Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
DOI : 10.47974/JIM-2161

Investigating topological spaces arising from commutative rings via associated zero divisor graphs : A holistic approach

Nasir Ali *
Department of Mathematics
COMSATS University Islamabad
Vehari Campus
Vehari 61100
Punjab
Pakistan

Maysoon Qousini †
Faculty of Science and Information Technology
Al-zaytoonah University of Jordan
Amman 11183
Jordan

Muhammad Imran Qureshi §
Department of Mathematics
COMSATS University Islamabad
Vehari Campus
Vehari 61100
Punjab
Pakistan

Hafiz Muhammad Afzal Siddiqui ‡
Department of Mathematics
COMSATS University Islamabad
Lahore Campus
Lahore
Punjab 54000
Pakistan

* *ORCID-ID:* 0000-0003-4116-9673
† *ORCID-ID:* 0000-0001-6658-2654
§ *ORCID-ID:* 0000-0002-0681-6313
‡ *ORCID-ID:* 0000-0003-1794-6460

* *E-mail:* nasirzawar@gmail.com (Corresponding Author)
† *E-mail:* M.qousini@zuj.edu.jo
§ *E-mail:* imranqureshi18@gmail.com
‡ *E-mail:* hmasiddiqui@gmail.com

Abstract

This research investigates the construction of topological spaces (TS) through zero divisor graphs (ZD-graphs) of finite commutative rings (FCR), denoted by $\Gamma(R)$, having order at most 10. The vertices of these graphs correspond to the non-zero divisors (ZD) within the commutative ring, where an edge connects two different vertices if their product in the ring results in zero. The focus lies on generating associative zero divisor graphs for orders 3 through 10, providing a comprehensive exploration of the intricate relationships within these rings. The study aims to reveal the topological characteristics inherent in these graphs, shedding light on the structural properties of commutative rings. The methodology involves a systematic examination of ZD-graphs and the subsequent derivation of topologies, offering a fresh perspective on the connections between algebraic structures and TS. Lately, we presented some real life applications of our work.

Subject Classification: *13A70, 54A10, 68R10.*

Keywords: *Commutative rings, Zero divisor graphs, Topological spaces, Algebraic topology, Graph theory, Commutative algebra.*

1. Introduction

In the realm of commutative algebra, the exploration of algebraic structures through graph-theoretic representations has emerged as a fertile area of study. This research endeavors to delve into the construction of TSs by investigating the ZD-graphs within small FCRs of order at most 10 [5]. To set the stage for our exploration, let us elucidate key concepts that form the foundation of our investigation. Graphs act as flexible frameworks for representing diverse relationships and phenomena encountered across social, Physical and information security.They prove instrumental in representing practical problems, with the term "network" often defining a graph where attributes are linked to vertices and edges. The study of real-world systems as networks falls under the purview of network science. Graph structures provide a powerful tool for modeling and analyzing relationships in various domains. They enable the visualization and study of communication systems, data frameworks, computational architectures, and process flows [1]. Graph theory plays a pivotal role in diverse fields, such as linguistics, where it captures the complexities of language structures, and in scientific disciplines like chemistry and physics, where it aids in understanding molecular arrangements. In condensed matter physics, graphs are essential for exploring atomic simulations, offering detailed insights into the spatial configurations of complex atomic networks [2].

Topology, a pivotal branch of mathematics, offers powerful tools that give rise to essential concepts such as connectivity, continuity, and homotopy. Its influence spans across various mathematical disciplines [4]. Topologizing discrete structures, including graph theory, is a subject of significant interest in published literature. The exploration of topology on graphs finds inspiration in digital image representation, where vertices and edges model the points and connectivity of an image [8]. Consequently, studying topological properties of digital images involves examining the topologies on the vertices of corresponding graphs [2, 3]. The study of ZD-graphs has gained significant attention in recent years due to its applications in various fields of mathematics and cryptography. Mukhtar et al. [14] explored the size of ZD-graphs and provided insights into their structural properties. Additionally, Annamalai and Durairajan [15] examined the relationship between codes and incidence matrices of ZD-graphs, contributing to the understanding of these graphs in the context of coding theory. Khandare et al. investigated the energy of ZD-graphs associated with CR, further expanding the applications of ZD-graphs in mathematical analysis [16]. These works lay a strong foundation for further research in the area of graph theory and its intersection with algebra.

A notable research avenue involves studying graph theory through the lens of topology. Researchers have employed diverse methods to create topologies from graphs. This paper aims to contribute to this area by creating a TS using rings and their associated undirected ZD-graph. We present distinct properties of the resulting topology. The intersection of graph theory, commutative algebra, and topology has opened avenues for exploring mathematical structures and their real-world applications. In this context, the construction of TSs from ZD-graphs within small FCRs presents an intriguing problem. While ZD-graphs have been extensively studied for their insights into algebraic properties, the exploration of TSs derived from these graphs within the specific context of small commutative rings of order at most 10 remains underdeveloped. The problem at hand involves systematically constructing associative ZD-graphs for orders 3 through 10 within small FCRs and subsequently generating topologies on these graphs. The challenge lies in understanding the topological implications of the underlying algebraic structures, unraveling how the connectivity patterns within ZD-graphs translate into meaningful TSs. The investigation encompasses not only the theoretical aspects of constructing these spaces but also the practical applicability of the derived topologies in real-world scenarios. Additionally, the literature review provider in later part of the introduction highlights the broader context of graph theory, network science, and topology, emphasizing the interconnectedness of these fields and their applications in diverse domains. While previous studies have explored related topics, the specific focus on small commutative rings and the systematic analysis of ZD-graphs with orders 3 through 10 represent novel and uncharted territory within the realm of mathematical research. Therefore, the problem statement encapsulates the need to bridge the gap between algebraic structures and TSs within the specified constraints of small FCRs, shedding light on the unexplored nuances and potential applications of the constructed topologies. The interplay between algebraic structures and TSs has garnered substantial attention in recent literature. The uniqueness of this study resides in its innovative approach of constructing TSs from ZD-graphs within small FCRs of order at most 10. While existing studies have examined ZD-graphs and their algebraic implications, this research contributes a unique perspective by systematically generating associative ZD-graphs for orders 3 through 10. The subsequent derivation of topologies on these graphs represents a pioneering effort to unveil the inherent topological characteristics within the specified context of small commutative rings.

Furthermore, the research addresses a notable gap in the current literature by delving into the interplay between graph theory, commutative algebra, and topology within the constraints of small FCRs. By exploring the orders 3 through 10, this study provides a nuanced understanding of how the algebraic structures of these rings translate into meaningful TSs. The investigation of topologies on graphs inspired by digital image representation adds a practical dimension to the research, extending its relevance to real-world applications [8]. The creation of a TS using cycles in an undirected graph introduces a novel approach, and the exploration of properties within this topology, such as obtaining the ideal TS through specific graph manipulations, contributes innovative insights to the field. In summary, this research not only advances the theoretical understanding of graph theory, commutative algebra, and topology but also pioneers the application of these concepts within the specific domain of small FCRs, offering a novel and valuable contribution to the broader mathematical community.

2. Preliminaries

A graph is defined as a mathematical structure represented by the ordered pair (V, E), where V is a set referred as vertices of the graph. Additionally, E is a subset of the Cartesian product $V \times V$, consisting of pairs (x, y) designated as edges of the graph G , and is represented as $E(G)$ [1]. A finite graph is formally defined as a graph characterized by a definite no. of nodes and edges. A graph that contains neither multiple edges nor loops is referred to as a simple graph. A path graph is a type of tree where there exists a only path connecting all the nodes. A path graph with n vertices is represented as P_n . Moreover, G is said to be connected if, for any two vertices, there exists at least one path linking them. The star graph S_n of order n , is a simple graph with n vertices having the following properties: one distinguished vertex has a degree of n - 1, while all other vertices each have a degree of 1 and are connected only to the distinguished vertex. A complete graph, denoted by K_n , is one in which vertices is directly connected by a single, unique edge. A bipartite graph is a type of graph where the vertex set can be divided into two non-overlapping subsets. say V_1 and V_2 , such that all edges connect vertices from V_1 to V_2, and no edges exist within the same set. If V_1 contains m vertices and V_2 contains n vertices, the complete bipartite graph is represented as $K_{\{m,n\}}$. A cyclic graph is a graph that contains at least one closed loop, or cycle, which is a path starting and ending at the same vertex without revisiting any vertex. It is symbolized as C_n . In a different context, let X be a non-empty set and a collection of subsets of X. The pair (X, T) is called a TS, where the following conditions hold::

1. Both X and the empty set are elements of T

2. The intersection of any two sets within T remains an element of T, and
3. The union of any number of sets from T also belongs to T.

For a TS (X, T) & a subset A$\subseteq X$ the closure of A, denoted as $cl(A)$, and int(A), i.e, interior of A, are defined using specific rules based on the topology T [8].

$$cl(A) = \cap \{V \subseteq X : V \text{ is a closed set where } A \subseteq V\}$$

$$\text{int}(A) = \cup \{W \subseteq X : W \text{ is an open set where } W \subseteq A\}$$

For a TS (X, τ) and $A \subseteq X$ is called, A is an open if and only if $A = \text{int}(A)$ and A is closed if and only if $A = \text{cl}(A)$. For a TS (X, τ) then the sub collection B of τ is called a base or bases or open base for τ if each member of τ can be defined by taking a union of members of B [6].

In other words let (X, τ) be a TS, then the sub collection B of τ is called a base if for a point x belonging to an open set U there exists $B \in B$ such that $x \in B \subseteq U$ [7]. Readers may study more basic definitions of graph theory, topology, and ring theory in [1,7,9]. In the context of a commutative ring R with a multiplicative identity denoted as 1 (where $1 \neq 0$), the ZD-graph of R is constructed as follows: the vertices represent the nonzero ZD of R, and between two differentvertices x and y ,and edge is formed if and only if $xy = 0$. We denote it as, $\Gamma(R)$ and its initial definition was introduced in [12], where various fundamental properties of $\Gamma(R)$ were thoroughly examined [10]. The basic definition, found in [11] and further explored in [12], initially considered all elements or members of the R as vertices in the graph. The exploration of ZD-graphs of CR and other algebraic structures has been the subject of investigation in numerous articles, such as [13]. For a CR, say R with a non-zero multiplicative identity ($1 \neq 0$), previous works, including [5], have provided a comprehensive list of graphs on $n = 1, 2, 3, or\ 4$ vertices that can be considered as $\Gamma(R)$. Additionally, a complete catalog of rings (considered up to isomorphism) generating these graphs has been presented in [12].

This research determined to develop topological spaces (TS) using zero divisor graphs (ZD-graphs) established from finite commutative rings (FCRs) with maximum orders of 10 for the study of algebra-topology relationships. ZD-graphs contain their non-zero divisor elements as vertices that form edges when both elements produce zero in the commutative ring. The structural framework shows both algebraic dependencies together with topological features that allow for novel understanding of ring structures. A methodical examination of FCR ZD-graphs from orders 3 to 10 revealed essential findings about the topological spaces they created. Each ring contains a ZD-graph which has unique features because its edges appear only when the ring contains zeros between pairs of elements. The graphs operate as bridges connecting abstract algebra with topology by presenting topological spaces that demonstrate both ring theory and topological structural interactions within the framework of algebraic configurations.

We observed changing topological features related to ring order expansion while studying the ring structure. When working with smaller ring structures the ZD-graphs created straightforward visual structures which displayed clear vertex-node relationships. When the ring order increased in number the ZD-graph complexity expanded dramatically until we noted the appearance of complex topological features. The graphs passed through multiple topological changes involving connection states and spatial simplicity alongside continuous features that corresponded to ring algebraic properties. The inherent algebraic properties of the ring interact deeply with the topological quality which emerges in the resulting graph. The article supplements the discussion with tables which unite vital properties of both ring structures and their corresponding ZD-graph representations allowing readers to understand the algebraic and topological landscape at work. Future studies depend on these tables because they present structured information about the effects of algebraic features on specific ZD-graph topologies. When ZD-graphs of various rings are matched against each other researchers gain insight into how changes in commutative ring structure impact topology through evolving patterns.

This study demonstrates ZD-graphs function as efficient analytical instruments for exploring connections between algebraic systems and their associated topologies. Research into algebraic-topological connections between algebraic structures and graph theoretic methods now possesses a solid basis through our topology derivation from FCR zero divisor graphs. The conclusions derived from this study create valuable advances for mathematical structural investigations throughout algebraic graph theory and topology alongside complex system examinations of similar algebraic and topological constructs. The research findings present both theoretical value and possible practical implementations. Scientists have used algebraic structures to construct topological spaces for analyzing real-world systems including networks alongside cryptographic systems and biological and social system structures. Recent real-world applications demonstrate how these mathematical tools generate valuable understanding about system connectivity alongside flow dynamics which enhances our knowledge of system interactions throughout evolution. The study provides foundational knowledge that future investigators can use to study progressively intricate rings of various types while examining the effect of these rings' algebraic properties on their associated topological descriptions. Additional research based on the fundamental outcomes of this investigation should pursue specialized implementation like the advancement of advanced topological modeling approaches for complex networks.

This research demonstrates essential progress toward understanding how algebraic systems relate to topological structures within finite commutative rings. Our study of zero divisor graphs has both revealed fresh insights about commutative ring structural properties alongside creating an original framework for developing topological spaces whose applications extend throughout theoretical and applied mathematics.

3. Main Results

This In this section, we delve deeply into the complex interplay between algebraic structures and graph-theoretic models, with a particular emphasis on the topologies that emerge from ZD-graphs associated with small FCRs. The primary aim of this investigation is to explore the underlying topological properties of these graphs, each characterized by a countable number of vertices, and to understand how these properties interact with the algebraic characteristics of the rings from which they are derived. We begin by constructing and analyzing these ZD-graphs, which form the foundation for the broader goal of constructing topological spaces (TSs) from these associative graphs within the context of small commutative rings.

As we proceed with this exploration, we provide a comprehensive set of tables detailing the various small FCRs and their corresponding ZD-graphs, each illustrating the algebraic and topological nuances inherent to these structures. These tables serve not only as a valuable reference for understanding the specific properties of individual rings but also as a means of mapping the connections between algebraic configurations and their graph-theoretic representations. In doing so, we aim to shed light on the deep relationships between the structure of commutative rings and the topological spaces they give rise to, contributing to the ongoing discourse in algebraic graph theory [8].

The subsequent analysis and discussion in this section will examine the implications of these findings and explore new dimensions in the understanding of topological structures, particularly focusing on the interactions between algebraic configurations and graph-theoretic representations. By considering these structures in detail, we hope to provide a more refined understanding of how algebraic properties can influence and shape the topology of associated graphs. This approach not only advances our knowledge in the field of algebraic graph theory but also lays the groundwork for further research into the rich and intricate connections between algebra, graph theory, and topology.

Definition 3.1. [8]: Let $G = (V, E)$ be agraph. The collection of vertices that are directly connected to a vertex v in a graph $G$is called the adjacency of v. Here it is shown by $M_G(v)$.. The Minimal adjacency of v is given by:

$$[v]_G = \cap_{v \in M_G(v)} M_G(v)$$

Theorem 3.2 [8]: *Let $G = (V, E)$ be a undirected graph. A class $\beta_G = \{[v]_G : v \in V\}$ is base for a topology on V.*

Corollary 3.3 [8]: *Each simple, connected, and undirected graph defines a topology on the set of its vertices.*

Definition 3.4 [8]: Let $G = (V, E)$ be simple, connected graph. The topology induced by the graph is defined by the collection $\beta_G = \{[v]_G : v \in V\}$ which serves as the basis for this topology. This topology is in the form of,

$$\tau_G = \left\{G \subseteq V: G \bigcup_{[v]_G \in \beta_G} [v]_G, v \in V\right.$$

Theorem 3.5: *Consider a FCR, denoted as R. If the isomorphism $\Gamma(R) \cong K_n$ holds true, then it is always possible to define a discrete topology on $\Gamma(R)$.*

Proof: Suppose $\Gamma(R) \cong K_n$ for some positive integer n. Let $\{v_1, v_2, \ldots, v_n\}$, represent the vertices of K_n, which correspond to ZD elements in the FCR R under the isomorphism. The vertices of $\Gamma(R)$ correspond to the ZD elements of R. The isomorphism implies that all nodes in $\Gamma(R)$ is connected to every other node, forming a fully connected graph since $\Gamma(R) \cong K_n$. Now the set of vertices $\{v_1, v_2, \ldots, v_n\}$ in $\Gamma(R)$ can serve as the bases sets for the topology. Since every vertex in $\Gamma(R)$ is connected to every other node (as $\Gamma(R) \cong K_n$), the collection of all singleton sets forms a bases for the topology. Therefore, when $\Gamma(R) \cong K_n$, then one can always define a discrete topology τ_G on $\Gamma(R)$. Which completes our proof.

Theorem 3.6: *Consider a FCR. If the isomorphism $\Gamma(R) \cong K_{m,n}$ holds true, then it is always possible to define a topology on $\Gamma(R)$ such that it consists of two subsets: one with m vertices and the other with n vertices.*

Proof: Let $\Gamma(R) \cong K_{m,n}$ for some positive integers m and n. Let $\{v_1, v_2, \ldots, v_n\}$, represent the vertices of $K_{m,n}$, which correspond to zero divisor elements in the FCR R under the isomorphism. Given that there exists an isomorphism between the ZD-graph Γ(R) and the complete bipartite graph $K_{m,n}$, we can represent this isomorphism as a bijective function $f: \Gamma(R) \rightarrow K_{m,n}$. This means that for every node in $\Gamma(R)$, there is a unique node in $K_{m,n}$ and vice versa, and the edges between vertices are preserved under this mapping. To define a topology on $\Gamma(R)$ with two subsets, one with m vertices and the other with n vertices, we will use the concept of open sets. We define two subsets of $\Gamma(R)$ as follows: U_1 = A (containing m vertices) U_2 = B (containing n vertices)

To ensure that U_1 and U_2 form a topology on $\Gamma(R)$, we need to verify that both U_1 and U_2 are open sets and Their complements are also open sets. To prove that U_1 and U_2 are open sets, we need to show that for any node x in U_1 and any node y in U_2 , there exist open neighborhoods around x and y contained entirely within U_1 and U_2, respectively. Since the isomorphism preserves edges, we know that if x is in U_1 and y is in U_2, there are no edges directly connecting x and y. Therefore, we can define open neighborhoods around x and y that do not include vertices from the other set. This ensures that both U_1 and U_2 are open sets. Now to prove that the complements of U_1 and U_2 are open sets, we need to show that for any vertex z not in U_1 (i.e., z is in U_2) and any vertex w not in U_2 (i.e., w is in U_1), there exist open neighborhoods around z and w contained entirely outside of U_1 and U_2, respectively. Similar to the previous argument, since the isomorphism preserves edges, there are no edges directly connecting z and w to the vertices in the other set. Therefore, we can define open neighborhoods around z and w that do not intersect with U_1 and U_2, respectively. Hence, we have defined a topology on $\Gamma(R)$ with two subsets, one containing m vertices and the other containing n vertices, satisfying the properties of a topology. Which completes the proof.

Note that to avoid the complexity $M_G(v)$ is calculated and written as The adjacency of vertex v is {X,Y,Z}.

Theorem 3.7: *If $\Gamma(R) \cong G$ having 3 vertices then their associated ZD-Graphs are given in the Table 1.*

Table 1
ZD-graphs with 3 vertices

No. Vertices	Ring (R)	|?|	Graph G
	$\mathbb{Z}_6$	6	
	$\mathbb{Z}_8$	8	
3	$\mathbb{Z}_2[x]/(x^3)$	8	$K_{1,2}$
	$\mathbb{Z}_4[x]/(2x, x^2-2)$	8	
	$\mathbb{Z}_2[t,s]\ /\ (t,s)^2$	8	
	$\mathbb{Z}_4[t]\ /\ (2,t)^2$	8	
	$F_4[t]\ /\ (t^2)$	16	K_3
	$\mathbb{Z}_4[t]\ /\ (t^2+t+1)$	16	

And , topologies defined on these ZD-graphs are as follows:

i. If ZD-graph shows following isomorphism, $\Gamma(R) \cong K_{1,2}$, then $\tau_G = \{V, \emptyset, \{a\}, \{b,c\}\}$

ii. If $\Gamma(R) \cong K_3$, then $\tau_G = Discrete\ Topology$

Proof: Case (a). When $\Gamma(R) \cong K_{1,2}$, as shown in Figure 1.

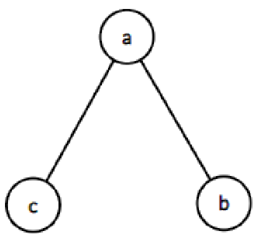


Figure 1

The graph has vertices set $V = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b,c}.

- The adjacency of vertex b is {a}, which is the same as the adjacency of vertex c.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a\}\} = \{a\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{b, c\}\} = \{b, c\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{b, c\}\} = \{b, c\}$$
$$\beta_G = \{\{a\}, \{b, c\}\}$$

and

$$\tau_G = \{V, \emptyset, \{a\}, \{b, c\}\}$$

Case (b): When $\Gamma(R) \cong K_3$, The proof can be concluded based on the result established in Theorem 3.5.

Theorem 3.8: *If $\Gamma(R) \cong G$ having 4 vertices then their associated ZD-Graphs are given in the Table 2.*

Table 2
ZD -graphs with 4 vertices

No. Vertices	**Ring (R)**	|⍰|	**Graph G**
	$\mathbb{Z}_2 \times F_4$	8	$K_{1,3}$
4	$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	$K_{2,2}$
	$\mathbb{Z}_{25}$	25	K_4
	$\mathbb{Z}_5[x]/(x^2)$	25	K_4

And, topologies defined on these ZD-graphs are as follows:

i. If there is following isomorphism $\Gamma(R) \cong K_{1,3}$, then $\tau_G = \{V, \emptyset, \{a\}, \{b, c, d\}\}$ and when $\Gamma(R) \cong K_{2,2}$ hen $\tau_G = \{V, \emptyset, \{a, b\}, \{c, d\}\}$

ii. If $\Gamma(R) \cong K_4$, then $\tau_G = Discrete\ Topology$

Proof: Case (a). When $\Gamma(R) \cong K_{1,3}$, as shown in Figure 2.

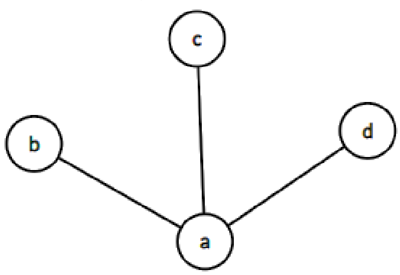


Figure 2

The graph has vertices set $V = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$. The adjacencies corresponding to each vertex are detailed below.

- The adjacency of vertex a is {b,c,d}.
- The adjacency of vertex b, c, and d is {a}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \big\{\{a\}\big\} = \{a\}$$

$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \big\{\{b, c, d\}\big\} = \{b, c, d\}$$

$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \big\{\{b, c, d\}\big\} = \{b, c, d\}$$

$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \big\{\{b, c, d\}\big\} = \{b, c, d\}$$

$$\beta_G = \big\{\{a\}, \{b, c, d\}\big\}$$

and

$$\tau_G = \{V, \emptyset, \{a\}, \{b, c, d\}\}$$

When $\Gamma(R) \cong K_{2,2}$, as shown in Figure 3.

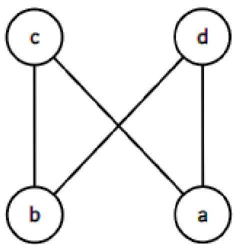


Figure 3

The graph has vertices set $V = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {c,d}, which is the same as the adjacency of vertex b.
- The adjacency of vertex c is {a,b}, which is the same as the adjacency of vertex d.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \big\{\{a, b\}\big\} = \{a, b\}$$

$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \big\{\{a, b\}\big\} = \{a, b\}$$

$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \big\{\{c, d\}\big\} = \{c, d\}$$

$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \big\{\{c, d\}\big\} = \{c, d\}$$

$$\beta_G = \big\{\{a, b\}, \{c, d\}\big\}$$

and

$$\tau_G = \{V, \emptyset, \{a,b\}, \{c,d\}\}$$

Case (b): When $\Gamma(R) \cong K_4$, The proof can be concluded based on the result established in Theorem 3.5.

Theorem 3.9: *If $\Gamma(R) \cong G$ having 5 vertices then their associated ZD-Graphs are given in the Table 3.*

Table 3
ZD -graphs with 5 vertices

No. Vertices	Ring (R)	|?|	Graph G
5	$\mathbb{Z}_2 \times \mathbb{Z}_5$	10	$K_{1,4}$
	$\mathbb{Z}_3 \times F_4$	12	$K_{2,3}$
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8	Fig. 6
	$\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$	8	Fig. 6

And topologies defined on these ZD-graphs are as follows:

i. If following isomorphism, $\Gamma(R) \cong K_{1,4}$, then $\tau_G = \{V, \emptyset, \{a\}, \{b,c,d,e\}\}$ and when $\Gamma(R) \cong K_{2,3}$ it follows $\tau_G = \{V, \emptyset, \{a,b\}, \{c,d,e\}\}$

ii. If $\Gamma(R) \cong$ Figure 6, then $\tau_G = \{V, \emptyset, \{b\}, \{d\}, \{b,d\}, \{b,e\}, \{b,d,e\}, \{a,c,d\}\}$

Proof: Case (a). When $\Gamma(R) \cong K_{1,4}$, as shown in Figure 4.

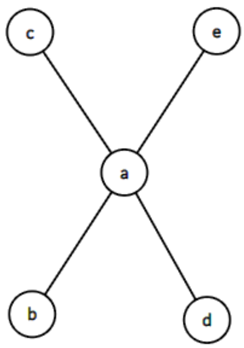


Figure 4

The graph has vertices set $V = \{a, b, c, d, e\}$.

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b,c,d,e}.
- The adjacency of vertex b, c, d, and e is {a}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a\}\} = \{a\}$$

$$[\mathrm{b}]_G = \bigcap_{v \in M_G(v)} = \{\{b,c,d,e\}\} = \{b,c,d,e\}$$
$$[\mathrm{c}]_G = \bigcap_{v \in M_G(v)} = \{\{b,c,d,e\}\} = \{b,c,d,e\}$$
$$[\mathrm{d}]_G = \bigcap_{v \in M_G(v)} = \{\{b,c,d,e\}\} = \{b,c,d,e\}$$
$$[\mathrm{e}]_G = \bigcap_{v \in M_G(v)} = \{\{b,c,d,e\}\} = \{b,c,d,e\}$$
$$\beta_G = \{\{a\},\{b,c,d,e\}\}$$

and

$$\tau_G = \{V, \emptyset, \{a\}, \{b,c,d,e\}\}$$

When $\Gamma(R) \cong K_{2,3}$, as shown in Figure 5.

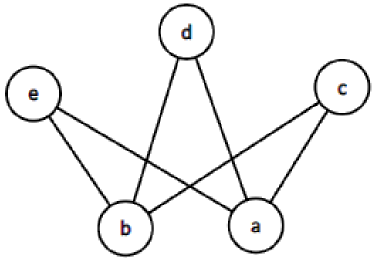


Figure 5
The graph has vertices set $V = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$.

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {c,d,e}, which is the same as the adjacency of vertex b.
- The adjacency of vertex c, d, and e is {a,b}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \bigcap_{v \in M_G(v)} = \{\{a,b\}\} = \{a,b\}$$
$$[\mathrm{b}]_G = \bigcap_{v \in M_G(v)} = \{\{a,b\}\} = \{a,b\}$$
$$[\mathrm{c}]_G = \bigcap_{v \in M_G(v)} = \{\{c,d,e\}\} = \{c,d,e\}$$
$$[\mathrm{d}]_G = \bigcap_{v \in M_G(v)} = \{\{c,d,e\}\} = \{c,d,e\}$$
$$[\mathrm{e}]_G = \bigcap_{v \in M_G(v)} = \{\{c,d,e\}\} = \{c,d,e\}$$
$$\beta_G = \{\{a,b\},\{c,d,e\}\}$$

and

$$\tau_G = \{V, \emptyset, \{a,b\}, \{c,d,e\}\}$$

Case (b): The ZD -graph for rings $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2\,[\mathrm{X}]/(x^2)$ is given in Figure 6.

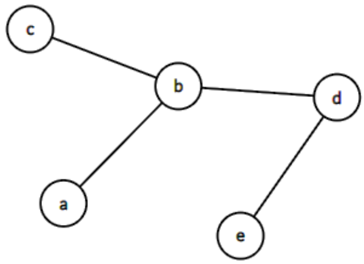


Figure 6

The graph has vertices set $V = \{\mathrm{a, b, c, d, e}\}$. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b}, which is the same as the adjacency of vertex c.
- The adjacency of vertex b is {a,c,d}.
- The adjacency of vertex d is {b,e}.
- The adjacency of vertex e is {d}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a, c, d\}\} = \quad \{a, c, d\}$$

$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{b\}, \{b, e\}\} = \ \{b\}$$

$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{a, c, d\},\} = \{a, c, d\}$$

$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{a, c, d\}, \{d\}\} = \{d\}$$

$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{b, e\}\} = \{b, e\}$$

$$\beta_G = \{\{b\}, \{d\}, \{b, e\}, \{a, c, d\}\}$$

and

$$\tau_G = \{V, \emptyset, \{b\}, \{d\}, \{b, d\}, \{b, e\}, \{b, d, e\}, \{a, c, d\}\}$$

Theorem 3.10: *If $\Gamma(R) \cong G$ having 6 vertices then their associated ZD-Graphs are given in the Table 4.*

Table 4
ZD -graphs with 6 vertices

No. Vertices	Ring ($\boldsymbol{R}$)	$\vert$⍰$\vert$	Graph $\boldsymbol{G}$
6	$\mathbb{Z}_3 \times \mathbb{Z}_5$	15	$K_{2,4}$
	$F_4 \times F_4$	16	$K_{3,3}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8	Fig. 9
	$\mathbb{Z}_7\,[x]/\,(x^2)$	49	K_6
	$\mathbb{Z}_{49}$	49	K_6

And topologies defined on these ZD-graphs are as follows:

i. If the isomorphism $\Gamma(R) \cong K_{2,4}$, then $\tau_G = \{V, \emptyset, \{a,b\}, \{c,d,e,f\}\}$ and when $\Gamma(R) \cong K_{3,3}$, so that $\tau_G = \{V, \emptyset, \{a,b,c\}, \{d,e,f\}\}$

ii. If $\Gamma(R) \cong$ Figure 9, then $\tau_G = \{V, \emptyset, \{b\}, \{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,c,d\}, \{b,c,e\}, \{a,b,c,d,e\}, \{b,d,f\}, \{a,b,c,d,f\}\}$

If $\Gamma(R) \cong K_6$, then $\tau_G =$ discrete topology

Proof: **Case (a).** When $\Gamma(R) \cong K_{2,4}$, as shown in Figure 7.

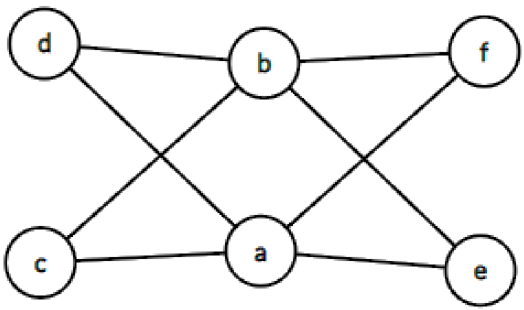


Figure 7
The graph have $V = \{a, b, c, d, e, f\}$.

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is{c,d,e,f}, which is the same as the adjacency of vertex b.
- The adjacency of vertex c, d, e, and f is {a,b}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a,b\}\} = \{a,b\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{a,b\}\} = \{a,b\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{c,d,e,f\}\} = \{c,d,e,f\}$$
$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{c,d,e,f\}\} = \{c,d,e,f\}$$
$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{c,d,e,f\}\} = \{c,d,e,f\}$$
$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \{\{c,d,e,f\}\} = \{c,d,e,f\}$$
$$\beta_G = \{\{a,b\}, \{c,d,e,f\}\}$$

and

$$\tau_G = \{V, \emptyset, \{a,b\}, \{c,d,e\}\}$$

When $\Gamma(R) \cong K_{3,3}$, as shown in Figure 8.

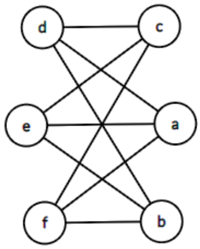


Figure 8

The graph have $V = \{\mathrm{a, b, c, d, e, f}\}$. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a, b, and c is {d,e,f}.
- The adjacency of vertex d, e, and f is {a,b,c}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a, b, c\}\} = \{a, b, c\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{a, b, c\}\} = \{a, b, c\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{a, b, c\}\} = \{a, b, c\}$$
$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{d, e, f\}\} = \{d, e, f\}$$
$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{d, e, f\}\} = \{d, e, f\}$$
$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \{\{d, e, f\}\} = \{d, e, f\}$$
$$\beta_G = \{\{a, b, c\}, \{d, e, f\}\}$$

and

$$\tau_G = \{V, \emptyset, \{a, b, c\}, \{d, e, f\}\}$$

Case (b): The ZD -graph for rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in Figure 9.

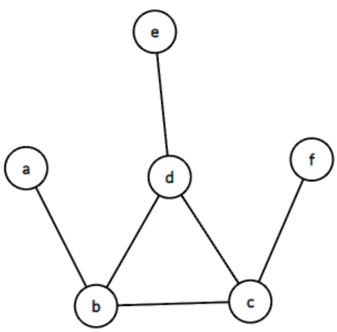


Figure 9
The graph has $V = \{\mathbf{a, b, c, d, e, f}\}$.

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b}.
- The adjacency of vertex b is {a,c,d}.
- The adjacency of vertex c is {b,d,f}.
- The adjacency of vertex d is {b,c,e}.
- The adjacency of vertex e is {d}.
- The adjacency of vertex f is {c}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a,c,d\}\} = \{a,c,d\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{b\},\{b,d,f\},\{b,c,e\}\} = \{b\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{a,c,d\},\{b,c,e\},\{c\}\} = \{c\}$$
$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{a,c,d\},\{b,d,f\},\{d\}\} = \{d\}$$
$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{b,c,e\}\} = \{b,c,e\}$$
$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \{\{b,d,f\}\} = \{b,d,f\}$$

$\beta_G = \{\{b\},\{c\},\{d\},\{a,c,d\},\{b,c,e\},\{b,d,f\}\}$

and

$$\tau_G = \{V,\emptyset,\{b\},\{c\},\{d\},\{b,c\},\{b,d\},\{c,d\},\{a,c,d\},\{b,c,e\}, \{a,b,c,d,e\},\{b,d,f\},\{a,b,c,d,f\}\}$$

Case (c): When $\Gamma(R) \cong K_6$, The proof can be concluded based on the result established in Theorem 3.5.

Theorem 3.11: *If* $\Gamma(R) \cong G$ *having* 7 *vertices then their associated ZD-Graphs are given in the Table 5.*

Table 5
***ZD*-graphs with 7 vertices**

No. Vertices	Ring (R)	$\|R\|$	Graph G
7	$\mathbb{Z}_2 \times \mathbb{Z}_7$	14	$K_{1,6}$
	$F_4 \times \mathbb{Z}_5$	20	$K_{3,4}$
	$\mathbb{Z}_3 \times \mathbb{Z}_4$	12	Figure 10
	$\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$	12	Figure 10
	$\mathbb{Z}_{16}$	16	Figure no 11
	$\mathbb{Z}_2[x]/(x^4)$	16	Figure no 11
	$\mathbb{Z}_4[x]/(x^2+2)$	16	Figure no 11
	$\mathbb{Z}_4[x]/(x^2+2x+2)$	16	Figure no 11
	$\mathbb{Z}_4[x]/(x^3-2, 2x^2, 2x)$	16	Figure no 11

	$\mathbb{Z}_2[x,y]\ /\ (\ x^3\ , xy\ , y^2)$	16	Figure no 12
	$\mathbb{Z}_8[x]/(2x, x^2)$	16	Figure no 12
	$\mathbb{Z}_4[x]/\ (x^3\ , 2x^2\ , 2x)$	16	Figure no 12
	$\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2, 2x, 2y)$	16	Figure no 12
	$\mathbb{Z}_4[x]/(x^2+2x)$	16	Figure 13
	$\mathbb{Z}_8[x]/(2x, x^2+4)$	16	Figure 13
	$\mathbb{Z}_2[x,y]/(x^2, y^2-xy)$	16	Figure 13
	$\mathbb{Z}_4[x,y]/(x^2, y^2-xy, xy-2, 2x, 2y)$	16	Figure no 13
	$\mathbb{Z}_4[x,y]/(x^2, y^2, xy-2, 2x, 2y)$	16	Figure no 14
	$\mathbb{Z}_2[x,y]\ /\ (x^2, y^2)$	16	Figure no 14
	$\mathbb{Z}_4[x]/(x^2)$	16	Figure no 14
	$\mathbb{Z}_2[x,y,z]/(x,y,z)^2$	16	K_7
	$\mathbb{Z}_4[x,y]/(x^2, y^2, xy, 2x, 2y)$	16	K_7
	$F_8[x]/(x^2)$	64	K_7
	$\mathbb{Z}_4[x]/(x^3+x+1)$	64	K_7

And topologies defined on these ZD-graphs are as follows:

i. If $\Gamma(R)\cong K_{1,6}$, $\Gamma(R)\cong K_{3,4}$ and $\Gamma(R)\cong K_7$ then $\tau_G=\{V,\emptyset,\{a\},\{b,c,d,e,f,g\}\}$, $\tau_G=\big\{V,\emptyset,\{a,b,c\},\{d,e,f,g\}\big\}$ and $\tau_G=$ discrete topology respectively.

ii. If $\Gamma(R)\cong$ Figure 10, then $\beta_G=\big\{\{c\},\{d\},\{e\},\{g\},\{d,f\},\{a,b,f,d\}\big\}$

iii. If $\Gamma(R)\cong$ Figure 11,then

$$\tau_G=\{\emptyset,\{a\},\{f\},\{g\},\{b,c,d,e,f,g\},\{a,f\},\{a,g\},\{a,b,c,d,e,f,g\},\{f,g\}\}$$

iv. If $\Gamma(R)\cong$ Figure 12, then

$$\tau_G=\{\emptyset,\{a,b,c\},\{b,c,d,e,f,g\},\{a,c,d,e,f,g\},\{a,b,d,e,f,g\},\{a,b,c,d,e,f,g\}\}$$

v. If $\Gamma(R)\cong$ Figure 13, then $\beta_G=\{\{e\},\{f\},\{g\},\{a,c\},\{b,d\}\}$.

vi. If $\Gamma(R)\cong$ Figure 14, then $\tau_G=$ Discrete Topology.

Proof:

(i) Let $\Gamma(R)$ has vertices set $V=\{\mathrm{a,b,c,d,e,f,g}\}$.

Case 1: Let $\Gamma(R)\cong K_{1,6}$, then we can partition the vertices into two subsets as follows $A=\{a\}\ and\ B=\{b,c,d,e,f,g\}$. Then by Theorem 3.6, $\beta_G=\big\{\{a\},\{b,c,d,e,f,g\}\big\}$ and

$$\tau_G=\big\{V,\emptyset,\{a\},\{b,c,d,e,f,g\}\big\}$$

Case 2: Let $\Gamma(R)\cong K_{3,4}$, then we can partition the vertices into two subsets as follows $A=\{a,b,c\}\ and\ B=\{d,e,f,g\}$. Then by Theorem 3.6, $\beta_G=\big\{\{a,b,c\},\{d,e,f,g\}\big\}$ and

$$\tau_G=\big\{V,\emptyset,\{a,b,c\},\{d,e,f,g\}\big\}$$

Case 3: Let $\Gamma(R) \cong K_7$, the proof follows from Theorem 3.5

(ii) The ZD -graph for rings $\mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[x] / (x^2)$ is given in Figure 10. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {c}, which is the same as the adjacency of vertex b.
- The adjacency of vertex c is {a,b,d,f}.
- The adjacency of vertex d is {c,e,g}.
- The adjacency of vertex e is {d,f,g}.
- The adjacency of vertex f is{c,e}.
- The adjacency of vertex g is {d,e}.

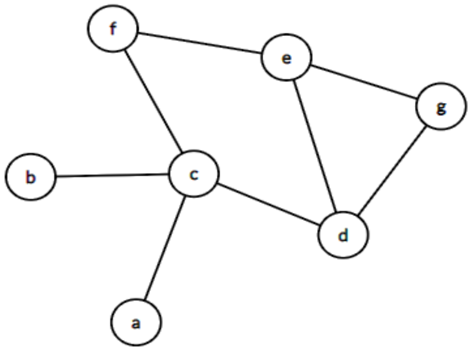


Figure 10

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a,b,d,f\}\} = \{a,b,d,f\}$$

$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{a,b,d,f\}\} = \{a,b,d,f\}$$

$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{c\},\{c,e,g\},\{c,e\}\} = \{c\}$$

$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{a,b,d,f\},\{d,f,g\},\{d,e\}\} = \{d\}$$

$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{c,e,g\},\{c,e\},\{d,e\}\} = \{e\}$$

$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \{\{a,b,d,f\},\{d,f,g\}\} = \{d,f\}$$

$$[\mathrm{g}]_G = \cap_{v \in M_G(v)} = \{\{c,e,g\},\{d,f,g\}\} = \{g\}$$

$$\beta_G = \{\{c\},\{d\},\{e\},\{g\},\{d,f\},\{a,b,f,d\}\}$$

And

$$\tau_G = \{\emptyset,\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\},\{d,f\},\{a,b,f,d\},\{c,d\},\{c,e\},\{c,g\},$$
$$\{c,d,f\},\{c,a,b,f,d\},\{d,e\},\{d,g\},\{d,f,g\},\{d,a,b,f,d\},\{e,g\},$$
$$\{e,d,f\},\{e,a,b,f,d\},\{g,d,f\},\{g,a,b,f,d\},\{d,f,a,b,d\},\{c,d,e\},\{c,d,g\},$$
$$\{c,d,f,g\},\{c,d,a,b,f,d\},\{c,e,g\},\{c,e,d,f\},\{c,e,a,b,f,d\},\{c,g,d,f\},$$
$$\{c,g,a,b,f,d\},\{c,d,f,a,b,d\},\{d,e,g\},\{d,e,d,f\},$$

$$\{d,e,a,b,f,d\},\{d,g,a,b,f,d\},\{e,g,d,f\},\{e,g,a,b,f,d\},\{g,d,f,a,b,d\},$$
$$\{c,d,e,g\},\{c,d,e,d,f\},\{c,d,e,a,b,f,d\},\{c,d,g,a,b,f,d\},\{c,e,g,d,f\},$$
$$\{c,e,g,a,b,f,d\},\{c,g,d,f,a,b,d\},\{d,e,g,a,b,f,d\},\{c,d,e,g,d,f\},$$
$$\{c,d,e,g,a,b,f,d\}\}$$

(iii) The ZD-graph for rings: $\mathbb{Z}_{16}, \mathbb{Z}_2[x]/(x^4), \mathbb{Z}_4[x]\,/(x^2+2), \mathbb{Z}_4[x]\,/(x^2+2x+2), \mathbb{Z}_4[x]\,/(x^3-2,2x^2,2x)$ is given in Figure 11.

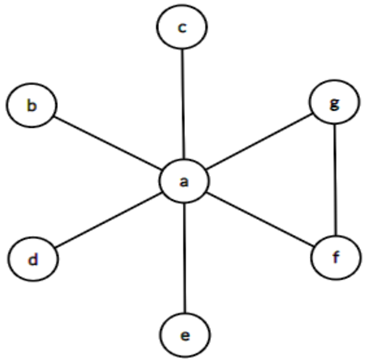


Figure 11

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b,c,d,e,f,g}.
- The adjacency of vertex b, c, d, and e is {a}.
- The adjacency of vertex f is {a,g}.
- The adjacency of vertex g is {a,f}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v\in M_G(v)} = \{\{a\},\{a,g\},\ \{a,f\}\} = \{a\}$$
$$[\mathrm{b}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{b,c,\ d,e\ ,f,\ g}\}\} = \{\mathrm{b,c\ ,d,\ e,f,g}\}$$
$$[\mathrm{c}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{b,c,d,e,f,g}\}\} = \{\mathrm{b,c,d,e,f,g}\}$$
$$[\mathrm{d}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{b,c,d,e,f,g}\}\} = \{\mathrm{b,c,d,e,f,g}\}$$
$$[\mathrm{e}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{b,c,d,e,f,g}\}\} = \{\mathrm{b,c,d,e,f,g}\}$$
$$[\mathrm{f}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{b,c,d,e,f,g}\},\{a,f\}\} = \{f\}$$
$$[\mathrm{g}]_G = \cap_{v\in M_G(v)} = \{\ \{\mathrm{b,c,d,e,f,g}\},\{a,g\}\} = \{g\}$$
$$\beta_G = \{\{a\},\{f\},\{g\},\{\mathrm{b,c,d,e,f,g}\}\}$$

and

$$\tau_G = \{\emptyset,\{a\},\{f\},\{g\},\{b,c,d,e,f,g\},\{a,f\},\{a,g\},\{a,b,c,d,e,f,g\},\{f,g\}\}$$

(iv) The ZD-graph for rings: $\mathbb{Z}_2[x,y]\,/\,(x^3, xy, y^2), \mathbb{Z}_8[x]/(2x, x^2), \mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$ is given in Figure 12.

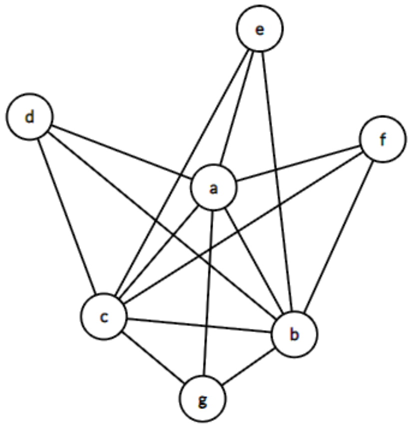


Figure 12

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b,c,d,e,f,g}.
- The adjacency of vertex b is {a,c,d,e,f,g}.
- The adjacency of vertex c is {a,b,d,e,f,g}.
- The adjacency of vertex d, e, f and g is {a,b,c}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{b, c, d, e, f, g}\}\} = \mathrm{aaa}\ \{\mathrm{b, c, d, e, f, g}\}$$
$$[\mathrm{b}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{a, c, d, e, f, g}\}\} = \text{is am}\ \{\mathrm{a, c, d, e, f, g}\}$$
$$[\mathrm{c}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{a, b, d, e, f, g}\}\} = \{\mathrm{a, b, d, e, f, g}\}$$
$$[\mathrm{d}]_G = \cap_{v\in M_G(v)} = \{\{a,b,c\}\} = \{a,b,c\}$$
$$[\mathrm{e}]_G = \cap_{v\in M_G(v)} = \{\{a,b,c\}\} = \{a,b,c\}$$
$$[\mathrm{f}]_G = \cap_{v\in M_G(v)} = \{\{a,b,c\}\} = \{a,b,c\}$$
$$[\mathrm{g}]_G = \cap_{v\in M_G(v)} = \{\ \{a,b,c\}\} = \{a,b,c\}$$

$$\beta_G = \{\{a,b,c\}, \{\mathrm{b, c, d, e, f, g}\}, iisiisi\ \{\mathrm{a, c, d, e, f, g}\}, \{\mathrm{a, b, d, e, f, g}\}\}$$

and

$$\tau_G = \{\emptyset, \{a,b,c\}, \{b,c,d,e,f,g\}, \{a,c,d,e,f,g\}, \{a,b,d,e,f,g\}, \{a,b,c,d,e,f,g\}\}$$

(v) The ZD-graph for rings: $\mathbb{Z}_4[x]\,/(x^2+2), \mathbb{Z}_4[x]/(x^2+2x), \mathbb{Z}_8[x]/(2x, x^2+4), \mathbb{Z}_2[x,y]/(x^2, y^2-xy),\ \ \mathbb{Z}_4[x]\,/(x^2, y^2-xy, xy-2, 2x, 2y)$ is given in Figure 13.

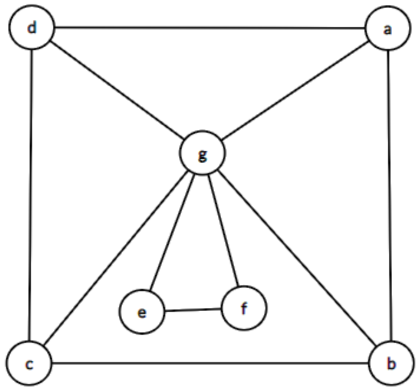


Figure 13

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b,d,g}, which is the same as the adjacency of vertex c.
- The adjacency of vertex b is {a,c,g}, which is the same as the adjacency of vertex d.
- The adjacency of vertex e is {f,g}.
- The adjacency of vertex f is {e,g}.
- The adjacency of vertex g is {a,b,c,d,e,f}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, d, e, f}\}, \{a, c, g\}\} = \{\mathrm{a, c}\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, d, e, f}\}, \{b, d, g\}\} = \{\mathrm{b, d}\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, d, e, f}\}, \{a, c, g\}\} = \{\mathrm{a, c}\}$$
$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, d, e, f}\}, \{b, d, g\}\} = \{b, d\}$$
$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, d, e, f}\}, \{e, g\}\} = \{e\}$$
$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, d, e, f}\}, \{f, g\}\} = \{f\}$$
$$[\mathrm{g}]_G = \cap_{v \in M_G(v)} = \{ \{\mathrm{b, d, g}\}, \{a, c, g\}, \{f, g\}, \{e, g\}\} = \{g\}$$
$$\beta_G = \{\{e\}, \{f\}, \{g\}, \{a, c\}, \{b, d\}\}$$

and

$$\tau_G = \{\emptyset, \{a, b, c, d, e, f, g\}, \{e\}, \{f\}, \{g\}, \{a, c\}, \{b, d\}, \{e, f\}, \{e, g\}, \{a, c, e\}, \{b, d, e\}, \{e, f, g\}, \{a, c, b, d\}, \{a, c, e, f\}, \{b, d, e, g\}, \{a, c, b, d, e\}, \{a, c, f, g\}, \{b, d, f, g\}, \{a, c, b, d, f\}, \{a, c, b, d, g\}, \{a, c, b, d, e, f\}, \{a, c, b, d, e, g\}, \{a, c, e, f, g\}, \{b, d, e, f, g\}, \{a, c, b, d, e, f, g\}\}$$

Case (vi): The ZD-graph for rings $\mathbb{Z}_4[x,y]/(x^2,y^2,xy-2,2x,2y), \mathbb{Z}_2[x,y]/(x^2,y^2), \mathbb{Z}_4[x]/(x^2)$ is given in Figure 14.

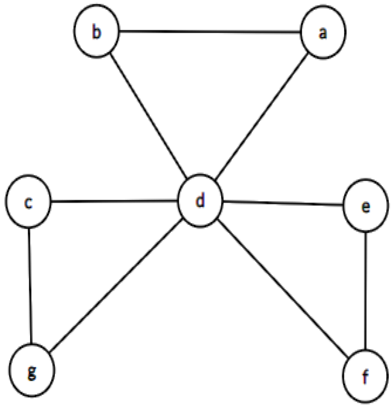


Figure 14

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {b,d}.
- The adjacency of vertex b is {a,d}.
- The adjacency of vertex c is {d,g}.
- The adjacency of vertex d is {a,b,c,e,f,g}.
- The adjacency of vertex e is {d,f}.
- The adjacency of vertex f is{d,e}.
- The adjacency of vertex g is {c,d}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{a},\mathrm{d}\},\{a,b,c,e,f,g\}\} = \{\mathrm{a}\}$$
$$[\mathrm{b}]_G = \cap_{v\in M_G(v)} = \{\{b,d\},\{a,b,c,e,f,g\}\} = \{\mathrm{b}\}$$
$$[\mathrm{c}]_G = \cap_{v\in M_G(v)} = \{\{\mathrm{c},\mathrm{d}\},\{a,b,c,e,f,g\}\} = \{\mathrm{c}\}$$
$$[\mathrm{d}]_G = \cap_{v\in M_G(v)} = \{\{b,d\},\{a,d\},\{d,g\},\{d,f\},\{d,e\},\{c,d\}\} = \{d\}$$
$$[\mathrm{e}]_G = \cap_{v\in M_G(v)} = \{\{d,e\},\{a,b,c,e,f,g\}\} = \{e\}$$
$$[\mathrm{f}]_G = \cap_{v\in M_G(v)} = \{\{d,f\},\{a,b,c,e,f,g\}\} = \{f\}$$
$$[\mathrm{g}]_G = \cap_{v\in M_G(v)} = \{\{d,g\},\{a,b,c,e,f,g\}\} = \{g\}$$
$$\beta_G = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\}\}$$

and

$$\tau_G = Discrete\ Topology$$

Theorem 3.12: *If $\Gamma(R) \cong G$ having* 8 *vertices then their associated ZD-Graphs are given in the Table 6.*

Table 6
***ZD*-graphs with 8 vertices**

No. Vertices	Ring (R)	|?|	Graph *G*
	$\mathbb{Z}_2 \times F_8$	16	$K_{1,7}$
	$\mathbb{Z}_3 \times \mathbb{Z}_7$	21	$K_{2,6}$
8	$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	$K_{4,4}$
	$\mathbb{Z}_{27}$	27	Figure 15
	$\mathbb{Z}_9[x] \,/\, (3x, x^2 - 3)$	27	Figure 15
	$\mathbb{Z}_9[x] \,/\, (3x, x^2 - 6)$	27	Figure 15
	$\mathbb{Z}_3[x]/(x^3)$	27	Figure 15
	$\mathbb{Z}^3[x, y]/(x, y)^2$	27	K_8
	$\mathbb{Z}_9[x]/(3, x)^2$	27	K_8
	$F_9[x]/(x^2)$	81	K_8
	$\mathbb{Z}_9[x]/(x^2 + 1)$	81	K_8

And topologies defined on these ZD-graphs are as follows:

i. If $\Gamma(R) \cong K_{1,7}$, $\Gamma(R) \cong K_{2,6}$, $\Gamma(R) \cong K_{4,4}$ and $\Gamma(R) \cong K_8$ then $\tau_G = \{V, \emptyset, \{a\}, \{b, c, d, e, f, g, h\}\}$, $\tau_G = \{V, \emptyset, \{a, b\}, \{c, d, e, f, g, e\}\}$, $\tau_G = \{V, \emptyset, \{a, b, c, d\}, \{e, f, g, e\}\}$ and, $\tau_G =$ Discrete Topology respectively.

ii. If $\Gamma(R) \cong$ Figure 15, then $\tau_G = \{\emptyset, V, \{g\}, \{h\}, \{a, b, c, d, e, f\}, \{g, h\}, \{g, a, b, c, d, e, f\}, \{h, a, b, c, d, e, f\}\}$

Proof:

(i) Let $\Gamma(R)$ has vertices set $V = \{\mathrm{a, b, c, d, e, f, g, h}\}$.

Case 1: Let $\Gamma(R) \cong K_{1,7}$, then we can partition the vertices into two subsets as follows $A = \{a\}$ *and* $B = \{b, c, d, e, f, g, h\}$. Then by Theorem 3.6, $\beta_G = \{\{a\}, \{b, c, d, e, f, g, h\}\}$

and

$$\tau_G = \{V, \emptyset, \{a\}, \{b, c, d, e, f, g, h\}\}$$

Case 2: Let $\Gamma(R) \cong K_{2,6}$, then we can partition the vertices into two subsets as follows $A = \{a, b\}$ *and* $B = \{c, d, e, f, g, h\}$. Then by Theorem 3.6, $\beta_G = \{\{a, b\}, \{c, d, e, f, g, h\}\}$

and

$$\tau_G = \{V, \emptyset, \{a, b\}, \{c, d, e, f, g, h\}\}$$

Case 3: Let $\Gamma(R) \cong K_{4,4}$, then we can partition the vertices into two subsets as follows $A = \{a, b, c, d\}$ *and* $B = \{e, f, g, h\}$. Then by Theorem 3.6, $\beta_G = \{\{a, b, c, d\}, \{e, f, g, h\}\}$

and

$$\tau_G = \{V, \emptyset, \{a,b,c,d\}, \{e,f,g,h\}\}$$

Case 4: Let $\Gamma(R) \cong K_8$, the proof follows from Theorem 3.5

(ii) The ZD-graph for ring $\mathbb{Z}_{27}, \mathbb{Z}_9[x] / (3x, x^2-3), \mathbb{Z}_9[x] / (3x, x^2-6), \mathbb{Z}_3[x] / (x^3)$ is given in Figure 15.

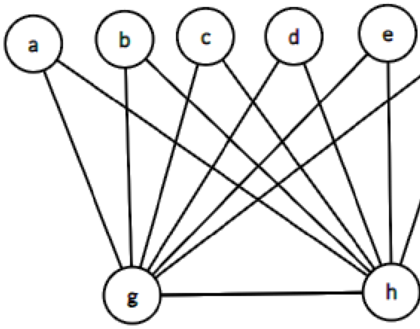


Figure 15

Consider $V = \{a,b,c,d,e,f,g,h\}$ be the vertex set. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a, b, c, d, e, and f is {g,h}.
- The adjacency of vertex g is {a,b,c,d,e,f,h}.
- The adjacency of vertex h is {a,b,c,d,e,f,g}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \{\{a,b,c,d,e,f,h\}, \{a,b,c,d,e,f,g\}\} = \{a,b,c,d,e,f\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \{\{a,b,c,d,e,f,h\}, \{a,b,c,d,e,f,g\}\} = \{a,b,c,d,e,f\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \{\{a,b,c,d,e,f,h\}, \{a,b,c,d,e,f,g\}\} = \{a,b,c,d,e,f\}$$
$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \{\{a,b,c,d,e,f,h\}, \{a,b,c,d,e,f,g\}\} = \{a,b,c,d,e,f\}$$
$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \{\{a,b,c,d,e,f,h\}, \{a,b,c,d,e,f,g\}\} = \{a,b,c,d,e,f\}$$
$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \{\{a,b,c,d,e,f,h\}, \{a,b,c,d,e,f,g\}\} = \{a,b,c,d,e,f\}$$
$$[\mathrm{g}]_G = \cap_{v \in M_G(v)} = \{\{g,h\}, \{a,b,c,d,e,f,g\}\} = \{g\}$$
$$[\mathrm{h}]_G = \cap_{v \in M_G(v)} = \{\{g,h\}, \{a,b,c,d,e,f,h\}\} = \{h\}$$
$$\beta_G = \{\{g\}, \{h\}, \{a,b,c,d,e,f\}\}$$

and

$$\tau_G = \{\emptyset, V, \{g\}, \{h\}, \{a,b,c,d,e,f\}, \{g,h\}, \{g,a,b,c,d,e,f\}, \{h,a,b,c,d,e,f\}\}$$

Theorem 3.13: *If* $\Gamma(R) \cong G$ *having* 9 *vertices then their associated ZD-Graphs are given in the Table 7.*

Table 7
***ZD*-graphs with 9 vertices**

No. Vertices	Ring (R)	|?|	Graph G
	$\mathbb{Z}_2 \times F_9$	18	$K_{1,8}$
	$\mathbb{Z}_3 \times F_8$	24	$K_{2,7}$
9	$F_4 \times \mathbb{Z}_7$	28	$K_{3,6}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	12	Figure 16
	$\mathbb{Z}_4 \times F_4$	16	Figure 17
	$\mathbb{Z}_2[x]/(x^2) \times F_4$	16	Figure 17

And topologies defined on these ZD-graphs are as follows:

i. If $\Gamma(R) \cong K_{1,8}$, $\Gamma(R) \cong K_{2,7}$, and $\Gamma(R) \cong K_{3,6}$ then $\tau_G = \{V, \emptyset, \{a\}, \{b, c, d, e, f, g, h, i\}\}$,$\tau_G = \{V, \emptyset, \{a, b\}, \{c, d, e, f, g, h, i\}\}$, and, $\tau_G = \{V, \emptyset, \{a, b, c\}, \{d, e, f, g, h, i\}\}$ respectively.

ii. If $\Gamma(R) \cong$ Figure 16, then $\tau_G = \{\emptyset, V, \{e\}, \{f\}, \{g, h\}, \{e, f, i\}, \{c, d, e, g, h\}, \{a, b, f, g, h\}, \{e, g, h\}, \{f, g, h\}, \{e, f, g, h, i\}, \{c, d, e, f, g, h\}, \{a, b, c, d, f, g, h\}\}$

iii. If $\Gamma(R) \cong$ Figure 17, then

$$\tau_G = \{\emptyset, V, \{d\}, \{e, f, g\}, \{d, i, h\}, \{a, b, c, e, f, g\}, \{d, e, f, g\}, \{d, a, b, c, e, f, g\}, \{e, f, g, i, h\}, \{d, i, h, a, b, c, e, f, g\}\}$$

Proof:

(i) Let $\Gamma(R)$ has vertices set $V = \{\mathrm{a, b, c, d, e, f, g, h, i}\}$.

Case 1: Let $\Gamma(R) \cong K_{1,8}$, then we can partition the vertices into two subsets as follows $A = \{a\}\ and\ B = \{b, c, d, e, f, g, h, i\}$. Then by Theorem 3.6, $\beta_G = \{\{a\}, \{b, c, d, e, f, g, h, i\}\}$

and

$$\tau_G = \{V, \emptyset, \{a\}, \{b, c, d, e, f, g, h, i\}\}$$

Case 2: Let $\Gamma(R) \cong K_{2,7}$, then we can partition the vertices into two subsets as follows $A = \{a, b\}\ and\ B = \{c, d, e, f, g, h, i\}$. Then by Theorem 3.6, $\beta_G = \{\{a, b\}, \{c, d, e, f, g, h, i\}\}$

and

$$\tau_G = \{V, \emptyset, \{a, b\}, \{c, d, e, f, g, h, i\}\}$$

Case 3: Let $\Gamma(R) \cong K_{3,6}$, then we can partition the vertices into two subsets as follows $A = \{a, b, c\}\ and\ B = \{d, e, f, g, h, i\}$. Then by Theorem 3.6, $\beta_G = \{\{a, b, c\}, \{d, e, f, g, h, i\}\}$

and

$$\tau_G = \{V, \emptyset, \{a, b, c\}, \{d, e, f, g, h, i\}\}$$

(ii) The ZD-graph for ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ is given in Figure 16.

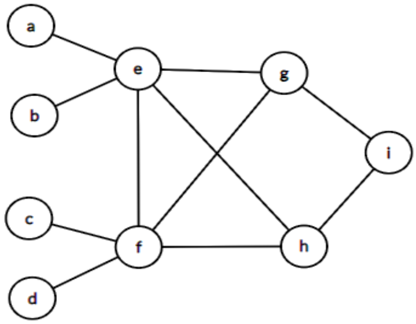


Figure 16

Consider $V = \{a, b, c, d, e, f, g, h, i\}$ be the vertex set. The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a is {e}, which is the same as the adjacency of vertex b.
- The adjacency of vertex c is {f}, which is the same as the adjacency of vertex d.
- The adjacency of vertex e is {a,b,f,g,h}.
- The adjacency of vertex f is{c,d,e,g,h}.
- The adjacency of vertex g is{e,f,i}, which is the same as the adjacency of vertex h.
- The adjacency of vertex i is{g,h}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \cap_{v \in M_G(v)} = \big\{\{a, b, f, g, h\}\big\} = \{a, b, f, g, h\}$$
$$[\mathrm{b}]_G = \cap_{v \in M_G(v)} = \big\{\{a, b, f, g, h\}\big\} = \{a, b, f, g, h\}$$
$$[\mathrm{c}]_G = \cap_{v \in M_G(v)} = \big\{\{c, d, e, g, h\}\big\} = \{c, d, e, g, h\}$$
$$[\mathrm{d}]_G = \cap_{v \in M_G(v)} = \big\{\{c, d, e, g, h\}\big\} = \{c, d, e, g, h\}$$
$$[\mathrm{e}]_G = \cap_{v \in M_G(v)} = \big\{\{c, d, e, g, h\}, \{e\}, \{e, f, i\}\big\} = \{e\}$$
$$[\mathrm{f}]_G = \cap_{v \in M_G(v)} = \big\{\{f\}, \{e, f, i\}, \{a, b, f, g, h\}\big\} = \{f\}$$
$$[\mathrm{g}]_G = \cap_{v \in M_G(v)} = \big\{\{a, b, f, g, h\}, \{c, d, e, g, h\}, \{g, h\}\big\} = \{g, h\}$$
$$[\mathrm{h}]_G = \cap_{v \in M_G(v)} = \big\{\{a, b, f, g, h\}, \{c, d, e, g, h\}, \{g, h\}\big\} = \{g, h\}$$
$$[\mathrm{i}]_G = \cap_{v \in M_G(v)} = \big\{\{e, f, i\}\big\} = \{e, f, i\}$$
$$\beta_G = \{\{e\}, \{f\}, \{g, h\}, \{e, f, i\}, \{c, d, e, g, h\}, \{a, b, f, g, h\}\}$$

and

$$\tau_G = \{\emptyset, V, \{e\}, \{f\}, \{g, h\}, \{e, f, i\}, \{c, d, e, g, h\}, \{a, b, f, g, h\}, \{e, g, h\}, \{f, g, h\},$$
$$\{e, f, g, h, i\}, \{c, d, e, f, g, h\}, \{a, b, c, d, f, g, h\}\}$$

(iii) The ZD-graph for ring $\mathbb{Z}_4 \times F_4), \mathbb{Z}_2[x]/(x^2) \times F_4$is given in Figure 17. Consider $V = \{a, b, c, d, e, f, g, h, i\}$ be the vertex set.

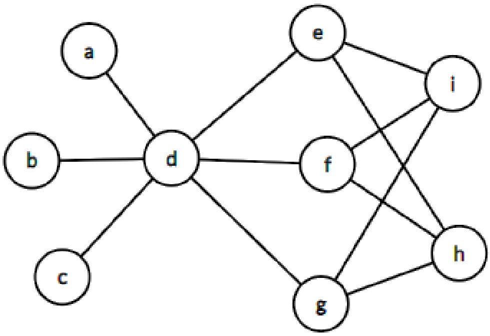


Figure 17

The adjacencies corresponding to each vertex are detailed below

- The adjacency of vertex a, b, and c is {d}.
- The adjacency of vertex d is{a,b,c,e,f,g}.
- The adjacency of vertex e is {d,i,h}.
- The adjacency of vertex f is {d,g,h}.
- The adjacency of vertex g is {d,f,h}.
- The adjacency of vertex h is {e,f,g}.
- The adjacency of vertex i is {e}.

The minimal adjacencies corresponding to each vertex are detailed below

$$[\mathrm{a}]_G = \bigcap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, e, f, g}\}\} = \{\mathrm{a, b, c, e, f, g}\}$$
$$[\mathrm{b}]_G = \bigcap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, e, f, g}\}\} = \{\mathrm{a, b, c, e, f, g}\}$$
$$[\mathrm{c}]_G = \bigcap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, e, f, g}\}\} = \mathrm{i}\{\mathrm{a, b, c, e, f, g}\}$$
$$[\mathrm{d}]_G = \bigcap_{v \in M_G(v)} = \{\{d\}, \{d, i, h\}\} = \{d\}$$
$$[\mathrm{e}]_G = \bigcap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, e, f, g}\}, \{e, f, g\}\} = \{e, f, g\}$$
$$[\mathrm{f}]_G = \bigcap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, e, f, g}\}, \{e, f, g\}\} = \{e, f, g\}$$
$$[\mathrm{g}]_G = \bigcap_{v \in M_G(v)} = \{\{\mathrm{a, b, c, e, f, g}\}, \{e, f, g\}\} = \{e, f, g\}$$
$$[\mathrm{h}]_G = \bigcap_{v \in M_G(v)} = \{\{d, i, h\}\} = \{d, i, h\}$$
$$[\mathrm{i}]_G = \bigcap_{v \in M_G(v)} = \{\{d, i, h\}\} = \{d, i, h\}$$
$$\beta_G = \{\{d\}, \{e, f, g\}, \{d, i, h\}, \{a, b, c, e, f, g\}\}$$

and

$$\tau_G = \{\emptyset, V, \{d\}, \{e, f, g\}, \{d, i, h\}, \{a, b, c, e, f, g\}, \{d, e, f, g\}, \{d, a, b, c, e, f, g\}, \{e, f, g, i, h\}, \{d, i, h, a, b, c, e, f, g\}\}$$

Theorem 3.14: *If $\Gamma(R) \cong G$ having* 10 *vertices then then their associated ZD-Graphs are given in the Table 8.*

Table 8
***ZD*-graphs with 10 vertices**

No. Vertices	Ring (R)	|?|	Graph G
10	$\mathbb{Z}_3 \times F_9$	27	$K_{2,8}$
	$F_4 \times F_8$	32	$K_{3,7}$
	$\mathbb{Z}_5 \times \mathbb{Z}_7$	35	$K_{4,6}$
	$\mathbb{Z}_{121}$	121	K_{10}
	$\mathbb{Z}_{11}[x]/(x^2)$	121	K_{10}

And topologies defined on these ZD-graphs are as follows:

i. If $\Gamma(R) \cong K_{2,8}$, $\Gamma(R) \cong K_{3,7}$, and $\Gamma(R) \cong K_{4,6}$ then $\tau_G = \{V, \emptyset, \{a,b\}, \{c,d,e,f,g,h,i,j\}\}$,$\tau_G$ consists of the elements $\{V, \{\}, \{a,b,c\}, \{d,e,f,g, h,i,j\}\}$, and, $\tau_G = \left\{V, \emptyset, \{a,b,c,d\}, \{e,f,g,h,i,j\}\right\}$ respectively.

ii. If $\Gamma(R) \cong K_{10}$, then, $\tau_G = Discrete\ Topology$

Proof:

(i) Let $\Gamma(R)$ have vertices set V consists of elements a, b, c, d, e, f, g, h ,i, j

Case I: Let $\Gamma(R) \cong K_{2,8}$, then we can partition the vertices into two subsets as follows $A = \{a,b\}$ *and* $B = \{c,d,e,f,g,h,i,j\}$. Then by Theorem 3.6, $\beta_G = \left\{\{a,b\}, \{c,d,e,f,g,h,i,j\}\right\}$

and

$$\tau_G = \left\{V, \emptyset, \{a,b\}, \{c,d,e,f,g,h,i,j\}\right\}$$

Case II: Let $\Gamma(R) \cong K_{3,7}$, then we can partition the vertices into two subsets as follows $A = \{a,b,c\}$ *and* $B = \{d,e,f,g,h,i,j\}$. Then by Theorem 3.6, $\beta_G = \left\{\{a,b,c\}, \{d,e,f,g,h,i,j\}\right\}$

and

$$\tau_G = \left\{V, \emptyset, \{a,b,c\}, \{d,e,f,g,h,i,j\}\right\}$$

Case III: Let $\Gamma(R) \cong K_{4,6}$, then we can partition the vertices into two subsets as follows $A = \{a,b,c,d\}$ *and* $B = \{e,f,g,h,i,j\}$. Then by Theorem 3.6, $\beta_G = \left\{\{a,b,c,d\}, \{e,f,g,h,i,j\}\right\}$

and

$$\tau_G = \left\{V, \emptyset, \{a,b,c,d\}, \{d,e,f,g,h,i,j\}\right\}$$

4. Applications

The findings of this research extend beyond the theoretical framework, presenting practical applications that showcase the utility of the constructed TSs arising from ZD-graphs in commutative rings.

- **Non-AI Detection in Security Systems:**

The distinctive topological patterns identified in our research can be leveraged for enhancing security systems to detect non-AI-generated threats. By applying the insights from the ZD-graph topologies, security algorithms can better discern non-artificial intelligence-based intrusions, thereby bolstering the resilience of digital security frameworks. See Figure 18.

Figure 18
AI Security Camera Intrusion Alarm / Person Detection

- **Plagiarism Detection in Academic Settings:**

The TSs derived from ZD-graphs offer a novel approach to improving plagiarism detection algorithms. The unique structures encoded in these spaces can be utilized to create advanced similarity metrics, enhancing the accuracy and efficiency of plagiarism detection systems in academic and professional environments. See Figure 19.

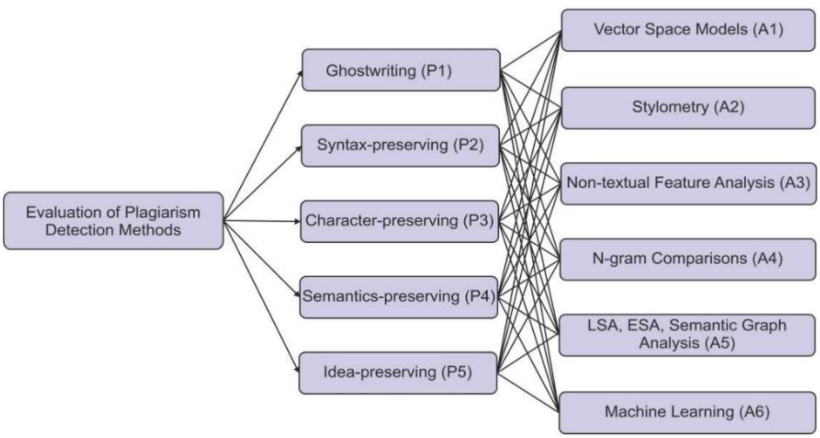


Figure 19
Plagiarism Detection in Academic

- **Supply Chain Optimization:**

Applying the principles of ZD-graphs, our research findings can contribute to optimizing supply chain networks. The TSs can assist in identifying critical nodes and potential disruptions within the supply chain, leading to more efficient resource allocation and improved overall resilience. See Figure 20.

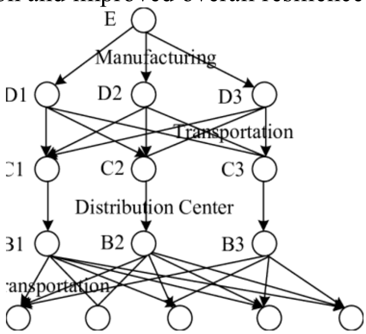


Figure 20
Topology of supply chain network

- **Biological Network Analysis:**

The topological structures emerging from ZD-graphs can find application in the analysis of biological networks. By representing biological interactions as ZD-graphs, our research may aid in deciphering complex molecular relationships, contributing to advancements in systems biology and drug discovery. See Figure 21.

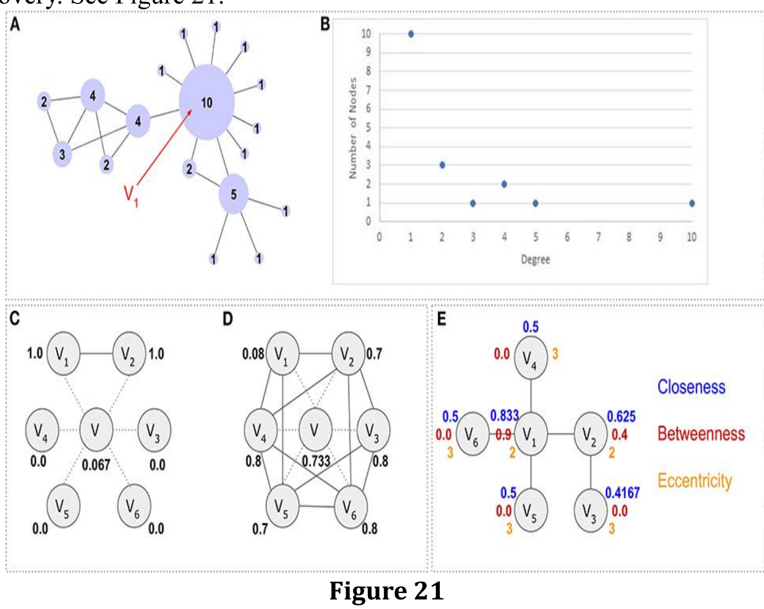
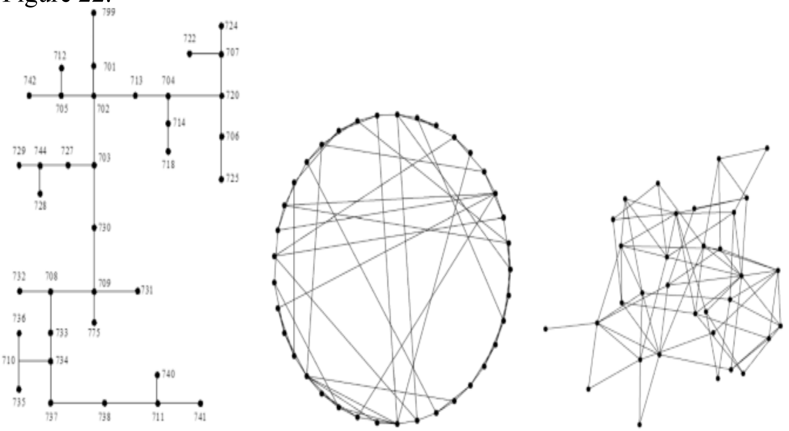


Figure 21
Biological network analysis through Graph and Topology defined on graphs

- **Smart Grid Management:**

Utilizing the insights gained from our study, the TSs derived from ZD-graphs can be applied to enhance the management of smart grids. The understanding of connectivity patterns can assist in optimizing energy distribution, improving grid reliability, and minimizing the impact of failures. See Figure 22.



(a) Radial test feeder with 37 nodes. (b) Graph of the small-world model with 37 nodes. (c) Random graph with 37 nodes.

Figure 22
Smart Grid Topology Example

In conclusion, the practical applications of our work span diverse domains, ranging from cybersecurity and academic integrity to supply chain management, biological network analysis, and smart grid optimization. These applications underscore the real-world impact of the TSs constructed from ZD-graphs, showcasing their relevance across a spectrum of industries and research fields.

5. Conclusion

In conclusion, our investigation into the construction of TSs through ZD-graphs in FCRs of order up to 10 has unveiled a rich tapestry of relationships and possibilities. The associative ZD-graphs presented for orders 3 through 10 serve as both a testament to the complexity of commutative rings and a roadmap for further exploration. Through a systematic examination of ZD-graphs, we have not only elucidated topological characteristics inherent in these structures but also laid the groundwork for novel topologies. The figures depicting ZD-graphs of associated rings offer a visual feast, enhancing the accessibility and appeal of our research. This work, as a pioneering effort in a relatively unexplored domain, not only expands our understanding of graph theory, commutative algebra, and topology but also invites researchers to embark on an exciting journey of discovery. By fostering a deeper connection between algebraic structures and TSs, our contribution aims to spark curiosity and inspire future investigations into the boundless realms of mathematical exploration. s

Declaration:

- Availability of data and materials: The data is provided on request to the authors.
- Conflicts of interest: The authors declare that they have no conflicts of interest and all the agree to publish this paper under academic ethics.
- Fundings: This received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.
- Author's contribution: All the authors equally contributed towards this work.

References

[1] A. De Schepper, J. Schillewaert, and H. Van Maldeghem, "On the Generating Rank and Embedding Rank of the Hexagonic Lie Incidence Geometries," *Combinatorica*, pp. 1–38 (2024).

[2] H. A.-H. Mahdi and S. N. F. Al-khafaji, "Construction A Topology On Graphs," *J. Al-Qadisiyah Comput. Sci. Math.*, vol. 5, no. 2, pp. 39–46 (2013).

[3] K. Kruse, "Mixed topologies on saks spaces of vector-valued functions," *Topol. Appl.*, vol. 345, p. 108843 (2024).

[4] M. Bonatto, D. Dikranjan, and D. Toller, "Groups with cofinite Zariski topology and potential density," *Topol. Appl.*, vol. 340, p. 108720 (2023).

[5] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *J. Algebra*, vol. 217, pp. 434–447 (1999).

[6] D. Dikranjan and W. Tholen, "Categorical Structure of Closure Operators". Dordrecht, Boston: Kluwer Academic Publishers (1995).

[7] S. A. Morris, "Topology Without Tears" (2017).

[8] J. Jokela, "Compatible topologies on mixed lattice vector spaces," Topol. Appl., vol. 320 (2022).

[9] L. H. Rowen, "Ring Theory". Academic Press, vol. 83 (2012).

[10] N. Ali, H. M. A. Siddiqui, and M. I. Qureshi, "A Graph-Theoretic Approach to Ring Analysis: Dominant Metric Dimensions in Zero-Divisor Graphs," arXiv preprint, arXiv:2312.16005 (2023).

[11] I. Beck, “Coloring of commutative rings,” *J. Algebra*, vol. 26, pp. 208–226 (1988).

[12] D. D. Anderson and M. Naseer, “Beck’s coloring of a commutative ring,” *J. Algebra*, vol. 159, pp. 500–514 (1993).

[13] S. P. Redmond, “On zero divisor graph of small finite commutative rings,” *Discrete Math.*, vol. 307, pp. 1155–1166 (2007).

[14] A. Mukhtar, R. Murtaza, S. U. Rehman, S. Usman, and A. Q. Baig, “Computing the size of zero divisor graphs,” *J. Inf. Optim. Sci.*, vol. 41, no. 4, pp. 855–864 (2020).

[15] N. Annamalai and C. Durairajan, “Codes from the Incidence Matrices of a zero-divisor Graphs,” *J. Discrete Math. Sci. Cryptogr.*, pp. 1–9 (2022).

[16] P. D. Khandare, S. M. Jogdand, and R. A. Muneshwar, “Energy and Wiener index of divisor graph of commutative ring.”

Received May, 2024