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Derivation of a new semi-imlicit Runge-Kutta method of contra-harmonic mean with forward-backward technique

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Abstract

In this article, the new method that solves Initial Value Problems IVPs is presented, where we derived our new method by using the Contra Harmonic CH mean instead of the classic arithmetic mean, more than the Forward Backward FB) technique to calculate both sides of the predictor (Forward) part and corrector (Backward) part of our semi-implicit

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method. The method was applied to a solved example with a different step size, and the results with the exact and the classical old method were compared with the old method.

Subject Classification: *65L06, 65L05.*

Keywords: *Initial value problems, Implicit Runge-Kutta method, Contra harmonic mean, Forward backward technique, Stability.*

1. Introduction

Numerical methods have been developed over time; one of the most popular methods for solving IVPs is the classical RK method. And the development of new methods continues to this day. Evans and Yaakub [1] used a CH idea in place of the classical (Arithmetic) mean and developed a new Contra Harmonic Runge-Kutta (CHRK) method. Badmos and Subair [2] introduced a comparison between the Implicit Block and the RK kind of solution for second-order Ordinary Differential Equations ODEs. Zhang, Wang, and Tang [3] presented an implicit RK Nyström method with Lagrange interpolation for nonlinear second-order IVPs with time-varying delay. Noar, Apandi, and Rosli [4] conducted a comparative study of the Taylor methods, the fourth-order RK method, and the RK Fehlberg method for solving ODEs. In (2021-2024) Jasim [5,6] presented new methods by using a different mean instead of the arithmetic mean with the FB technique, to get new RK methods that work in parallel. In our recent study, we developed a novel technique that differs from Jasim's binding [7]. We dubbed this method CHRK2, and it produced better results than Jasim's [7]

2. The general form of CHRK2 means:

It takes a formula

$$y_{n+1} - y_n = \frac{h}{n}\sum_{j=1}^{s}\left(\frac{k_j^2 + k_{j+1}^2}{k_j + k_{j+1}}\right) J = 1,2,\dots,s \tag{1}$$

$$k_j = f(y_n + h\sum_{i=1}^{j-1} u_i\, k_i) \tag{2}$$

Where $h\ step\ size\ , u_i\ parameters$

2.1. *Definition:*

A front for computation is "an imaginary line which separates a previously calculated value from the values to be calculated later" [6,8].

Our new method is modified to follow Jasim's [7] idea, in which a connection between the predictor and corrector parts is made, giving us greater flexibility between the two parts and enabling faster calculations than in the old method. This hold by using an expansion to the front of the computation (as shown in figures 1 and 2), where we were able to make a connection between parts that makes y_{n+1}^p depends on y_n^p and y_{n-1}^c, while y_n^c depends on both y_{n-1}^p and y_n^p. (see [9-13])

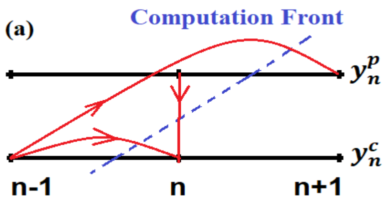

Figure 1
Represents a method of Jasim's

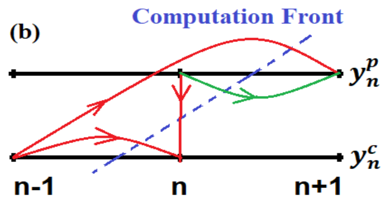

Figure 2
Represents our CHRK2

3. Deriving of CHRK2

First, we start to drive the forward part of CHRK2

Suppose that

$$y_{n+1}^p - y_n^p = h\left(\frac{ak_1^2+bk_2^2}{ck_1+dk_2}\right) \tag{3}$$

$$k_1 = f(y_{n-1}^c),\ k_2 = f(y_{n-1}^c + ehk_1) \tag{4}$$

$$k_1^2 = f^2\ ,k_2^2 = f^2 + 2ehf^2 f_{y_{n-1}^c} + o(h^2)$$

$$ak_1^2 + bk_2^2 = (a+b)f^2 + 2behf^2 f_{y_{n-1}^c} + o(h^2) \tag{5}$$

$$ck_1 + dk_2 = (c+d)f + dehff_{y_{n-1}^c} + o(h^2) \tag{6}$$

Divide (5) by (6)

$$\frac{ak_1^2+bk_2^2}{ck_1+dk_2} = h\left(\frac{(a+b)f^2+2behf^2 f_{y_{n-1}^c}+o(h^2)}{(c+d)f+dehff_{y_{n-1}^c}+o(h^2)}\right) \tag{7}$$

Substitute (7) in (3) and (4) we get,

$$y_{n+1}^p - y_n^p = \left(\frac{(a+b)hf^2+2beh^2f^2 f_{y_{n-1}^c}+o(h^3)}{(c+d)f+dehff_{y_{n-1}^c}+o(h^2)}\right) \tag{8}$$

Now, let:

$$I = (a+b)hf^2 + 2beh^2 f^2 f_{y_{n-1}^c} + o(h^3) \tag{9}$$

$$II = (c+d)f + dehff_{y_{n-1}^c} + o(h^2) \tag{10}$$

Then, (8) becomes

$$y_{n+1}^p - y_n^p = \left(\frac{I}{II}\right) \tag{11}$$

Know that,

$$ERROR = TAYLOR\ SERIES - \frac{I}{II} \tag{12}$$

can write (12) as

$$II * ERROR = II * TAYLOR\ SERIES - I \tag{13}$$

By using the formula

$$y_n - y_{n-1} = hf + h^2 ff_{y_{n-1}} + o(h^3) \tag{14}$$

of TAYLOR SERIES [11] Substitute (9), (10) and (14) in (13),

$$(c+d)hf^2 + deh^2f^2f_{y_{n-1}^c} + o(h^3) + (c+d)h^2f^2f_{y_{n-1}} = (a+b)hf^2 + 2beh^2f^2f_{y_{n-1}^c} \tag{15}$$

Comparing both sides of equation (15) we get two equations

$(c+d) = (a+b)$ and $de + (c+d) = 2be$. By choosing $a = c = d = 1 \rightarrow e = 2, b = 1$. So, (3) and (4) becomes $y_{n+1}^p - y_n^p = h\left(\frac{k_1^2+k_2^2}{k_1+k_2}\right)$,$k_1 = f(y_{n-1}^c)$,$k_2 = f(y_{n-1}^c + 2hk_1)$

Now, to derive the backward part of CHRK2

$$y_{n-1}^p - y_n^c = -h\left(\frac{aL_1^2+bL_2^2}{cL_1+dL_2}\right) \tag{16}$$

$$L_1 = f(y_n^p), L_2 = f(y_n^p - ehL_1) \tag{17}$$

Multiply (16) by -1 getting

$$y_n^c - y_{n-1}^p = h\left(\frac{aL_1^2+bL_2^2}{cL_1+dL_2}\right) \tag{18}$$

Expansion of L_1 and L_2 we get, $L_1^2 = f^2$,$L_2^2 = f^2 - 2ehf^2f_{y_n^p} + o(h^2)$

$$aL_1^2 + bL_2^2 = (a+b)f^2 - 2behf^2f_{y_n^p} + o(h^2) \tag{19}$$

$$cL_1 + dL_2 = (c+d)f - dehff_{y_n^p} + o(h^2) \tag{20}$$

Dividing (19) on (20)

$$\frac{aL_1^2+bL_2^2}{cL_1+dL_2} = \left(\frac{(a+b)f^2 - 2behf^2f_{y_n^p} + o(h^2)}{(c+d)f - dehff_{y_n^p} + o(h^2)}\right) \tag{21}$$

Substitute (21) in (18) we get

$$y_n^c - y_{n-1}^p = \left(\frac{(a+b)hf^2 - 2beh^2f^2f_{y_n^p} + o(h^3)}{(c+d)f - dehff_{y_n^p} + o(h^2)}\right) \tag{22}$$

Now, suppose,

$$I = (a+b)hf^2 - 2beh^2f^2f_{y_n^p} + o(h^3) \tag{23}$$

$$II = (c+d)f - dehff_{y_n^p} + o(h^2) \tag{24}$$

So, (22) becomes,

$$y_n^c - y_{n-1}^p = \left(\frac{I}{II}\right) \tag{25}$$

Since,

$$II * ERROR = II * TAYLOR\ SERIES - I \tag{26}$$

And using TAYLOR SERIES of the formula [11]

$$y_{n-1} - y_n = -hf + \frac{h^2}{2}ff_{y_n} - o(h^3) \tag{27}$$

Multiply equation by -1, getting:

$$y_n - y_{n-1} = hf - \frac{h^2}{2}ff_{y_n} + o(h^3) \tag{28}$$

Substitute (23), (24) and (28) in (26) getting

$$o(h^3) = \big((c+d)-(a+b)\big)hf^2 + \left(\frac{-(c+d)}{2} - de + 2be\right)h^2 f^2 f_{y_n^p} + o(h^3) \tag{29}$$

Comparing both sides of equation (29) we get $(c+d) = (a+b)$, $(c+d) = (4be - 2de)$. By choosing $c = d = a = 1 \rightarrow b = 1, e = 1$. So, (16, 17) becomes, $y_n^c - y_{n-1}^p = h\left(\frac{L_1^2+L_2^2}{L_1+L_2}\right)$,$L_1 = f(y_n^p)$, $L_2 = f(y_n^p - hL_1)$

4. Stability analysis of CHRK2

To specify the interval of stability for the forward part of our method, using the test equation $\frac{dy}{dt} = y\lambda \; where \; \lambda = \frac{\partial f}{\partial y} \; constant$ [14,15].

Now, since

$$y_{n+1}^p - y_n^p = h\left(\frac{k_1^2+k_2^2}{k_1+k_2}\right) \tag{30}$$

$$k_1 = f(y_{n-1}^c) \text{ ,} k_2 = f(y_{n-1}^c + 2hk_1) \text{ So, } k_1 = f(y_{n-1}^c) = \lambda y_{n-1}^c \tag{31}$$

$$k_2 = f(y_{n-1}^c + 2hk_1) = (\lambda y_{n-1}^c + 2h\lambda^2 y_{n-1}^c) \tag{32}$$

Substitute (31), (32) in (30)

$$y_{n+1}^p - y_n^p = \left(\frac{(y_{n-1}^c)(2h\lambda+4h^2\lambda^2+4h^3\lambda^3)}{(2+h\lambda)}\right), M = \frac{y_{n+1}^p - y_n^p}{y_{n-1}^c} = \left(\frac{2h\lambda+4h^2\lambda^2+4h^3\lambda^3}{2+h\lambda}\right) \tag{33}$$

Now (33) satisfy the absolute stability interval when $|M| < 1$, where this hold when $h\lambda \in (-0.8250\,, 0)$.

5. Numerical examples

Test example1: Appling CHRK2 on IVP $\acute{y} = y, y(0) = 1$, exact solution is $y = e^t$, with step size $h =, 0.05, 0.1$. The results observed in table 1,2.

Table 1

CHRK2 applied to IVP $\acute{y} = y, y(0) = 1$, within steps size $h = 0.05$.

t=	Exact sol.	Predictor sol.	Corrector sol.	$\lvert Exa.-Pre.\rvert$	$\lvert Exa.-Cor.\rvert$
0	1	1	1	0	0
0.05	1.05127109	1.020201340	1.051010067	0.031069756	0.000261029
0.10	1.10517091	1.072820387	1.104651086	0.032350530	0.000519831
0.15	1.16183424	1.128123536	1.161057263	0.033710706	0.000776979
0.20	1.22140275	1.186249224	1.220369724	0.035153533	0.001033033
0.25	1.28402541	1.247342951	1.282736872	0.036682464	0.001288544
….	….	…..	…..	…..	…..
1.95	7.02868758	6.844850626	7.015317389	0.183836954	0.013370191
2	7.38905609	7.195981470	7.375116462	0.193074628	0.013939636

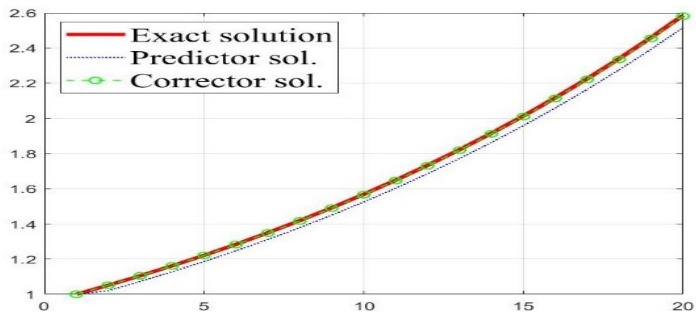

Figure 3

Represents solutions with h=0.05, on $\acute{y} = y, y(0) = 1$

Table 2

CHRK2 applied to IVP $\acute{y} = y, y(0) = 1$, & step sizes $h = 0.1$.

t=	Exact sol.	Pre. sol.	Cor. sol.	$\lvert Exa. - Pre.\rvert$	$\lvert Exa. - Cor.\rvert$
0	1	1	1	0	0
0.1	1.105170918	1.020201340	1.102020134	0.084969578	0.003150784
0.2	1.221402758	1.131110430	1.215131177	0.090292327	0.006271581
0.3	1.349858807	1.253334482	1.340464625	0.096524325	0.009394182
0.4	1.491824697	1.388103576	1.479274982	0.103721121	0.012549714
0.5	1.648721270	1.536773289	1.632952311	0.111947981	0.015768958
..	..	…	….	….	…..
1.9	6.685894442	6.283819967	6.599382946	0.402074474	0.086511495
2	7.389056098	6.946058254	7.293988772	0.442997844	0.095067326

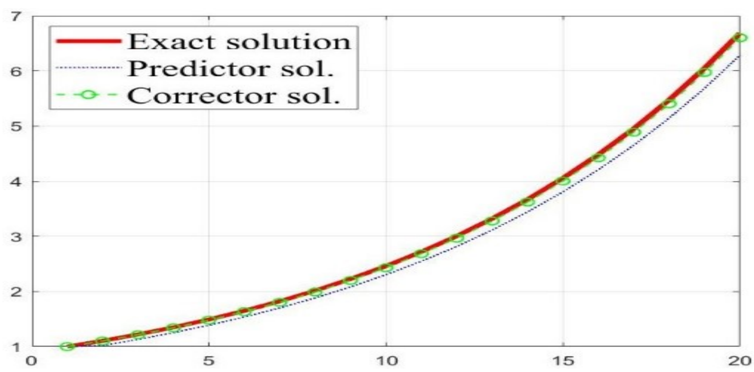

Figure 4

Represents solutions with steps of the size h=0.1, on $\acute{y} = y, y(0) = 1$

Test example2: Appling CHRK2 on IVP $\acute{y} = -2ty^2, y(0) = 1$, exact solutions is $y = 1/(1 + t^2)$,with step size $h = 0.04, 0.05$. Results observed in table 3, 4.

Table 3

CHRK2 applied to IVP $\acute{y} = -2ty^2, y(0) = 1$, with step size $h = 0.02$.

t=	Exact sol.	Pred. sol.	Corr. sol.	$\lvert Ex. - Pre \rvert$	$\lvert Ex. - Cor \rvert$
0	1	1	1	0	0
0.02	0.9996001	1.001000	0.999198	0.001399	0.000401
0.04	0.9984025	0.999400	0.997866	0.000997	0.000535
0.06	0.9964129	0.997409	0.995797	0.000996	0.000615
0.08	0.9936406	0.994770	0.992970	0.001129	0.000674
0.10	0.9900990	0.991426	0.989387	0.001327	0.000711
0.12	0.9858044	0.987364	0.985062	0.001560	0.000741
0.14	0.9807767	0.982591	0.980012	0.001815	0.000764
0.16	0.9750390	0.977120	0.974258	0.002081	0.000780
…	….	…..	…..	…..	…..
0.98	0.5100999	0.514813	0.509979	0.004713	0.000120
1	0.5000000	0.504623	0.499894	0.004623	0.000105

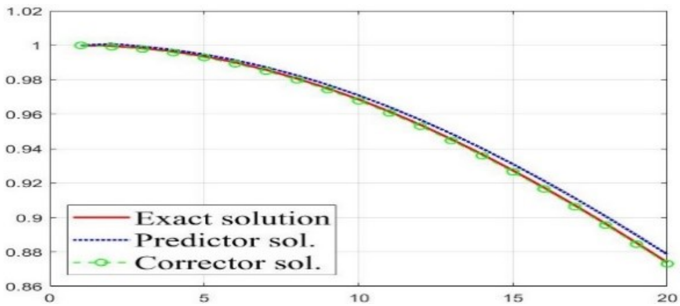

Figure 5

Represents solutions with h=0.02, on $\acute{y} = -2ty^2, y(0) = 1$

Table 4

CHRK2 applied to IVP $\acute{y} = -2ty^2, y(0) = 1$, with step size $h = 0.05$.

t=	Exact sol.	Pred. sol.	Corr. sol.	$\lvert Ex. - Pre \rvert$	$\lvert Ex. - Cor. \rvert$
0	1	1	1	0	0
0.05	0.99750623	1.00100000	0.9949899	0.00349376	0.002516239
0.10	0.99009900	0.99100000	0.9868059	0.00090099	0.003293023
0.15	0.97799511	0.97888167	0.9743492	0.00088656	0.003645844
0.20	0.96153846	0.96323939	0.9577808	0.00170093	0.003757591
0.25	0.94117647	0.94403390	0.9374813	0.00285743	0.003695156
0.30	0.91743119	0.92155813	0.9139333	0.00412693	0.003497802
0.35	0.89086859	0.89624275	0.8876732	0.00537416	0.003195337
0.40	0.86206896	0.86858119	0.8592562	0.00651223	0.002812693
0.45	0.83160083	0.83908659	0.8292299	0.00748576	0.002370917
….	….	…..	…..	…..	…..
0.95	0.52562417	0.53251871	0.5287926	0.00689453	0.003168470
1	0.50000000	0.5061032	0.5038023	0.00610328	0.003802528

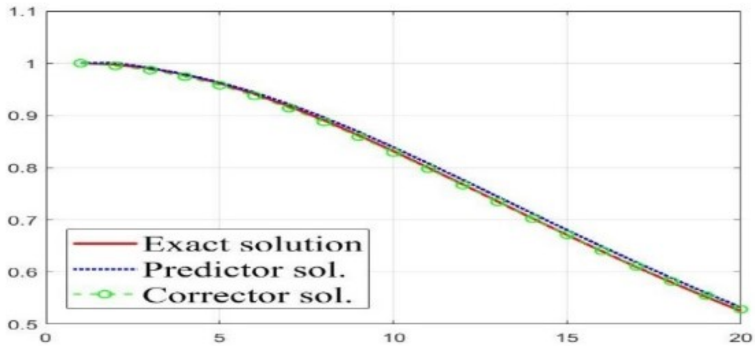

Figure 6
represents solutions with h=0.05, on $\acute{y} = -2ty^2, y(0) = 1$

6. Conclusions

The results presented in Tables 1 through 4, along with Figures (3 to 6), clearly indicate that the corrector solutions outperform the predictor solutions. This is particularly evident when using an appropriate step length, which is a crucial condition for ensuring the stability of Runge-Kutta methods.

7. Discuss the results.

The tables show first columns represent values of t, a second columns an exact solutions, a third columns forward (predictor) solutions, and the fourth columns a backward (corrector) solutions. A fifth and sixth column display absolute difference among an exact and predictor-corrector solutions, respectively. These findings highlight the superiority of corrector method over the classical predictor method, especially when a suitable step size is employed. This aligns with the principal condition for stability in Runge-Kutta methods.

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