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Hybrid block method with quasilinearization for the dynamical simulation of a methamphetamine abuse model

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Abstract

We propose a single-step hybrid block approach that incorporates three strategically chosen fixed off-grid points to tackle a system of methamphetamine abuse models governed by first-order equations. The method is formulated using a polynomial basis formed via a power-series representation and is proven to be both convergent and A-stable. Numerical experiments demonstrate that the proposed method effectively approximates the solution, highlighting its capability to solve the methamphetamine abuse model and other first-order initial value problems. Additionally, we analyze the effect of immigration on the dynamics of methamphetamine abuse and find that controlling user influx is a

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crucial strategy for mitigating the spread of substance use. These findings emphasize the significance of incorporating immigration control measures into intervention policies to curb methamphetamine abuse effectively.

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1. Introduction

Physical problems in science and medicine, such as modeling animal populations and diseases, are frequently modeled using systems of first-order ordinary differential equations (ODEs). Many of these ODEs lack analytical solutions, making numerical methods essential for obtaining approximate solutions. Over the years, numerous researchers have proposed numerical methods for solving first-order ODEs [1-5], among others.

Shateyi [6] presented a unified hybrid block method (HBM) that integrates uniformly distributed off-grid nodes to solve both linear and nonlinear first-order ODEs. The outcomes generated with Shateyi's method surpassed the performance of current methods. In a similar vein, Rufai et al. [7] proposed a one-step HBM that employs regularly spaced off-grid points for addressing first-order initial value problems (IVPs) and time-varying partial differential equations (PDEs). The approach incorporates an adaptive step-size mechanism to ensure the estimated error remains within a predefined tolerance at each step, yielding results that demonstrated greater efficiency compared to other methods.

In their study, [8] proposed an HBM featuring three optimized parameters for solving first-order initial value problems. The proposed schemes were reformulated to reduce the number of function evaluations, and the numerical experiments demonstrated better performance compared to the reviewed methods. Adeyefa and Olagunju [9] designed a one-step HBM with fixed off-grid points to solve initial value problems (IVPs) of first, second, and third order. Furthermore, other researchers [10-12] have introduced similar methods to address second-order ODEs, with results showing improved accuracy over existing techniques.

In pursuit of even improved accuracy for HBM, Ramos [13] proposed a two-step optimized approach incorporating two off-grid points to solve first-order initial value problems (IVPs), yielding superior results compared to several existing techniques. Qureshi et al. [14] proposed an optimized fifth-order one-step method for solving high-order IVPs. The authors implemented the method for both fixed and adaptive step-size settings, leading to enhanced performance. Additionally, the authors in [15] introduced a single-step computational algorithm for first-order stiff and singular systems, operating with variable step-size adjustment. More recently, the authors in [16] developed a single-step HBM integrating two optimal nodes to examine the dynamics of infectious disease models. These adaptive techniques produced better results than other methods.

Despite extensive research in this area, there remains a need to further explore advancements in numerical methods for solving first-order ODEs, particularly to address both existing and emerging problems more accurately. The objective of this study is to design a one-step HBM that leverages three well-selected off-grid points to achieve higher accuracy and efficiency in solving first-order ordinary differential equations, demonstrated through its application to the methamphetamine abuse model.

2. Mathematical Model Description

We consider an $S-U-T-R$ model for methamphetamine (meth) abuse dynamics, where the population $N(t)$ comprises four distinct compartments: $S(t)$, $U(t)$, $T(t)$, and $R(t)$. The compartment $S(t)$ comprises individuals who are susceptible to substance abuse due to interactions with current users. Individuals actively abusing methamphetamine, including recent initiates and those who have relapsed, are classified under $U(t)$. The group $T(t)$ consists of individuals undergoing rehabilitation programs aimed at overcoming methamphetamine dependency. Finally, $R(t)$ denotes individuals who were previously addicted to methamphetamine but have successfully stopped using the drug, though they remain at risk of relapse.

The susceptible group, $S(t)$, grows due to population influx at a rate k_1. Individuals in this group may start using methamphetamine at a rate proportional to β, which depends on their interactions with current users. Recovery can occur either naturally, at a rate τ, or through treatment programs, at a rate α. Entry into treatment programs occurs at a rate η. The natural mortality rate is denoted by θ, while an additional mortality rate specific to active users is represented by ω. Individuals in the recovery compartment $(R(t))$ may relapse into methamphetamine use at a rate λ. Based on these assumptions, we obtain the following differential equations, which describe the dynamics of the population:

$$\left.\begin{aligned}
S'(t) &= k_1 - \tfrac{2\beta SU}{S+U} - \theta S,\\
U'(t) &= \tfrac{2\beta SU}{S+U} + \lambda R - (\theta+\omega+\tau+\eta)U,\\
T'(t) &= \eta U - (\alpha+\theta)T,\\
R'(t) &= \alpha T + \tau U - (\lambda+\theta)R,
\end{aligned}\right\}. \tag{1}$$

To incorporate modern realities such as globalization and population movement, immigration is included in all compartments of the model. Specifically, individuals are assumed to immigrate directly into the $U(t)$, $T(t)$, and $R(t)$ compartments at rates k_2, k_3, and k_4, respectively. The modified equations are expressed as follows, as reported in [17]

$$\left.\begin{aligned}
S'(t) &= k_1 - \tfrac{2\beta SU}{S+U} - \theta S,\\
U'(t) &= k_2 + \tfrac{2\beta SU}{S+U} + \lambda R - (\theta+\omega+\tau+\eta)U,\\
T'(t) &= k_3 + \eta U - (\theta+\alpha)T,\\
R'(t) &= k_4 + \alpha T + \tau U - (\lambda+\theta)R,
\end{aligned}\right\}, \tag{2}$$

subject to the conditions $S(0) = S_0, U(0) = U_0, T(0) = T_0, R(0) = R_0$ where S_0, U_0, T_0 and R_0 are strictly positive constants.

A schematic diagram of the model dynamics is presented in Figure 1, illustrating the interactions among the sub-populations and the transitions between compartments.

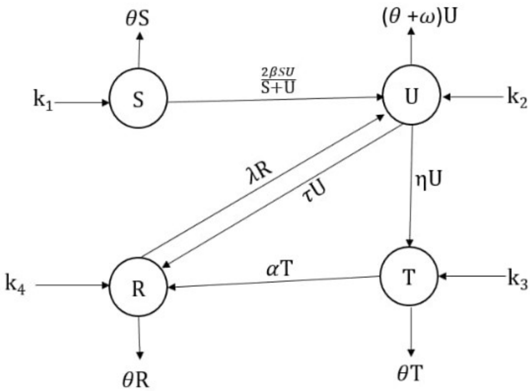

Figure 1
Interaction dynamics between sub-populations in the meth abuse model.

2.1 *Model Feasibility Analysis*

It is straightforward to observe that the vector function

$$f = \begin{pmatrix} k_1 - \frac{2\beta SU}{S+U} - \theta S, \\ k_2 + \frac{2\beta SU}{S+U} + \lambda R - (\theta + \omega + \eta + \tau)U, \\ k_3 + \eta U - (\theta + \alpha)T, \\ k_4 + \alpha T + \tau U - (\lambda + \theta)R \end{pmatrix}$$

is continuous with respect to the variables $S, U, T,$ and R. In addition, given the positive initial conditions $S_0, U_0, T_0, R_0,$ the Jacobian J_f

$$J_f = \begin{pmatrix} -\frac{2\beta SU^2}{(S+U)^2} - \theta & -\frac{2\beta S^2 U}{(S+U)^2} & 0 & 0 \\ \frac{2\beta SU^2}{(S+U)^2} & \frac{2\beta S^2 U}{(S+U)^2} - (\theta + \omega + \eta + \tau) & 0 & -\lambda \\ 0 & \eta & -(\theta + \alpha) & 0 \\ 0 & \tau & \alpha & -(\lambda + \theta) \end{pmatrix}$$

is bounded. Therefore, the dynamical system described by equation (2) has a unique solution for $t \geq 0$. In addition, the solution remains positive for positive initial conditions within an invariant region as stated below.

Theorem 2.1: *The solutions $S(t), U(t), T(t), R(t)$ of the system described by equation (2), when starting from positive initial conditions, will stay positive for all future time $t > 0$ within an invariant region*

$$\Omega = \left\{(S,U,T,R) \in \mathbb{R}_+^4 \middle| 0 \leq N(t) \leq \frac{k_1+k_2+k_3+k_4}{\theta}\right\}$$

where $N(t) = S(t) + U(t) + T(t) + R(t)$.

Proof: Let us observe the dynamics of the solutions as they progress along the S-axis $S_0 > 0, U_0 = 0, T_0 = 0, R_0 = 0$. On this axis, it follows from equation (2) that $S'(t) \geq -\theta S, S(0) = S_0$ which yields $S(t) = S_0 e^{-\theta t} > 0$ for all $t > 0$. Similar arguments applied to the other equations yield positivity of the solutions as required.

Let $\bar{k} = k_1 + k_2 + k_3 + k_4$. As $N = S + U + T + R$, it follows from equation (2) that

$$N'(t) = \bar{k} - \theta N(t) - \omega U(t) \leq \bar{k} - \theta N(t). \tag{3}$$

By the strong comparison theorem [18], it follows that $N(t) \leq N_0 e^{-\theta t} + \frac{\bar{k}}{\theta}(1 - e^{-\theta t})$ and $\limsup_{t\to\infty} N(t) \leq \frac{\bar{k}}{\theta}$. This completes the proof.

Furthermore, the model possesses an abuse reproduction ratio, denoted as R_a, which is given by

$$R_a = \frac{2\beta(\theta+\lambda)(\theta+\alpha)}{K},$$

where

$$K = \theta(\theta+\alpha)(\theta+\eta+\tau+\omega) + \lambda(\theta+\omega)(\theta+\alpha) + \theta\lambda\eta.$$

For additional details on the model, see [17].

3. Development of the Hybrid Block Approach

This section describes the derivation of a single-step HBM that employs three strategically chosen off-grid nodes for solving first-order problems

$$y'(t) = f\big(t, y(t)\big), \quad y(t_0) = y_0, \quad t \in [t_0, T], \tag{4}$$

where f is a function that is differentiable to a sufficient degree and satisfies the Lipschitz condition.

In developing the scheme, the approximate solution $Z(t)$ is written as:

$$y(t) \approx Z(t) = \sum_{i=0}^{5} c_i t^i, \tag{5}$$

in which the coefficients c_i are determined through a set of interpolation and collocation requirements enforced at designated nodes within the interval $a \leq t \leq b$, where

$$b = t_N > t_{N-1} > \cdots > t_2 > t_1 > t_0 = a.$$

The first derivative of the solution $y(t)$ in equation (5) is given by

$$y'(t) = \sum_{i=1}^{5} i c_i t^{i-1}. \tag{6}$$

Equation (5) is interpolated at $t = t_n$, while equation (6) is collocated at $t = t_{n+i}$, where $i = 0, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, 1$. This setup yields a set of six equations involving six variables, from which the coefficients c_i are determined. The resulting expression is represented in matrix equation form

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 1 & 2t_{n+\frac{1}{6}} & 3t^2_{n+\frac{1}{6}} & 4t^3_{n+\frac{1}{6}} & 5t^4_{n+\frac{1}{6}} \\ 0 & 1 & 2t_{n+\frac{1}{2}} & 3t^2_{n+\frac{1}{2}} & 4t^3_{n+\frac{1}{2}} & 5t^4_{n+\frac{1}{2}} \\ 0 & 1 & 2t_{n+\frac{5}{6}} & 3t^2_{n+\frac{5}{6}} & 4t^3_{n+\frac{5}{6}} & 5t^4_{n+\frac{5}{6}} \\ 0 & 1 & 2t_{n+1} & 3t^2_{n+1} & 4t^3_{n+1} & 5t^4_{n+1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{pmatrix}. \tag{7}$$

Using Equation (7), the values of c_i are determined and subsequently inserted into Equation (5) to construct the solution. A continuous approximation equation is formulated by introducing a new variable $x = x_n + th$, expressed as

$$y(x_n + th) = \alpha_0 y_n + h[\eta_0(t) f_n + \eta_{\frac{1}{6}}(t) f_{n+\frac{1}{6}} + \eta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \eta_{\frac{5}{6}}(t) f_{n+\frac{5}{6}} + \eta_1(t) f_{n+1}], \tag{8}$$

where n is the grid index, and $h = t_{n+1} - t_n$.

The continuous coefficients are given by

$$\begin{aligned} \alpha_0 &= 1, \\ \eta_0(t) &= \frac{h}{150}(432t^5 - 1350t^4 + 1540t^3 - 765t^2 + 150t), \\ \eta_{\frac{1}{6}}(t) &= -\frac{27h}{100}(24t^5 - 70t^4 + 70t^3 - 25t^2), \\ \eta_{\frac{1}{2}}(t) &= \frac{h}{30}(216t^5 - 540t^4 + 410t^3 - 75t^2), \\ \eta_{\frac{5}{6}}(t) &= -\frac{27h}{100}(24t^5 - 50t^4 + 30t^3 - 5t^2), \\ \eta_1(t) &= \frac{h}{150}(432t^5 - 810t^4 + 460t^3 - 75t^2). \end{aligned}$$

The hybrid block method is derived by evaluating equation (8) at $t = t_{n+i}, i = \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, 1$. This process produces the following schemes:

$$y_{n+\frac{1}{6}} = y_n + \frac{h\left(4274f_n + 7371f_{n+\frac{1}{6}} - 1240f_{n+\frac{1}{2}} + 621f_{n+\frac{5}{6}} - 226f_{n+1}\right)}{64800}, \tag{9}$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h\left(86f_n + 729f_{n+\frac{1}{6}} + 440f_{n+\frac{1}{2}} - 81f_{n+\frac{5}{6}} + 26f_{n+1}\right)}{2400}, \tag{10}$$

$$y_{n+\frac{5}{6}} = y_n + \frac{5h\left(26f_n + 135f_{n+\frac{1}{6}} + 200f_{n+\frac{1}{2}} + 81f_{n+\frac{5}{6}} - 10f_{n+1}\right)}{2592}, \tag{11}$$

$$y_{n+1} = y_n + \frac{h\left(14f_n + 81f_{n+\frac{1}{6}} + 110f_{n+\frac{1}{2}} + 81f_{n+\frac{5}{6}} + 14f_{n+1}\right)}{300}. \tag{12}$$

4. Analysis of the Method

Here, we discuss the evaluation of the scheme's characteristics, covering convergence and absolute stability analysis.

4.1 *Convergence Analysis*

By the Dahlquist equivalence theorem [19], a method that is consistent and zero-stable is convergent. Our goal is to show that the derived scheme is both consistent and zero-stable. Firstly, the scheme is reformulated into the matrix equation

$$A_1 Z_{n+1} = A_0 Z_n + h(B_0 G_n + B_1 G_{n+1}), \tag{13}$$

where the matrices are

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{2137}{32400} \\ 0 & 0 & 0 & \frac{43}{1200} \\ 0 & 0 & 0 & \frac{65}{1296} \\ 0 & 0 & 0 & \frac{7}{150} \end{pmatrix}, B_1 = \begin{pmatrix} \frac{91}{800} & -\frac{31}{1620} & \frac{23}{2400} & -\frac{113}{32400} \\ \frac{243}{800} & \frac{11}{60} & -\frac{27}{800} & \frac{13}{1200} \\ \frac{25}{96} & \frac{125}{324} & \frac{5}{32} & -\frac{25}{1296} \\ \frac{27}{100} & \frac{11}{30} & \frac{27}{100} & \frac{7}{150} \end{pmatrix}.$$

The vectors Z_n, Z_{n+1}, G_n, and G_{n+1} are defined as

$$Z_n = \begin{pmatrix} y_{n-\frac{5}{6}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{6}} \\ y_n \end{pmatrix}, Z_{n+1} = \begin{pmatrix} y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{5}{6}} \\ y_{n+1} \end{pmatrix},$$

$$G_n = \begin{pmatrix} f_{n-\frac{5}{6}} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{6}} \\ f_n \end{pmatrix}, G_{n+1} = \begin{pmatrix} f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{pmatrix}.$$

4.1.1 Consistency of the Method

Associated with the scheme is a difference operator $\mathcal{L}$

$$\mathcal{L}(y(t_n); h) = \sum_{i=0,\frac{1}{6},\frac{1}{2},\frac{5}{6},1} (\alpha_i y(t_n + ih) - h\eta_i y'(t_n + ih)). \tag{14}$$

Assuming that $y(t_n)$ is sufficiently differentiable so that Taylor series expansion of $\mathcal{L}(y(t_n); h)$ yields

$$\mathcal{L}(y(t_n); h) = \chi_0 y(t_n) + \chi_1 h y'(t_n) + \cdots + \chi_p h^p y^{(p)}(t_n) + \chi_{p+1} h^{p+1} y^{(p+1)}(t_n),$$

where χ_j (for $j = 0,1,\ldots,p+1$). The scheme achieves order p if $\chi_0 = 0, \chi_1 = 0, \ldots, \chi_p = 0$ and $\chi_{p+1} \neq 0$.

For the method, we obtain $\chi_0 = \chi_1 = \cdots = \chi_5 = 0$ and the error constants

$$\chi_6 = \left(\frac{25}{13436928}, -\frac{1}{276480}, \frac{25}{13436928}, \frac{1}{21772800}\right).$$

Thus, the method has at least order 5, confirming consistency.

4.1.2 Zero-Stability

This property is determined by the homogeneous system obtained from Equation (13) as $h \to 0$:

$$A_1 Z_{n+1} = A_0 Z_n.$$

The characteristic polynomial of the system is

$$\rho(z) = \det(zA_1 - A_0) = z^3(z-1)$$

whose roots all lie within or on the unit disk, and the root $z = 1$ is simple. Therefore, the method satisfies the root condition for zero stability and is zero-stable.

4.2 *Linear Stability and Order Stars*

This property is analyzed using the test problem $y'(t) = \sigma y(t)$, where the real component of σ is strictly negative. Setting $z = \sigma h$, the amplification matrix is

$$H(z) = (A_1 - zB_1)^{-1}(A_0 + zB_0)$$

whose spectral radius is

$$\rho(z) = \frac{5z^4+102z^3+924z^2+4320z+8640}{5z^4-102z^3+924z^2-4320z+8640}.$$

The region in the complex plane where the method is absolutely stable, denoted by R, is characterized as

$$R = \{z \in \mathcal{C} : |\rho(z)| < 1\}.$$

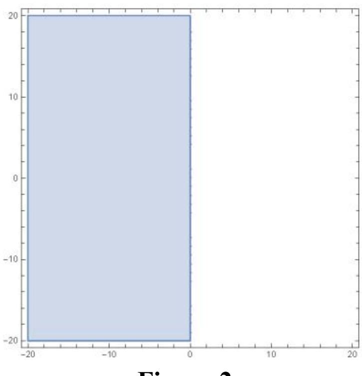

Figure 2
A-stability region of the scheme.

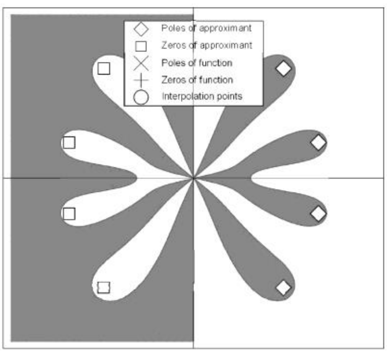

Figure 3
Order star plot showing absolute stability of the method.

The scheme will be considered to have absolute stability or A-stable provided its region of absolute stability R encompasses the entire left half of the complex plane [20]. The region of absolute stability for the proposed scheme is shown in Figure 2.

Figure 2 shows that the region of stability for the method is wholly situated in the left half of the complex plane, confirming A-stability [20]. Furthermore, it can also be seen in the order star plot Figure 3 that no pole of $\rho(z)$ is in the left-half plane of $\mathbb{C}$.

5. Application of the Method to the Methamphetamine Abuse Model (2)

In this section, we apply the schemes in equations (9)-(10) to the model equation (2), written in the form $y'(t) = f(t, y(t))$, where $y = (S, U, T, R)$ and f corresponds to the right portion of equation (2).

5.1 *Quasilinearization*

The function f is nonlinear for the methamphetamine abuse model. To reduce the computational complexity of the schemes in equations (9)-(10), we employ the Quasi-Linearization Method (QLM) proposed in [21].

If y_n is the current approximation and $y_{n+1} - y_n$ is assumed to be small, such that higher-order terms in $(y_{n+1} - y_n)$ are negligible, then

$$f(t, y_{n+1}) \approx N(t, y_n) + L(t, y_n)y_{n+1} \tag{15}$$

where $N(t, y_n) = f(t, y_n) - y_n \frac{\partial f}{\partial y}(t, y_n)$ and $L(t, y_n) = \frac{\partial f}{\partial y}(t, y_n)$. Applying the Quasi-Linearization Method (QLM) to the function $f_1(t, S, U, T, R) = k_1 - 2\beta \frac{SU}{S+U} - \theta S$ in system (2) gives

$$\begin{aligned} L(t, S, U, T, R) &= -2\beta \frac{U}{S+U} - \theta, \\ N(t, S, U, T, R) &= k_1 - 2\beta \frac{US}{U+S} + 2\beta \frac{U^2 S}{(U+S)^2}. \end{aligned}$$

The remaining components of equation (2) are similarly linearized and used in equations (9)-(12).

5.2 *Simulation Results*

To analyze the dynamical behavior of the methamphetamine abuse model with immigration effects, we perform numerical simulations using the specified initial conditions and parameter values. The system is initialized with the following values:

$$S(0) = 100, \quad U(0) = 10, \quad T(0) = 25, \quad R(0) = 25.$$

The parameter values used in the simulations are given as:

$$\beta = 0.0145,\ \theta = 0.0293,\ k_1 = 0.95,\ k_3 = 0.05,\ k_4 = 0.03,\ \lambda = 0.95,$$
$$k_2 = 0.15,\ \omega = 0.03,\ \eta = 0.32,\ \tau = 0.5,\ \alpha = 0.699.$$

The presence of immigration into all compartments ensures that the system does not have an abuse-free equilibrium unless the immigration parameters

k_2, k_3, k_4 are all set to zero. To examine the influence of immigration on system dynamics, we conduct simulations under two different scenarios:

No Immigration Scenario:

Here we investigate the natural progression of the system in the absense of external immigration into the U, T and R compartments by setting $k_2 = k_3 = k_4 = 0$ while other parameters retain their specified values. In Figure 4, a rapid decline in meth abuse is observed in all the compartments as the dynamics approach an abuse-free equilibrium point of the form $(S^*, 0,0,0)$**.** Figure 5 shows rapid convergence of the proposed method as applied to the meth abuse model.

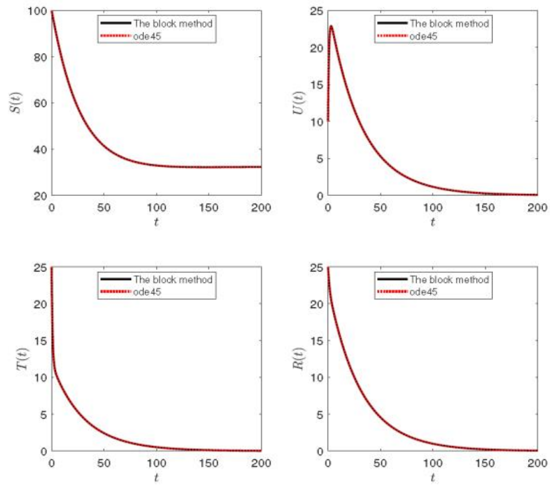

Figure 4
The dynamics of different compartments when there is no external influence. The results also demonstrate that our method is effective in comparison to the ode45 solver.

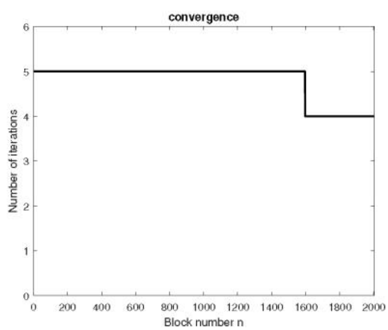

Figure 5
The number of iterations per block showing excellent convergence of the method for no immigration scenario.

Full Immigration Scenario:

Here, all parameters take their given non-zero values, representing a continuous influx into all compartments. Immigration into the susceptible and user compartments significantly increases the number of users, reinforcing the persistence of meth abuse in the system as can be seen in Figure 6. Figure 7 shows rapid convergence of the proposed method as applied to the meth abuse model.

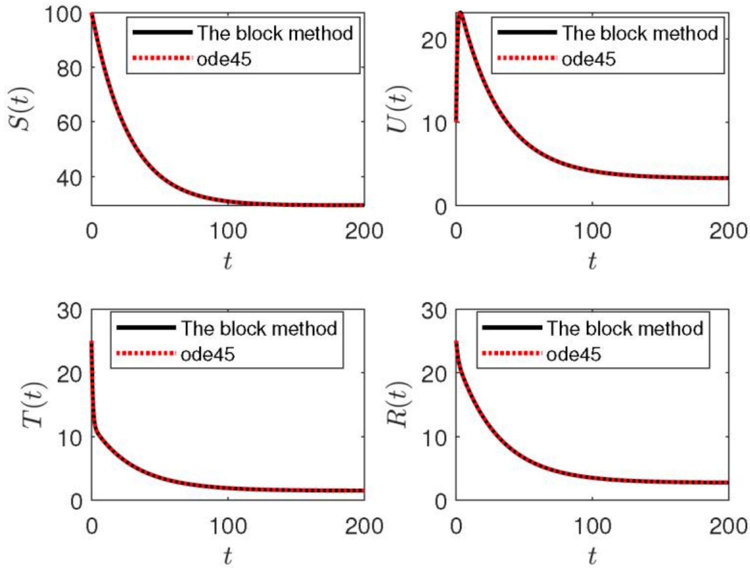

Figure 6
The dynamics of different compartments under full immigration into all compartments. The results illustrate the influence of immigration rates on the prevalence of substance abuse, with no abuse-free equilibrium.

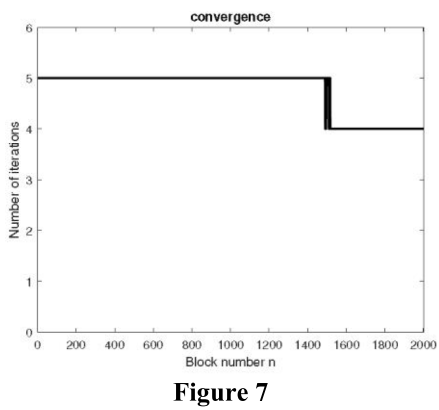

Figure 7
Per block iteration count showing excellent convergence of the method for full immigration scenario.

6. Conclusion

We have introduced a single-step hybrid block numerical scheme in this study, specifically designed to simulate the methamphetamine abuse model, which is a first-order system. A thorough numerical analysis confirmed that the scheme demonstrated zero-stability, consistency, convergence, and A-stablility. The numerical simulations performed on the methamphetamine abuse model yielded results comparable to those obtained using ode45, illustrated by Figure 4 and Figure 6. These findings show the scheme's reliability and effectiveness in solving such models and indicate that the proposed algorithm offers an efficient computational framework for solving the targeted class of problems and reliable for solving the methamphetamine abuse model as shown in Figure 5 and Figure 7. Finally, simulation findings emphasize the critical role of immigration in methamphetamine abuse dynamics and suggest that controlling user influx while enhancing treatment access could be an effective intervention strategy.

Conflict of Interest

The authors affirm that there are no conflicts of interest.

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