteq \mathbb{R}^d$ as a countable dense set and $(r_j)_j$ as a positive sequence of rational numbers. For $i, j, \gamma \in \mathbb{N}$, we denote

$$\begin{aligned} \Gamma &= \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : J \subseteq K\}, \\ \Gamma_i^j &= \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : J \cap U^{x_i}(r_j) \neq \emptyset\}, \\ \Theta_i^j(\gamma) &= \{(K, \mu, \theta, q, J) \in \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) : \Delta_{\mu,\theta}^q(J \cap B^{x_i}(r_j))c - \frac{1}{\gamma}\} \end{aligned}$$

and the projection

$$\begin{aligned} \pi : \mathbb{A} \times \mathcal{K}(\mathbb{R}^d) &\rightarrow \mathbb{A} \\ (K, \mu, \theta, q, J) &\mapsto (K, \mu, \theta, q). \end{aligned}$$

By using Lemma 4.2, we get

$$\begin{aligned} & \{(K, \mu, \theta, q) \in \Omega_1 : \text{Dim}_{\mu,\theta}^q(K) \geq c\} \\ &= \cap_{\gamma} \{(K, \mu, \theta, q) \in \Omega_1 : \exists J \subseteq K \text{ in such away} \\ & \quad 1^\circ) J \text{ is compact and non - empty} \end{aligned}$$

$$\begin{aligned}
& 2^\circ) \text{ If } G \text{ is an open subset of } \mathbb{R}^d \text{ and } G \cap J \neq \emptyset, \\
& \text{ then, } \Delta_{\mu,\theta}^q(J \cap \overline{G}) \geq c - \frac{1}{\gamma} \} \\
& = \bigcap_{\gamma} \pi(\{(K, \mu, \theta, q, J) \in \Omega_1 \times \mathcal{K}(\mathbb{R}^d) : J \subseteq K\} \\
& \quad \cap \{(K, \mu, \theta, q, J) \in \Omega_1 \times \mathcal{K}(\mathbb{R}^d) : \text{if } i, j \in \mathbb{N} \\
& \quad \text{with } J \cap U^{x_i}(r_j) \neq \emptyset \text{ then} \\
& \quad \Delta_{\mu,\theta}^q(J \cap B^{x_i}(r_j)) \geq c - \frac{1}{\gamma}\}) \\
& = \Omega_1 \cap \bigcap_{\gamma} \pi(\Gamma \cap \bigcap_i \bigcap_j (\Gamma_i^j \cup \Theta_i^j(\gamma))) \\
& = \Omega_1 \cap \Sigma,
\end{aligned}$$

where

$$\Sigma = \bigcap_{\gamma} \pi(\Gamma \cap \bigcap_i \bigcap_j (\Gamma_i^j \cup \Theta_i^j(\gamma))).$$

We can see that, Γ and Γ_i^j are closed sets. Since $\Delta_{\mu,\theta}^q$ is Borel measurable and from [33, Proposition 4.6], we get $\Theta_i^j(\gamma)$ is Borel set, so Σ is an analytic subset of $\mathbb{A}$, then, $\Omega_1 \cap \Sigma$ is an analytic subset of Ω_1 .

Question 1: Does the $\mathbb{A} \rightarrow \overline{\mathbb{R}}_+ : (K, \mu, \theta, q) \mapsto \mathcal{P}_{\mu,\theta}^{q,s}(K)$ is $\mathfrak{B}(\mathfrak{A}(\mathbb{A}))$ -measurable.

Question 2. Does the $\mathbb{A} \rightarrow \mathbb{R} : (K, \mu, \theta, q) \mapsto \text{Dim}_{\mu,\theta}^q(K)$ is $\mathfrak{B}(\mathfrak{A}(\mathbb{A}))$ -measurable.

5. An example: Measurability of Multifractal slices

The results presented in this section hold considerable importance in the exploration of multifractal slices and as demonstrated in [33, 34].

Given $d, k \in \mathbb{N}$ with $d \geq k$ and $G(d, k)$ the Grassmannian manifold consisting of vector sub-spaces of $\mathbb{R}^d$ with dimension k . For $W \in G(d, k)$, we write $W^\perp$ the orthogonal complement of W and π_W the orthogonal projector onto W . Moreover, for all $W \in G(d, k)$, $a \in \mathbb{R}^d$ and $\delta > 0$, put

$$B(W_a, \delta) = \{x \in \mathbb{R}^d : \rho(W_a, x) \leq \delta\},$$

where

$$W_a = \{w + a : w \in W\}.$$

Consider a measure space $(X, \mathcal{C}, \gamma)$, the restriction of γ to $A \in \mathcal{C}$ is denoted by

$$\gamma \llcorner A(B) = \gamma(A \cap B), \text{ for all } B \in \mathcal{C}.$$

Let $\mathcal{P}_R(\mathbb{R}^d)$ the collection of Radon probability measures on $\mathbb{R}^d$. For a fixed $\gamma \in \mathcal{P}_R(\mathbb{R}^d)$ and $W \in G(d, k)$, the slice of γ by the sub-space W_a is indicated as follows

$$\gamma_{W_a} = \lim_{\delta \searrow 0} \frac{1}{(2\delta)^{d-k}} \gamma \llcorner B(W_a, \delta).$$

Notice that, this limit with respect the weak topology exists for $\mathcal{H}^{d-k}$ almost all $a \in W^\perp$ [30, Chapter 10]. Otherwise, for $a \in W^\perp$, we simply put

$\gamma_{W_x} = \gamma_{W_a}$, for any $x \in \pi_{W^\perp}^{-1}(\{a\})$. Also, we put $Z = \mathbb{R}^d \times G(d, k)$ and $\mathcal{Y} = \mathcal{K}(\mathbb{R}^d) \times \mathcal{P}_R(\mathbb{R}^d) \times \mathcal{P}_R(\mathbb{R}^d) \times \mathbb{R}$.

Theorem 5.1: Let $\mu, \theta \in \mathcal{P}_R(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $F \subset \mathbb{R}^d$ be a closed set.

1. The function

$$\begin{aligned} \varphi: Z \times \mathbb{R} &\rightarrow \overline{\mathbb{R}}_+ \\ (a, W, q) &\mapsto \begin{cases} \mathcal{H}_{\mu_{W_a}, \theta_{W_a}}^{q, s}(F \cap W_a) & \text{if } \mu_{W_a} \text{ and } \theta_{W_a} \text{ exist} \\ 0 & \text{if } \mu_{W_a} \text{ or } \theta_{W_a} \text{ does not exist.} \end{cases} \end{aligned}$$

is $\mathfrak{B}(\mathfrak{A}(Z \times \mathbb{R}))$ -measurable.

2. The function

$$\begin{aligned} \psi: Z \times \mathbb{R} &\rightarrow \overline{\mathbb{R}} \\ (a, W, q) &\mapsto \begin{cases} \dim_{\mu_{W_a}, \theta_{W_a}}^q(F \cap W_a) & \text{if } \mu_{W_a} \text{ and } \theta_{W_a} \text{ exist} \\ 0 & \text{if } \mu_{W_a} \text{ or } \theta_{W_a} \text{ doesn't exist.} \end{cases} \end{aligned}$$

is $\mathfrak{B}(\mathfrak{A}(Z \times \mathbb{R}))$ -measurable.

Proof: Let

$$\Lambda_1 = \{(a, W) \in Z: \mu_{W_a} \text{ exists}\}$$

and

$$\Lambda_2 = \{(a, W) \in Z: \theta_{W_a} \text{ exists}\}.$$

Λ_1 and Λ_2 are Borel subsets of Z [29]. Given $i \in \mathbb{N}$ and $\delta > 0$, let

$$f: \mathcal{Y} \rightarrow \overline{\mathbb{R}}_+: (K, \mu, \theta, q) \mapsto \mathcal{H}_{\mu, \theta}^{q, s}(K),$$

$$g: \mathcal{Y} \rightarrow \overline{\mathbb{R}}: (K, \mu, \theta, q) \mapsto \dim_{\mu, \theta}^q(K),$$

$$k_i: Z \rightarrow \mathcal{K}(\mathbb{R}^d): (a, W) \mapsto W_a \cap F \cap B^0(i),$$

$$\lambda_{1, \delta}: Z \rightarrow \mathcal{P}_R(\mathbb{R}^d): (a, W) \mapsto \frac{1}{(2\delta)^{d-k}} \mu \llcorner B(W_a, \delta),$$

$$\lambda_{2, \delta}: Z \rightarrow \mathcal{P}_R(\mathbb{R}^d): (a, W) \mapsto \frac{1}{(2\delta)^{d-k}} \theta \llcorner B(W_a, \delta),$$

$$\lambda_1: Z \rightarrow \mathcal{P}_R(\mathbb{R}^d)$$

$$(a, W) \mapsto \begin{cases} \mu_{W_a}(F \cap W_a) = \lim_{\delta \searrow 0} \lambda_{1, \delta}(a, W) & \text{if } (a, W) \in \Lambda_1 \\ \mathcal{H}^d & \text{if } (a, W) \notin \Lambda_1. \end{cases}$$

and

$$\lambda_2: Z \rightarrow \mathcal{P}_R(\mathbb{R}^d)$$

$$(a, W) \mapsto \begin{cases} \theta_{W_a}(F \cap W_a) = \lim_{\delta \searrow 0} \lambda_{2, \delta}(a, W) & \text{if } (a, W) \in \Lambda_2 \\ \mathcal{H}^d & \text{if } (a, W) \notin \Lambda_2. \end{cases}$$

The map k_i is Borel function [23], $\lambda_{1, \delta}$ and $\lambda_{2, \delta}$ are Borel measurable functions [29], so λ_1 and λ_2 are also. Then, from [21, 37.3], we can see that the function

$$\begin{aligned}\Gamma_i: \mathcal{Z} \times \mathbb{R} &\rightarrow \mathcal{Y} \\ (a, W, q) &\mapsto (k_i(a, W), \lambda_1(a, W), \lambda_2(a, W), q)\end{aligned}$$

is $\mathfrak{B}(\mathfrak{A}(\mathcal{Z} \times \mathbb{R}))$ - $\mathfrak{B}(\mathfrak{A}(\mathcal{Y}))$ -measurable. As an application of Theorem 3.1, we have that

$$\varphi = \mathbb{I}_{\Lambda_1 \cap \Lambda_2 \times \mathbb{R}} \sup_i f \circ \Gamma_i \text{ and } \psi = \mathbb{I}_{\Lambda_1 \cap \Lambda_2 \times \mathbb{R}} \sup_i g \circ \Gamma_i$$

are $\mathfrak{B}(\mathfrak{A}(\mathcal{Z} \times \mathbb{R}))$ -measurable.

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