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Generalized fractional Hankel-Clifford convolution, translation, and norm equalities in Sobolev spaces : A Pseudo-differential operator approach

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Abstract

This paper introduces the generalized fractional Hankel-Clifford convolution and translation. It also focuses on pseudo-differential operators that satisfy a certain $L^1_{\alpha,\beta,a,b}$-norm inequality when using the fractional power of generalized Hankel- Clifford transforms. And the properties of Sobolev-type space $\mathcal{G}^s_{\alpha,\beta,a,b}$ are studied.

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1. Introduction

Integral transforms have been essential analytical tools in mathematics. They are widely used to solve problems in differential equations, function spaces, and operator analysis. Past research has built a strong foundation for these mathematical transformations along with the pseudo-differential operator. One such transform is the fractional Hankel-Clifford transform, which extends the classical Hankel-Clifford transform and interest in research due to its applications in advanced mathematical analysis and applied sciences.Some scholars who have made significant contributions are Kerr [4], Malgonde and Bandewar [5], Prasad et al. [7],[8], Waphare et al. [6],[10],[13] and Zemanian [15], etc. They expanded

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the classical Hankel transform to include fractional powers, distributions, and generalization. Which motivates this study.

Our recent studies have introduced fractional generalized Hankel-Clifford transforms and their differential operators[11]. Also, examine their characteristics[12]. We also study pseudo-differential operator and their properties associated with the same transform (under communication).

In this study, we introduce the idea of fractional generalized Hankel-Clifford convolution and translation. These operations are key to studying pseudo-differential operators in certain $L^1_{\alpha,\beta,a,b}$-norm inequalities when analyzed using the fractional power of the generalized Hankel-Clifford transform. We also studied Sobolev-type spaces, denoted $\mathcal{G}^s_{\alpha,\beta,a,b}$.

For additional background on operator-based studies in analytic function theory, see [1],[3].

The future possibilities for this research are broad, with potential applications in areas like harmonic analysis, solving differential equations, and mathematical physics. This study will open the door to new ideas and practical methods for addressing complex problems in modern Mathematics and Science.

Let us begin directly by introducing the fractional power of generalized Hankel-Clifford transformations [11] corresponding to the order $a-b$ where the parameters α and β determine the structure of the transformation

$$(h^\theta_{1,\alpha,\beta,a,b}\,\varphi)(y) = \hat{\varphi}^\theta_{1,\alpha,\beta,a,b}\,(y) = \int_0^\infty K^\theta_1(\eta,y)\,\varphi(\eta)\,d\eta, \tag{1}$$

where,

$$K^\theta_1(\eta,y) = \begin{cases} \gamma_{a,b,\theta}\, y^{\alpha-\beta}\, e^{\left(\frac{i}{2}(\eta^2+y^2)\cot\theta\right)} C_{\alpha,\beta,a,b}(\eta y \csc\theta) & \theta \neq n\pi, \\ y^{\alpha-\beta}\, C_{\alpha,\beta,a,b}(\eta y) & \theta = \pi/2, \\ \delta(\eta - y) & \theta = n\pi, \end{cases} \tag{2}$$

and $C_{\alpha,\beta,a,b}(\eta) = (\eta)^{(\beta-\alpha)/2} J_{a-b}(2\sqrt{\eta})$; $\gamma_{a,b,\theta} = \exp[i(a-b+1)(\pi/2-\theta)]$.

Similarly,

$$(h^\theta_{2,\alpha,\beta,a,b}\,\psi)(y) = \tilde{\psi}^\theta_{2,\alpha,\beta,a,b}\,(y) = \int_0^\infty K^\theta_2(\eta,y)\,\psi(\eta)\,d\eta, \tag{3}$$

where,

$$K^\theta_2(\eta,y) = \begin{cases} \gamma_{a,b,\theta}\, \eta^{\alpha-\beta}\, e^{\left(\frac{i}{2}(\eta^2+y^2)\cot\theta\right)} C_{\alpha,\beta,a,b}(\eta y \csc\theta) & \theta \neq n\pi, \\ \eta^{\alpha-\beta}\, C_{\alpha,\beta,a,b}(\eta y) & \theta = \pi/2, \\ \delta(\eta - y) & \theta = n\pi. \end{cases} \tag{4}$$

Note that

$$(h^\theta_{2,\alpha,\beta,a,b}\,\psi)(y) = y^{\alpha-\beta}\,(h^\theta_{1,\alpha,\beta,a,b}\,\varphi)(y). \tag{5}$$

The inverse are defined as,

$$\varphi = \left((h^\theta_{1,\alpha,\beta,a,b})^{-1}\,\hat{\varphi}^\theta_{1,\alpha,\beta,a,b}\right)(\eta) = \int_0^\infty \overline{K^\theta_1}(y,\eta)\,\hat{\varphi}^\theta_{1,\alpha,\beta,a,b}\,(y)\,dy, \tag{6}$$

$$\psi = \left((h^\theta_{2,\alpha,\beta,a,b})^{-1}\,\tilde{\psi}^\theta_{2,\alpha,\beta,a,b}\right)(\eta) = \int_0^\infty \overline{K^\theta_2}(y,\eta)\,\tilde{\psi}^\theta_{2,\alpha,\beta,a,b}\,(y)\,dy, \tag{7}$$

where, $\overline{K_1^\theta}$ and $\overline{K_2^\theta}$ are conjugates of K_1^θ and K_2^θ respectively.

1.1 *Differential Operators*

Let us introduce some differential operators associated with the fractional power of generalized Hankel-Clifford transforms from [11], followed by Torre [9] as

$$R_{1,\alpha,\beta,a,b,\theta} = \eta^{\frac{\alpha+a-\beta-b+1}{2}} e^{\left(\frac{i}{2}\eta^2 \cot\theta\right)} D_\eta\, \eta^{\frac{-\alpha-a+\beta+b}{2}} e^{\left(-\frac{i}{2}x^2 \cot\theta\right)},$$
$$S_{1,\alpha,\beta,a,b,\theta} = \eta^{\frac{\alpha-a-\beta+b}{2}} e^{\left(\frac{i}{2}\eta^2 \cot\theta\right)} D_\eta\, \eta^{\frac{-\alpha+a+\beta-b+1}{2}} e^{\left(-\frac{i}{2}\eta^2 \cot\theta\right)}.$$

Also, the Kepinski-type differential operator is defined as

$$\Delta_{1,\alpha,\beta,a,b,\theta} = S_{1,\alpha,\beta,a,b,\theta}\, R_{1,\alpha,\beta,a,b,\theta}$$

Similarly,

$$R_{2,\alpha,\beta,a,b,\theta} = -\eta^{\frac{-\alpha-a+\beta+b}{2}} e^{\left(\frac{i}{2}\eta^2 \cot\theta\right)} D_\eta\, \eta^{\frac{\alpha+a-\beta-b+1}{2}} e^{\left(-\frac{i}{2}\eta^2 \cot\theta\right)},$$
$$S_{2,\alpha,\beta,a,b,\theta} = -\eta^{\frac{-\alpha+a+\beta-b+1}{2}} e^{\left(\frac{i}{2}\eta^2 \cot\theta\right)} D_\eta\, \eta^{\frac{\alpha-a-\beta+b}{2}} e^{\left(-\frac{i}{2}\eta^2 \cot\theta\right)}.$$
$$\Delta_{2,\alpha,\beta,a,b,\theta} = R_{2,\alpha,\beta,a,b,\theta}\, S_{2,\alpha,\beta,a,b,\theta}$$

Also, We work within the Zemanian-type spaces $\mathcal{H}^\theta_{1,\alpha,\beta,a,b}(I)$ and $\mathcal{H}^\theta_{2,\alpha,\beta,a,b}(I)$ as motivated by [15] and introduced in our previous work [14].

1.2 *Preliminary*

Here, we state definition, properties and results essential for this paper's main conclusions. These are derived from our research on fractional powers of the generalized Hankel-Clifford transformation, as detailed in our paper titled Pseudo Differential Operator involving Fractional Power of Generalized Hankel-Clifford Type Transformations which is under communication.

Proposition 1.1: For $\alpha, \beta \in \mathbb{R}, a \geq b$ and all $\varphi \in \mathcal{H}^\theta_{1,\alpha,\beta,a,b}(I)$

i. $h^\theta_{1,\alpha,\beta,a,b}(\overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\, \varphi)(y) = -(y\csc\theta)\, h^\theta_{1,\alpha,\beta,a,b}\varphi(y)$,

ii. $h^\theta_{1,\alpha,\beta,a,b}\big((\overline{\Delta}_{1,\alpha,\beta,a,b,\theta})^r\, \varphi\big)(y) = (-y\csc\theta)^r\, h^\theta_{1,\alpha,\beta,a,b}\varphi(y)$.

Definition 1.2: Given a function $\varphi \in \mathcal{H}^\theta_{1,\alpha,\beta,a,b}(I)$, we define the pseudo-differential type operator $h^\theta_{1,\alpha,\beta,a,b,p}$, associated with the symbol $p(\eta, y)$, as follows:

$$\big(h^\theta_{1,\alpha,\beta,a,b,p}\, \varphi\big)(\eta) = \int_0^\infty \overline{K_1^\theta}(y,\eta)\, p(\eta,y)\hat{\varphi}^\theta_{1,\alpha,\beta,a,b}\,(y)\, dy.$$

Similarly,

Definition 1.3: Given a function $\psi \in \mathcal{H}^\theta_{2,\alpha,\beta,a,b}(I)$, we define the pseudo-differential type operator $h^\theta_{2,\alpha,\beta,a,b,p}$, associated with the symbol $p(\eta, y)$, as follows:

$$\left(h^{\theta}_{2,\alpha,\beta,a,b,p}\,\psi\right)(\eta)=\int_0^\infty \overline{K_2^{\theta}}(y,\eta)\,p(\eta,y)\tilde{\psi}^{\theta}_{2,\alpha,\beta,a,b}\,(y)\,dy.$$

Theorem 1.4: *For* $\varphi\in\mathcal{H}^{\theta}_{1,\alpha,\beta.a,b}(I)$, *The pseudo-differential operator* $h^{\theta}_{1,\alpha,\beta,a,b,p}$ *associated with the symbol* $p(\eta,y)\in\mathcal{H}^{l}$ *can be represented as*

$$\left(h^{\theta}_{1,\alpha,\beta,a,b,p}\,\varphi\right)=\overline{\gamma}_{a,b,\theta}\,\eta^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a,b}(\eta y\csc\theta)e^{-\frac{i}{2}(\eta^2+y^2)\cot\theta}$$
$$\times\left\{\overline{\gamma}_{a,b,\theta}\int_0^\infty \mathcal{Q}^{\theta}_{1,\alpha,\beta.a,b}(y,z)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}\,(z)\,dz\right\}dy.$$

Where all the integrals are convergent.

2. Generalized fractional Hankel-Clifford convolution and translation

Consider $\Delta(a,b,c)$ area of a triangle with sides a,b,c from [2],[14]. If Δ exists then for $\alpha,\beta\in\mathbb{R}, a-b\geq 0$

$$D^{\theta}_{\alpha,\beta,a,b}(x,y,z)=\frac{\overline{\gamma}_{a,b,\theta}\,\Delta^{2(a-b)-1}\,e^{-\frac{i}{2}(x^2+y^2+z^2)\cot\theta}}{2^{2(a-b)}\,(\csc\theta)^{\frac{3\alpha-3\beta-a+b+2}{2}}\,(xyz)^{\frac{\alpha-\beta+a-b}{2}}\,\Gamma(a-b+1/2)\sqrt{\pi}},\tag{8}$$

otherwise, it's zero. Observe that $D^{\theta}_{\alpha,\beta}(x,y,z)$ is symmetric in x,y,z and positive.

Lemma 2.1: *Consider* $a-b\geq 0$, *then*

$$\gamma_{a,b,\theta}\int_0^\infty C_{\alpha,\beta,a,b}(z\zeta\csc\theta)\,e^{\frac{i}{2}(z^2+\zeta^2)\cot\theta}\,D^{\theta}_{\alpha,\beta,a,b}(x,y,z)\,z^{\alpha-\beta}dz$$
$$=e^{-\frac{i}{2}(x^2+y^2-\zeta^2)\cot\theta}\,\zeta^{\alpha-\beta-a+b}C_{\alpha,\beta,a,b}(x\zeta\csc\theta)\,C_{\alpha,\beta,a,b}(y\zeta\csc\theta).$$

Also

$$\gamma_{a,b,\theta}\Gamma(a-b+1)\int_0^\infty e^{\frac{i}{2}z^2\cot\theta}\,D^{\theta}_{\alpha,\beta,a,b}(x,y,z)z^{\frac{\alpha-\beta+a-b}{2}}dz$$
$$=e^{-\frac{i}{2}(x^2+y^2)\cot\theta}\,(xy\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}.$$

Proof: Following Watson [14], let

$$\int_0^\infty J_{a,b}(x\zeta)J_{a,b}(y\zeta)J_{a,b}(z\zeta)\zeta^{1+a-b}d\zeta=\frac{2^{a-b-1}\,\Delta^{2a-2b-1}}{(xyz)^{a-b}\,\Gamma(a-b+1/2)\sqrt{\pi}}.$$

Substituting , $x=2\sqrt{p}, y=2\sqrt{q}, z=2\sqrt{r}$, we get

$$\int_0^\infty J_{a,b}(2\sqrt{p}\,\zeta)J_{a,b}(2\sqrt{q}\,\zeta)J_{a,b}(2\sqrt{r}\,\zeta)\zeta^{1-a+b}d\zeta=\frac{2^{a-b-1}\,\Delta^{2a-2b-1}}{2^{3a-3b}(pqr)^{(a-b)/2}\,\Gamma(a-b+1/2)\sqrt{\pi}}.$$

Also $\zeta=\sqrt{s}$,

$$\int_0^\infty J_{a,b}(2\sqrt{ps})J_{a,b}(2\sqrt{qs})J_{a,b}(2\sqrt{rs})s^{(-a+b)/2}ds=\frac{\Delta^{2a-2b-1}}{2^{2a-2b}(pqr)^{(a-b)/2}\,\Gamma(a-b+1/2)\sqrt{\pi}}.$$

Next,

$$\int_0^\infty J_{a,b}(2\sqrt{ps\csc\theta})\,(\sqrt{ps\csc\theta})^{(-\alpha+\beta)/2}\,J_{a,b}(2\sqrt{qs\csc\theta})\,(\sqrt{qs\csc\theta})^{(-\alpha+\beta)/2}$$

$$J_{a,b}(2\sqrt{rs\ \csc\theta})\ (\sqrt{rs\ \csc\theta})^{(-\alpha+\beta)/2}\ s^{(-a+b)/2}ds$$
$$= \frac{\Delta^{2a-2b-1}}{2^{2a-2b}(\csc\theta)^{\frac{3\alpha-3\beta-a+b+2}{2}}(pqr)^{(\alpha-\beta+a-b)/2}\ \Gamma(a-b+1/2)\sqrt{\pi}}.$$

From (2), we get

$$\overline{\gamma}_{a,b,\theta}\ e^{-\frac{i}{2}(p^2+q^2+r^2)\cot\theta}\int_0^\infty C_{\alpha,\beta,a,b}(ps\ \csc\theta)\ C_{\alpha,\beta,a,b}(qs\ \csc\theta)\ C_{\alpha,\beta,a,b}(rs\ \csc\theta)\ s^{\frac{3\alpha-3\beta-a+b}{2}}ds$$
$$= \frac{\overline{\gamma}_{a,b,\theta}\ e^{-\frac{i}{2}(p^2+q^2+r^2)\cot\theta}\Delta^{2a-2b-1}}{2^{2a-2b}(\csc\theta)^{\frac{3\alpha-3\beta-a+b+2}{2}}(pqr)^{(\alpha-\beta+a-b)/2}\ \Gamma(a-b+1/2)\sqrt{\pi}}$$
$$= D^{\theta}_{\alpha,\beta,a,b}(p,q,r).$$

By inversion formula (6) gives,

$$\gamma_{a,b,\theta}\int_0^\infty C_{\alpha,\beta,a,b}(z\zeta\csc\theta)\ e^{\frac{i}{2}(z^2+\zeta^2)\cot\theta}\ D^{\theta}_{\alpha,\beta,a,b}(x,y,z)\ z^{\alpha-\beta}dz$$
$$= e^{-\frac{i}{2}(x^2+y^2-\zeta^2)\cot\theta}\ \zeta^{\alpha-\beta-a+b}C_{\alpha,\beta,a,b}(x\zeta\csc\theta)\ C_{\alpha,\beta,a,b}(y\zeta\csc\theta),$$

setting ζ at zero , we get

$$\gamma_{a,b,\theta}\Gamma(a-b+1)\int_0^\infty e^{\frac{i}{2}z^2\cot\theta}\ D^{\theta}_{\alpha,\beta,a,b}(x,y,z)z^{\frac{\alpha-\beta+a-b}{2}}dz$$
$$= e^{-\frac{i}{2}(x^2+y^2)\cot\theta}\ (xy\ \csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}.$$

Hence Proved.

Definition 2.2: For $1 \le p < \infty, L^p_{\alpha,\beta,a,b}(I)$ is the space of all Lebesgue measurable function φ on I such that

$$||\varphi||_{L^p_{\alpha,\beta,a,b}} = \left(\int_0^\infty \eta^{\frac{\alpha+a-\beta-b}{2}}\ |\varphi(\eta)|^p\ d\eta\right)^{1/p} < \infty.$$

Definition 2.3: Let $\phi \in L^1_{\alpha,\beta,a,b}(I)$ and $0 < x, y < \infty$ then Hankel translation is denoted by $(\tau^{\theta}_x\phi)(y)$ defined as

$$(\tau^{\theta}_x\varphi)(y) = \gamma_{a,b,\theta}\int_0^\infty e^{\frac{i}{2}z^2\cot\theta}\ D^{\theta}_{\alpha,\beta,a,b}(x,y,z)\ \varphi(z)z^{\frac{\alpha-\beta+a-b}{2}}dz.$$

Observe that $(\tau^{\theta}_x\varphi)(y) = (\tau^{\theta}_y\varphi)(x)$.

Definition 2.4: Let $\phi, \psi \in L^1_{\alpha,\beta,a,b}(I)$ and $0 < x < \infty$ then Hankel convolution denoted by $(\phi\ \#_{\theta}\ \psi)(x)$ is

$$(\varphi\ \#_{\theta}\ \psi)(x) = \gamma_{a,b,\theta}\int_0^\infty e^{\frac{i}{2}y^2\cot\theta}(\tau^{\theta}_x\varphi)(y)\ \psi(y)y^{\frac{\alpha-\beta+a-b}{2}}dy.$$

Lemma 2.5: *Let* $(.)^{\frac{-\alpha+a\beta-b}{2p}}\varphi \in L^p_{\alpha,\beta,a,b}(I)$ *then*

$$\left\| y^{\frac{\alpha-a-\beta+b}{2q}} (\tau_x^\theta \varphi)(y) \right\|_{L^p_{\alpha,\beta,a,b}} \le \frac{(x\csc\theta)^{\frac{-\alpha+a+\beta-b}{2}}}{\Gamma(a-b+1)} \left\| z^{\frac{-\alpha+\beta+a-b}{2p}} \varphi(z) \right\|_{L^p_{\alpha,\beta,a,b}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: From Lemma 2.1, Definition 2.3 and Hölder's inequality, we have

$$\begin{aligned}
\left|(\tau_x^\theta \varphi)(y)\right| &= \left|\gamma_{a,b,\theta} \int_0^\infty z^{\frac{\alpha-\beta+a-b}{2}} e^{\frac{i}{2}z^2\cot\theta} D^\theta_{\alpha,\beta,a,b}(x,y,z)\, \varphi(z) dz\right| \\
&= \left|\int_0^\infty \left(z^{\frac{\alpha-\beta+a-b}{2}} e^{\frac{i}{2}z^2\cot\theta} D^\theta_{\alpha,\beta,a,b}(x,y,z)\right)^{1/p}\right. \\
&\quad \times \left.\left(z^{\frac{\alpha-\beta+a-b}{2}} e^{\frac{i}{2}z^2\cot\theta} D^\theta_{\alpha,\beta,a,b}(x,y,z)\right)^{1/q} \varphi(z) dz\right| \\
&\le \left(\int_0^\infty z^{\frac{\alpha-\beta+a-b}{2}} D^\theta_{\alpha,\beta,a,b}(x,y,z) |\varphi(z)|^p\right)^{1/p} \\
&\quad \times \left(\int_0^\infty z^{\frac{\alpha-\beta+a-b}{2}} D^\theta_{\alpha,\beta,a,b}(x,y,z)\right)^{1/q}.
\end{aligned}$$

Again using Lemma 2.1

$$\left| y^{\frac{\alpha-\beta-a+b}{2q}} (\tau_x^\theta \varphi)(y) \right|^p \le \left(\frac{(x\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\right)^{p/q} \int_0^\infty z^{\frac{\alpha-\beta+a-b}{2}} D^\theta_{\alpha,\beta,a,b}(x,y,z)\, |\varphi(z)|^p\, dz.$$

Hence,

$$\begin{aligned}
&\int_0^\infty \left| y^{\frac{\alpha-\beta-a+b}{2q}} (\tau_x^\theta \varphi)(y) \right|^p y^{\frac{\alpha-\beta+a-b}{2}} dy \\
&\le \left(\frac{(x\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\right)^{p/q} \int_0^\infty |\varphi(z)|^p z^{\frac{\alpha-\beta+a-b}{2}} \int_0^\infty y^{\frac{\alpha-\beta+a-b}{2}} D^\theta_{\alpha,\beta,a,b}(x,y,z)\, dy\, dz \\
&\le \left(\frac{(x\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\right)^{\frac{p}{q}+1} \int_0^\infty |z^{\frac{-\alpha+\beta+a-b}{2p}} \varphi(z)|^p z^{\frac{\alpha-\beta+a-b}{2}} dz.
\end{aligned}$$

Therefore, we conclude

$$\left\| y^{\frac{\alpha-\beta-a+b}{2q}} (\tau_x^\theta \varphi)(y) \right\|_{L^p_{\alpha,\beta,a,b}} \le \frac{(x\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)} \left\| z^{\frac{-\alpha+\beta+a-b}{2p}} \varphi(z) \right\|_{L^p_{\alpha,\beta,a,b}}.$$

Lemma 2.6: *Let* $(.)^{\frac{-\alpha+\beta+a-b}{2p}} \varphi \in L^p_{\alpha,\beta,a,b}(I)$ *then*

$$\left\| x^{\frac{\alpha-\beta-a+b}{2q}} (\tau_y^\theta \varphi)(x) \right\|_{L^p_{\alpha,\beta,a,b}} \le \frac{(y\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)} \left\| z^{\frac{-\alpha+\beta+a-b}{2p}} \varphi(z) \right\|_{L^p_{\alpha,\beta,a,b}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 2.7: *Let* $(.)^{\frac{-\alpha+\beta+a-b}{2}}\varphi \in L^1_{\alpha,\beta,a,b}(I)$ *and* $(.)^{\frac{-\alpha+\beta+a-b}{2p}}\psi \in L^p_{\alpha,\beta,a,b}(I)$ *then*

$$\left\| x^{\frac{\alpha-\beta-a+b}{2q}}(\varphi \#_\theta \psi)\right\|_{L^p_{\alpha,\beta,a,b}} \leq$$

$$\frac{(\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\left\| z^{\frac{-\alpha+\beta+a-b}{2}}\varphi(z)\right\|_{L^1_{\alpha,\beta,a,b}}\left\| y^{\frac{-\alpha+\beta+a-b}{2p}}\psi(y)\right\|_{L^p_{\alpha,\beta,a,b}},$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof: Using Lemma 2.5 for p=1 on Definition 2.4 and Holder's inequality, we obtain

$$|(\varphi \#_\theta \psi)(x)| = \left|\gamma_{a,b,\theta}\int_0^\infty y^{\frac{\alpha-\beta+a-b}{2}} e^{\frac{i}{2}y^2\cot\theta}(\tau_x^\theta\varphi)(y)\,\psi(y)dy\right|$$

$$= \left|\int_0^\infty \left(y^{\frac{\alpha-\beta+a-b}{2}} e^{\frac{i}{2}y^2\cot\theta}(\tau_x^\theta\varphi)(y)\right)^{1/p}\left(y^{\frac{\alpha-\beta+a-b}{2}} e^{\frac{i}{2}y^2\cot\theta}(\tau_x^\theta\varphi)(y)\right)^{1/q}\psi(y)dy\right|$$

$$\leq \left(\int_0^\infty |\psi(y)|^p(\tau_x^\theta\varphi)(y)y^{\frac{\alpha-\beta+a-b}{2}}dy\right)^{1/p}\left(\int_0^\infty (\tau_x^\theta\varphi)(y)y^{\frac{\alpha-\beta+a-b}{2}}dy\right)^{1/q}$$

$$\leq \left(\int_0^\infty |\psi(y)|^p(\tau_x^\theta\varphi)(y)y^{\frac{\alpha-\beta+a-b}{2}}dy\right)^{1/p}$$

$$\times\left(\frac{(x\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\left\| z^{\frac{-\alpha+\beta+a-b}{2}}\varphi(z)\right\|_{L^1_{\alpha,\beta,a,b}}\right)^{1/q}.$$

By Lemma 2.6

$$\int_0^\infty \left| x^{\frac{\alpha-\beta-a+b}{2q}}(\varphi \#_\theta \psi)(x)\right|^p x^{\frac{\alpha-\beta+a-b}{2}}dx$$

$$\leq \left(\frac{(\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\left\| z^{\frac{-\alpha+\beta+a-b}{2}}\varphi(z)\right\|_{L^1_{\alpha,\beta,a,b}}\right)^{p/q}\int_0^\infty\int_0^\infty |\psi(y)|^p(\tau_x^\theta\varphi)(y)(xy)^{\frac{\alpha-\beta+a-b}{2}}dydx$$

$$\leq \left(\frac{(\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\left\| z^{\frac{-\alpha+\beta+a-b}{2}}\varphi(z)\right\|_{L^1_{\alpha,\beta,a,b}}\right)^{p/q}\int_0^\infty |\psi(y)|^p y^{\frac{\alpha-\beta+a-b}{2}}dy$$

$$\times\int_0^\infty (\tau_x^\theta\varphi)(x)x^{\frac{\alpha-\beta+a-b}{2}}dx.$$

Hence, from Lemma 2.5,

$$\left\| x^{\frac{\alpha-\beta-a+b}{2q}}(\varphi \#_\theta \psi)\right\|_{L^p_{\alpha,\beta,a,b}} \leq$$

$$\frac{(\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\left\| z^{\frac{-\alpha+\beta+a-b}{2}}\varphi(z)\right\|_{L^1_{\alpha,\beta,a,b}}\left\| y^{\frac{-\alpha+\beta+a-b}{2p}}\psi(y)\right\|_{L^p_{\alpha,\beta,a,b}}.$$

Proposition 2.8: Let $\varphi,\psi \in L^1_{\alpha,\beta,a,b}(I)$ then

$$h^{\theta}_{1,\alpha,\beta,a,b}\left(y^{\alpha-\beta}\,(\tau^{\theta}_{x}\varphi)\right)(\zeta) = e^{-\frac{i}{2}x^2\cot\theta}\,\zeta^{\frac{\alpha-\beta-a+b}{2}}C_{\alpha,\beta,a,b}(x\zeta\csc\theta) \times h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\varphi(z)\right)(\zeta),$$

and

$$h^{\theta}_{2,\alpha,\beta,a,b}\left(\tau^{\theta}_{x}\psi\right)(\zeta) = e^{-\frac{i}{2}x^2\cot\theta}\,\zeta^{\frac{\alpha-\beta-a+b}{2}}C_{\alpha,\beta,a,b}(x\zeta\csc\theta) \times h^{\theta}_{2,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{-\alpha+\beta+a-b}{2}}\psi(z)\right)(\zeta).$$

Proof: Given that $\varphi \in L^1_{\alpha,\beta,a,b}(I)$,we apply Lemma 2.1 in conjunction with Definition 2.3 to derive the following result

$$\begin{aligned}
&h^{\theta}_{1,\alpha,\beta,a,b}\left(y^{\alpha-\beta}\,(\tau^{\theta}_{x}\varphi)\right)(\zeta)\\
&= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta y\csc\theta)e^{\left(\frac{i}{2}(\zeta^2+y^2)\cot\theta\right)}\,y^{\alpha-\beta}(\tau^{\theta}_{x}\varphi)(y)dy\\
&= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta y\csc\theta)e^{\left(\frac{i}{2}(\zeta^2+y^2)\cot\theta\right)}\,y^{\alpha-\beta}\\
&\quad\times\gamma_{a,b,\theta}\int_0^\infty z^{\frac{\alpha-\beta+a-b}{2}}e^{\frac{i}{2}z^2\cot\theta}\,D^{\theta}_{\alpha,\beta,a,b}(x,y,z)\,\varphi(z)dz\\
&= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^\infty\left[\gamma_{a,b,\theta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta y\csc\theta)e^{\left(\frac{i}{2}(\zeta^2+y^2)\cot\theta\right)}\,y^{\alpha-\beta}D^{\theta}_{\alpha,\beta,a,b}(x,y,z)dy\right]\\
&\quad\times e^{\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\varphi(z)dz\\
&= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^\infty e^{\left(-\frac{i}{2}(z^2+x^2)\cot\theta\right)}C_{\alpha,\beta,a,b}(\zeta y\csc\theta)C_{\alpha,\beta,a,b}(\zeta x\csc\theta)C_{\alpha,\beta,a,b}(\zeta z\csc\theta)\\
&\quad\times e^{\frac{i}{2}\zeta^2\cot\theta}\zeta^{\frac{\alpha-\beta-a+b}{2}}\,e^{\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\varphi(z)dz\\
&= e^{-\frac{i}{2}x^2\cot\theta}\,\zeta^{\frac{\alpha-\beta-a+b}{2}}C_{\alpha,\beta,a,b}(x\zeta\csc\theta)h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\varphi(z)\right)(\zeta).
\end{aligned}$$

Hence done. Clearly, the second part of this proposition can be proved using (5).

Proposition 2.9: Let $\varphi,\psi \in L^1_{\alpha,\beta,a,b}(I)$ then

$$\begin{aligned}
&h^{\theta}_{1,\alpha,\beta,a,b}\left(x^{\alpha-\beta}\,(\varphi\,\#_{\theta}\,\psi)\right)(\zeta) = e^{-\frac{i}{2}\zeta^2\cot\theta}\,\zeta^{-\frac{\alpha-\beta-a+b}{2}}\\
&\times h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}\varphi(y)\right)(\zeta)h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\psi(z)\right)(\zeta),
\end{aligned}$$

and

$$\begin{aligned}
&h^{\theta}_{2,\alpha,\beta,a,b}(\varphi\,\#_{\theta}\,\psi)(\zeta) = e^{-\frac{i}{2}\zeta^2\cot\theta}\,\zeta^{-\frac{\alpha-\beta-a+b}{2}}\\
&\times h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}\varphi(y)\right)(\zeta)h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\psi(z)\right)(\zeta),
\end{aligned}$$

Proof: Given that $\varphi, \psi \in L^1_{\alpha,\beta,a,b}(I)$ then by using Lemma 2.1 and Definition 2.4, we obtain

$$
\begin{aligned}
&h^{\theta}_{1,\alpha,\beta,a,b}\big(x^{\alpha-\beta}\,(\varphi\,\#_\theta\,\psi)\big)(\zeta)\\
&= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta x\csc\theta)e^{\left(\frac{i}{2}(\zeta^2+x^2)\cot\theta\right)}\,x^{\alpha-\beta}(\varphi\,\#_\theta\,\psi)(x)dx\\
&= (\gamma_{a,b,\theta})^3\zeta^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta x\csc\theta)e^{\left(\frac{i}{2}(\zeta^2+x^2)\cot\theta\right)}\,x^{\alpha-\beta}\int_0^\infty\int_0^\infty D^{\theta}_{\alpha,\beta,a,b}(x,y,z)\\
&\quad\times e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}\varphi(y)z^{\frac{\alpha-\beta+a-b}{2}}e^{\frac{i}{2}z^2\cot\theta}\,\psi(z)dydzdx\\
&= (\gamma_{a,b,\theta})^3\zeta^{\alpha-\beta}\int_0^\infty\int_0^\infty\left[\int_0^\infty C_{\alpha,\beta,a,b}(\zeta x\csc\theta)e^{\frac{i}{2}(\zeta^2+x^2)\cot\theta}x^{\alpha-\beta}D^{\theta}_{\alpha,\beta,a,b}(x,y,z)dx\right]\\
&\quad\times e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}\varphi(y)z^{\frac{\alpha-\beta+a-b}{2}}e^{\frac{i}{2}z^2\cot\theta}\,\psi(z)dydz\\
&= (\gamma_{a,b,\theta})^2\zeta^{\alpha-\beta}\int_0^\infty\int_0^\infty\left[e^{-\frac{i}{2}(y^2+x^2)\cot\theta}C_{\alpha,\beta,a,b}(\zeta y\csc\theta)C_{\alpha,\beta,a,b}(\zeta z\csc\theta)e^{\frac{i}{2}\zeta^2\cot\theta}\right.\\
&\left.\times\,\zeta^{\frac{\alpha-\beta-a+b}{2}}\right]e^{\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}\varphi(y)z^{\frac{\alpha-\beta+a-b}{2}}e^{\frac{i}{2}z^2\cot\theta}\,\psi(z)dydz\\
&= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\zeta^{-\frac{\alpha-\beta+a-b}{2}}e^{-\frac{i}{2}\zeta^2\cot\theta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta y\csc\theta)e^{\frac{i}{2}(y^2+\zeta^2)\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}e^{\frac{i}{2}y^2\cot\theta}\\
&\times\varphi(y)dy\,\gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^\infty C_{\alpha,\beta,a,b}(\zeta z\csc\theta)e^{\frac{i}{2}(z^2+\zeta^2)\cot\theta}e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\psi(z)dz\\
&= e^{-\frac{i}{2}\zeta^2\cot\theta}\,\zeta^{-\frac{\alpha-\beta-a+b}{2}}h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}y^2\cot\theta}y^{\frac{\alpha-\beta+a-b}{2}}\varphi(y)\right)(\zeta)\\
&\times h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}z^2\cot\theta}z^{\frac{\alpha-\beta+a-b}{2}}\psi(z)\right)(\zeta),
\end{aligned}
$$

Hence done. Clearly, the second part of this proposition can be proved using (5).

3. $L^1_{\alpha,\beta,a,b}$-norm inequality and Sobolev-type spaces

Let us define $\mathcal{A}^{\theta}_{1,\alpha,\beta,a,b}(\zeta,y)$ for $0<x<\infty$ as

$$\mathcal{A}^{\theta}_{1,\alpha,\beta,a,b}(\zeta,y) = h^{\theta}_{1,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}x^2\cot\theta}x^{\frac{\alpha-\beta+a-b}{2}}p(x,y)\right)(\zeta).$$

Similarly

$$\mathcal{A}^{\theta}_{2,\alpha,\beta,a,b}(\zeta,y) = h^{\theta}_{2,\alpha,\beta,a,b}\left(e^{-\frac{i}{2}x^2\cot\theta}x^{\frac{-\alpha+\beta+a-b}{2}}p(x,y)\right)(\zeta), x\in(0,\infty).$$

Lemma 3.1: *For $a-b\geq 0$ and $r,l\in\mathbb{N}_0$, there exists a constant $K_{1,a,b,r,m,l}>0$ such that*

$$\left|\mathcal{A}^{\theta}_{1,\alpha,\beta,a,b}(\zeta,y)\right|\leq K_{1,a,b,r,m,l}\,(1+y)^l\,(1+(\zeta\csc\theta)^r)^{-1}\,\zeta^{\frac{\alpha-\beta+a-b}{2}}.$$

Proof: Using Proposition 1.1(ii), Theorem 1 of Sec. 3 of [11] and for $r\in\mathbb{N}_0$, we have

$(-\zeta\ \mathrm{csc}\theta)^r\ \mathcal{A}^{\theta}_{1,\alpha,\beta.a,b}(\zeta,y)$

$= h^{\theta}_{1,\alpha,\beta,a,b}\left(\left(\overline{\Delta}_{1,\alpha,\beta,a,b,\theta}\right)^r\ e^{-\frac{i}{2}x^2\mathrm{cot}\theta}x^{\frac{\alpha-\beta+a-b}{2}}p(x,y)\right)(y)$

$= \gamma_{a,b,\theta}\ \zeta^{\alpha-\beta}\int_0^{\infty}\ C_{\alpha,\beta,a,b}(\zeta x\ \mathrm{csc}\theta)e^{\left(\frac{i}{2}(x^2+\zeta^2)\ \mathrm{cot}\theta\right)}\sum_{i=0}^{n}\ a_i\ e^{\left(-\frac{i}{2}x^2\ cot\theta\right)}\ x^{\frac{\alpha+a-\beta-b}{2}+i}\ D_x^{r+i}$

$\times\, x^{-\frac{\alpha+a-\beta-b}{2}}\ e^{\left(\frac{i}{2}x^2\ cot\theta\right)}\ e^{-\frac{i}{2}x^2\mathrm{cot}\theta}x^{\frac{\alpha-\beta+a-b}{2}}p(x,y)dx$

$= \gamma_{a,b,\theta}\zeta^{\alpha-\beta}\int_0^{\infty}\ (\zeta x\mathrm{csc}\theta)^{\frac{-\alpha+a+\beta-b}{2}}C_{a,b}(\zeta x\mathrm{csc}\theta)e^{\frac{i}{2}\zeta^2\mathrm{cot}\theta}\sum_{i=0}^{n}\ a_i x^{\frac{\alpha+a-\beta-b}{2}+i}D_x^{r+i}p(x,y)dx.$

By Definition 2.2 form [12]

$\left|(\zeta\mathrm{csc}\ \theta)^r\mathcal{A}^{\theta}_{1,\alpha,\beta.a,b}(\zeta,y)\right|$

$\leq |\zeta|^{\frac{\alpha-\beta+a-b}{2}}|\mathrm{csc}\ \theta|^{\frac{-\alpha+\beta+a-b}{2}}\int_0^{\infty}\left|C_{a,b}(\zeta x\ \mathrm{csc}\ \theta)\right|\sum_{i=0}^{n} a_i x^{\frac{\alpha+a-\beta-b}{2}+i}\left|D_x^{r+i}p(x,y)\right|dx$

$\leq |\zeta|^{\frac{\alpha-\beta+a-b}{2}}|\mathrm{csc}\ \theta|^{\frac{-\alpha+\beta+a-b}{2}}\sum_{i=0}^{n} a_i L'_{l,m,i+n,a,b,\theta}(1+y)^l\int_0^{\infty}x^{a-b+i}(1+x)^{-m}dx$

$\leq |\zeta|^{\frac{\alpha-\beta+a-b}{2}}|\mathrm{csc}\ \theta|^{\frac{-\alpha+\beta+a-b}{2}}\sum_{i=0}^{n} a_i L'_{l,m,i+n,a,b,\theta}(1+y)^l$

$\beta(a-b+i+1, m-a+b-i-1),$

where, β is Beta function and $L'_{l,m,i+n,a,b,\theta}$ is positive constant.

Provided that $m > r+1$ one can find a constant $K_{1,a,b,r,m,l} > 0$ such that

$$\left|\mathcal{A}^{\theta}_{1,\alpha,\beta,a,b}(\zeta,y)\right| \leq K_{1,a,b,r,m,l}\ (1+y)^l\ (1+(\zeta\mathrm{csc}\theta)^r)^{-1}\ \zeta^{\frac{\alpha-\beta+a-b}{2}}.$$

Hence proved.

An analogous argument leads to the result below.

Lemma 3.2: *For* $a-b \geq 0$ *and* $r', l \in \mathbb{N}_0$, *there exists a constant* $K_{2,a,b,r,m,l} > 0$ *such that*

$$\left|\mathcal{A}^{\theta}_{2,\alpha,\beta,a,b}(\zeta,y)\right| \leq K_{2,a,b,r,m,l}\ (1+y)^l\ (1+(\zeta\mathrm{csc}\theta)^{r'})^{-1}\ \zeta^{\frac{-\alpha+\beta+a-b}{2}}.$$

Definition 3.3: For $s \in \mathbb{R}$, the Sobolev-type space is denoted by $\mathcal{G}^s_{1,\alpha,\beta,a,b}(I)$ and defined as the set of all $\varphi \in \mathcal{H}^{\theta}_{1,\alpha,\beta,a,b}(I)$ such that

$$\|\varphi\|_{\mathcal{G}^s_{1,\alpha,\beta,a,b}} = \left\|y^{\frac{2s-\alpha-a+\beta+b}{2}}h^{\theta}_{1,\alpha,\beta,a,b}(\varphi)\right\|_{L^1_{\alpha,\beta,a,b}} < \infty.$$

Definition 3.4: For $s \in \mathbb{R}$, the Sobolev-type space is denoted by $\mathcal{G}^s_{2,\alpha,\beta,a,b}(I)$ and defined as the set of all $\varphi \in \mathcal{H}^{\theta}_{2,\alpha,\beta,a,b}(I)$ such that

$$\|\psi\|_{\mathcal{G}^s_{2,\alpha,\beta,a,b}} = \left\|y^{\frac{2s+\alpha-a-\beta+b}{2}}h^{\theta}_{2,\alpha,\beta,a,b}(\varphi)\right\|_{L^1_{\alpha,\beta,a,b}} < \infty.$$

Theorem 3.5: *For all* $l \in \mathbb{N}_0$ *and* $a-b \geq 0$, *there exist* $M_{1,a,b,r,m,l}$ *such that*

$$\left\|h^{\theta}_{1,\alpha,\beta,a,b,p}(\varphi)\right\|_{\mathcal{G}^{\frac{a-b-\alpha+\beta}{2}}_{1,\alpha,\beta,a,b}} \leq \frac{M_{1,a,b,r,m,l}\,(\csc\theta)^{\frac{a-b-\alpha+\beta}{2}}}{\Gamma(a-b+1)}\sum_{k=0}^{l}\binom{l}{k}\|\varphi\|_{\mathcal{G}^{\frac{a-b-\alpha+\beta}{2}+k}_{1,\alpha,\beta,a,b}}.$$

Proof: From Theorem 1.4,

$$\left(h^{\theta}_{1,\alpha,\beta,a,b,p}\,\varphi\right)(x) = \overline{\gamma}_{a,b,\theta}\, x^{\alpha-\beta}\int_0^{\infty} C_{\alpha,\beta,a,b}(\zeta x\csc\theta)e^{-\frac{i}{2}(x^2+\zeta^2)\cot\theta} \times \left\{\overline{\gamma}_{a,b,\theta}\int_0^{\infty} \mathcal{Q}^{\theta}_{1,\alpha,\beta.a,b}(y,\zeta)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\,dy\right\}d\zeta.$$

Using Lemma 2.1 and (6), we get

$$\gamma_{a,b,\theta}\,\zeta^{\alpha-\beta}\int_0^{\infty} C_{\alpha,\beta,a,b}(\zeta x\csc\theta)e^{\frac{i}{2}(x^2+\zeta^2)\cot\theta}\left(h^{\theta}_{1,\alpha,\beta,a,b,p}\,\varphi\right)(x)dx$$
$$= \overline{\gamma}_{a,b,\theta}\int_0^{\infty} \mathcal{Q}^{\theta}_{1,\alpha,\beta.a,b}(y,\zeta)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\,dy$$
$$= \overline{\gamma}_{a,b,\theta}\gamma_{a,b,\theta}\,\zeta^{\alpha-\beta}\int_0^{\infty}\int_0^{\infty} C_{\alpha,\beta,a,b}(\zeta x\csc\theta)e^{\frac{i}{2}(x^2+\zeta^2)\cot\theta}C_{\alpha,\beta,a,b}(xy\csc\theta) \times e^{-\frac{i}{2}(x^2+y^2)\cot\theta}p(x,y)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\,dxdy$$
$$= \gamma_{a,b,\theta}\,\zeta^{\alpha-\beta}\csc\theta\int_0^{\infty}\int_0^{\infty} e^{\frac{i}{2}(y^2+\zeta^2-x^2)\cot\theta}x^{\frac{a-b-\alpha+\beta}{2}}\int_0^{\infty} z^{\alpha-\beta}C_{\alpha,\beta,a,b}(xz\csc\theta) \times e^{\frac{i}{2}(x^2+z^2)\cot\theta}D^{\theta}_{\alpha,\beta.a,b}(y,\zeta,z)x^{\alpha-\beta}e^{\frac{i}{2}(\zeta^2-y^2)\cot\theta}p(x,y)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\,dzdxdy$$
$$= \zeta^{\alpha-\beta}\csc\theta\int_0^{\infty}\int_0^{\infty} e^{\frac{i}{2}\zeta^2\cot\theta}\left[\gamma_{a,b,\theta}z^{\alpha-\beta}\int_0^{\infty} C_{\alpha,\beta,a,b}(xz\csc\theta)e^{\frac{i}{2}(x^2+z^2)\cot\theta} \times \left(x^{\frac{a-b+\alpha-\beta}{2}}e^{-\frac{i}{2}x^2\cot\theta}p(x,y)\right)dx\right]\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)D^{\theta}_{\alpha,\beta.a,b}(y,\zeta,z)dzdy$$
$$= \zeta^{\alpha-\beta}\csc\theta\int_0^{\infty}\int_0^{\infty} e^{\frac{i}{2}\zeta^2\cot\theta}\,h^{\theta}_{1,\alpha,\beta,a,b}\left(x^{\frac{a-b+\alpha-\beta}{2}}e^{-\frac{i}{2}x^2\cot\theta}p(x,y)\right)(z)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y) \times D^{\theta}_{\alpha,\beta.a,b}(y,\zeta,z)dzdy$$
$$= \zeta^{\alpha-\beta}\csc\theta\int_0^{\infty}\int_0^{\infty} e^{\frac{i}{2}\zeta^2\cot\theta}\mathcal{A}^{\theta}_{1,\alpha,\beta,a,b}(z,y)\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)D^{\theta}_{\alpha,\beta.a,b}(y,\zeta,z)dzdy$$

From Definition 2.3, Definition 2.4 and Lemma 3.1, we obtain

$$\left|\zeta^{-\alpha+\beta}h^{\theta}_{1,\alpha,\beta,a,b}(h^{\theta}_{1,\alpha,\beta,a,b,p}\,\varphi)(\zeta)\right|$$
$$\leq K_{1,a,b,r,m,l}\int_0^{\infty}(1+y)^l\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\int_0^{\infty}(1+(z\csc\theta)^r)^{-1}z^{\frac{\alpha-\beta+a-b}{2}}D^{\theta}_{\alpha,\beta.a,b}(y,\zeta,z)dzdy$$
$$\leq K_{1,a,b,r,m,l}\int_0^{\infty}(1+y)^l\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)\int_0^{\infty}z^{\frac{\alpha-\beta+a-b}{2}}e^{\frac{i}{2}z^2\cot\theta}D^{\theta}_{\alpha,\beta.a,b}(y,\zeta,z)e^{-\frac{i}{2}z^2\cot\theta} \times (1+(z\csc\theta)^r)^{-1}\,dzdy.$$

Let $\psi_{\theta}(x) = e^{-\frac{i}{2}x^2\cot\theta}(1+(x\csc\theta)^r)^{-1}$ and $\varphi^{\theta,k}_{1,\alpha,\beta,a,b}(x) = x^{\frac{2k-\alpha+\beta-a+b}{2}}e^{-\frac{i}{2}x^2\cot\theta}\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(x)$ then

$$\left|\zeta^{-\alpha+\beta}h^{\theta}_{1,\alpha,\beta,a,b}(h^{\theta}_{1,\alpha,\beta,a,b,p}\varphi)(\zeta)\right|$$
$$\leq K_{1,a,b,r,m,l}(\gamma_{a,b,\theta})^{-1}\sum_{k=0}^{l}\binom{l}{k}y^k\hat{\varphi}^{\theta}_{1,\alpha,\beta,a,b}(y)(\tau^{\theta}_x\psi_{\theta})(y)dy$$

$$\leq K_{1,a,b,r,m,l}(\gamma_{a,b,\theta})^{-2}\sum_{k=0}^{l}\binom{l}{k}\left(\psi_\theta \#_\theta\, \varphi_{1,\alpha,\beta,a,b}^{\theta,k}\right)(\zeta).$$

Now we apply Theorem 2.7 for $p = 1$,

$$\left\|\zeta^{-\alpha+\beta} h_{1,\alpha,\beta,a,b}^{\theta}(h_{1,\alpha,\beta,a,b,p}^{\theta}\varphi)(\zeta)\right\|_{L_{1,\alpha,\beta,a,b}^{1}}$$

$$\leq K_{1,a,b,r,m,l}\sum_{k=0}^{l}\binom{l}{k}\left\|\left(\psi_\theta \#_\theta\, \varphi_{1,\alpha,\beta,a,b}^{\theta,k}\right)(\zeta)\right\|_{L_{1,\alpha,\beta,a,b}^{1}}$$

$$\leq K_{1,a,b,r,m,l}\sum_{k=0}^{l}\binom{l}{k}\frac{(\csc\theta)^{\frac{-\alpha+\beta+a-b}{2}}}{\Gamma(a-b+1)}\left\|z^{\frac{-\alpha+\beta+a-b}{2}}\psi_\theta(z)\right\|_{L_{\alpha,\beta,a,b}^{1}}$$

$$\left\|y^{\frac{-\alpha+\beta+a-b}{2}}\varphi_{1,\alpha,\beta,a,b}^{\theta,k}(y)\right\|_{L_{\alpha,\beta,a,b}^{1}}$$

$$\leq \frac{K_{1,a,b,r,m,l}(\csc\theta)^{\frac{\beta-\alpha+a-b}{2}}}{\Gamma(a-b+1)}\sum_{k=0}^{l}\binom{l}{k}\left\|y^{k-\alpha+\beta}\hat{\varphi}_{1,\alpha,\beta,a,b}^{\theta}(y)\right\|_{L_{\alpha,\beta,a,b}^{1}}$$

$$\times\int_0^\infty\left|z^{\frac{\beta-\alpha+a-b}{2}}e^{-\frac{i}{2}z^2\cot\theta}(1+(z\csc\theta)^r)^{-1}\right|dz$$

$$\leq \frac{K_{1,a,b,r,m,l}(\csc\theta)^{\frac{\beta-\alpha+a-b}{2}}}{\Gamma(a-b+1)}\sum_{k=0}^{l}\binom{l}{k}\left\|y^{k-\alpha+\beta}\hat{\varphi}_{1,\alpha,\beta,a,b}^{\theta}(y)\right\|_{L_{\alpha,\beta,a,b}^{1}}$$

$$\int_0^\infty |z^{a-b}(1+(z\csc\theta)^r)^{-1}|dz.$$

For conclusion, we use Definition 3.3. Also to make integration convergent assume $r > 1 + a - b$.

$$\left\|h_{1,\alpha,\beta,a,b,p}^{\theta}(\varphi)\right\|_{\mathcal{G}_{1,\alpha,\beta,a,b}^{\frac{a-b-\alpha+\beta}{2}}} \leq \frac{M_{1,a,b,r,m,l}\,(\csc\theta)^{\frac{a-b-\alpha+\beta}{2}}}{\Gamma(a-b+1)}\sum_{k=0}^{l}\binom{l}{k}\|\varphi\|_{\mathcal{G}_{1,\alpha,\beta,a,b}^{\frac{a-b-\alpha+\beta}{2}+k}}.$$

For some constant $M_{1,a,b,r,m,l} > 0$. Hence proved.

Analogously, the above theorem also holds for $h_{2,\alpha,\beta,a,b}^{\theta}$ transform.

Theorem 3.6: *For all* $l \in \mathbb{N}_0$ *and* $a - b \geq 0$, *there exist* $M_{2,a,b,r,m,l}$ *such that*

$$\left\|h_{2,\alpha,\beta,a,b,p}^{\theta}(\psi)\right\|_{\mathcal{G}_{1,\alpha,\beta,a,b}^{\frac{a-b-\alpha+\beta}{2}}} \leq \frac{M_{2,a,b,r,m,l}\,(\csc\theta)^{\frac{a-b-\alpha+\beta}{2}}}{\Gamma(a-b+1)}\sum_{k=0}^{l}\binom{l}{k}\|\psi\|_{\mathcal{G}_{2,\alpha,\beta,a,b}^{\frac{a-b-\alpha+\beta}{2}+k}}.$$

Remark 3.7: The study is motivated by Prasad and Kumar [8] and we strongly believe that the result obtained in our study is stronger than it and can be reduced to [8] by putting the value $\alpha = \frac{1}{4} + \frac{\mu}{2}, \beta = \frac{1}{4} - \frac{\mu}{2}$ and $a = \frac{1}{4} + \frac{\nu}{2}, b = \frac{1}{4} - \frac{\nu}{2}$.

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