*Journal of Interdisciplinary Mathematics*
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 2, pp. 265–277
DOI : 10.47974/JIM-2153

# Characterization of nilpotent Lie algebras by their multipliers, $t(L) = 13$

Zari Ahmadieh [†]
*Department of Mathematics and Computer Sciences*
*Ma. C., Islamic Azad University*
*Mashhad*
*Iran*

Hamid Darabi *
*Department of Mathematics*
*Esfarayen University of Technology*
*Esfarayen*
*Iran*

Farshid Saeedi [§]
*Department of Mathematics and Computer Sciences*
*Ma. C., Islamic Azad University*
*Mashhad*
*Iran*

**Abstract**

For every $n$-dimensional nilpotent Lie algebra $L$, let $\mathcal{M}(L)$be the Schur multiplier of $L$. Then we define $t(L)$ by $t(L) = 1/2(n^2 - n) - dim\mathcal{M}(L)$. We classify nilpotent Lie algebras $L$ satisfying $t(L) = 13$.

***Subject Classification (2020):*** *17B05, 17B30.*

***Keywords:*** *Lie algebras, Schur multiplier, Nilpotent Lie algebras.*

[†] *ORCID-ID:* `0009-0006-4003-1575`
* *ORCID-ID:* `0000-0001-8465-1756`
[§] *ORCID-ID:* `0000-0002-7627-0707`

[†] *E-mail:* **zari.ahmadieh@iau.ac.ir**
* *E-mail:* **darabi@esfarayen.ac.ir (Corresponding Author)**
[§] *E-mail:* **saeedi@mshdiau.ac.ir**

**1. Introduction and preliminaries**

The concept of Schur multiplier was first introduced to a group in famous article of Schur [19]. Later, this concept was proposed for Lie algebras [1], Lie superalgebras [14], $n$-Lie algebras [7], and $n$-Lie superalgebras [18]. Nilpotent Lie algebras (NLAs) play a crucial role in the study of algebraic structures and their applications in various fields, including geometry, physics, and representation theory. The classification of these algebras is a fundamental problem in the theory of Lie algebras, as it provides insights into their structure and behavior. One of the key invariants associated with a Lie algebra is the Schur multiplier, denoted by $\mathcal{M}(L)$, which measures the extent to which the algebra can be embedded into a larger structure. For an $n$-dimensional NLA $L$, we define the function $t(L)$ as $t(L) = \frac{1}{2}(n^2 - n) - \dim\mathcal{M}(L)$. This function serves as a critical tool in understanding the relationship between the dimension of the algebra and its Schur multiplier. The classification of NLAs based on the value of $t(L)$ has been extensively studied, particularly for small values of $t(L)$.

In this paper, we focus on the classification of NLAs for which $t(L) = 13$. This specific case presents unique challenges and opportunities for exploration, as it lies at the intersection of known classifications in the theory of NLAs. By analyzing the properties and structures of these algebras, we aim to contribute to the broader understanding of NLAs and their classifications. The organization of this paper is as follows: We begin with a review of the relevant concepts and previous classifications of NLAs. We then present our main results regarding the classification of NLAs with $t(L) = 13$, followed by examples and discussions of the implications. It was proved that $\frac{1}{2}(n-1)(n-2)+1$ is an upper bound for $\dim\mathcal{M}(L)$ for any $n$-dimensional NLA $L$, and according to this, $s(L)$ is defined by $s(L) = \frac{1}{2}(n-1)(n-2) - \dim\mathcal{M}(L) + 1$. There are many theorems and results regarding to the classification of NLAs $L$ with dimension finite for small values of $t(L)$ and $s(L)$; for example, see [8, 9, 15, 17, 20, 21] for $s(L)$. In literature some special cases is studied such as the especial case where $A$ is simple and has the orthogonal involution abelian non-Jordan-Lie inner ideals of $K$ in [22]. Finite-dimensional NLAs $L$ with $0 \le t(L) \le 12$ have been classified by several papers; see [2, 3, 5, 6, 10, 12, 13]. In the present paper, we attempt to classify NLAs $L$ that satisfy $t(L) = 13$.

Now, we recall some algebras used in this paper. The subalgebra $L^2 = [L, L]$ is called the derived subalgebra of $L$. If $L^2 = 0$, then clearly $L$ is abelian and denoted by $A(n)$. If $\dim L^2 = 1$, then there is $k \ge 1$ such that $L$ is isomorphic to $H(k) \oplus A(n-2k-1)$. We recall that $H(k)$ is the Heisenberg Lie algebra of dimension $2k+1$ and is given by $H(k) = \langle t, t_1, \dots, t_{2k}\colon [t_{2i-1}, t_{2i}] = x, i = 1, \dots, k\rangle$. A key issue of the Lie algebra theory is the classification of low-dimensional Lie algebras. This issue is studied in several papers and books. There exists only one abelian Lie algebra of each dimension. In an arbitrary field, nonabelian NLAs up to dimension five are $H(1)$, $H(2)$, $L_{4,3}$,

$L_{5,i}$, for all $i = 5,6,7,8,9$, $H(1) \oplus A(1)$, $H(1) \oplus A(2)$, and $L_{4,3} \oplus A(1)$; see Table 1 for their multiplications.

Cicalo, Graaf, and Schneider [4] classified the 6-dimensional NLAs on the arbitrary field. Nonabelian NLAs of dimension six over a field $\mathcal{K}$ with $char(\mathcal{K}) \neq 2$ are $H(1) \oplus A(3)$, $H(2) \oplus A(1)$, $L_{4,3} \oplus A(2)$,

$$L_{5,i} \oplus A(1), 5 \leq i \leq 9,$$

$$L_{6,i}, i = 10, \dots, 18, 20, 23, 25, 26, 27, 28,$$

$$\text{or } L_{6,i}(\varepsilon), i = 19, 21, 22, 24.$$

We refer the interested reader to Table 1. This table is used in this paper several times. The Lie algebras of type Nilpotent with dimensions 7, 8, and 9 are used in this article are classified over fields of characteristic not equal to 2. Therefore, we will carry out all classifications in this article over fields of characteristic not equal to 2. Now, we recall the following theorems, which are useful for proving our main results, which are stated without proof.

**Table 1**
**Indecomposable NLAs up to dimension six.**

| Name | Nonzero Lie multiplications |
|---|---|
| $L_{6,28}$ | $[t_1,t_3]=t_4, [t_2,t_3]=t_6, [t_1,t_2]=t_3, [t_1,t_4]=t_5$ |
| $L_{6,27}$ | $[t_1,t_3]=t_5, [t_1,t_2]=t_3, [t_2,t_4]=t_6$ |
| $L_{6,26}$ | $[t_1,t_3]=t_5, [t_1,t_2]=t_4, [t_2,t_4]=t_6$ |
| $L_{6,25}$ | $[t_1,t_3]=t_5, [t_1,t_2]=t_3, [t_1,t_4]=t_6$ |
| $L_{6,24}(\epsilon)$ | $[t_2,t_3]=t_6, [t_1,t_2]=t_3, [t_1,t_4]=\epsilon t_6, [t_1,t_3]=[t_2,t_4]=t_5$ |
| $L_{6,23}$ | $[t_2,t_4]=[t_1,t_3]=t_5, [t_1,t_4]=t_6, [t_1,t_2]=t_3$ |
| $L_{6,22}(\epsilon)$ | $[t_3,e_4]=[t_1,t_2]=t_5, [t_2,t_4]=\epsilon t_6, [t_1,t_3]=t_6$ |
| $L_{6,21}(\epsilon)$ | $[t_1,t_3]=t_4, [t_1,t_4]=t_6, [t_1,t_2]=t_3, [t_2,t_3]=t_5, [t_2,t_5]=\epsilon t_6$ |
| $L_{6,20}$ | $[t_1,t_3]=t_5, [t_1,t_2]=t_4, [t_2,t_4]=[t_1,t_5]=t_6$ |
| $L_{6,19}(\epsilon)$ | $[t_1,t_3]=t_5, [t_2,t_4]=[t_1,t_5]=t_6, [t_1,t_2]=t_4, [t_3,t_5]=\epsilon t_6$ |
| $L_{6,18}$ | $[t_1,t_3]=t_4, [t_1,t_5]=t_6, [t_1,t_4]=t_5, [t_1,t_2]=t_3$ |
| $L_{6,17}$ | $[t_1,t_3]=t_4, [t_1,t_4]=t_5, [t_2,t_3]=[t_1,t_5]=t_6, [t_1,t_2]=t_3$ |
| $L_{6,16}$ | $[t_1,t_3]=t_4, [t_2,t_5]=t_6, [t_1,t_4]=t_5, [t_3,t_4]=-t_6, [t_1,t_2]=t_3$ |
| $L_{6,15}$ | $[t_1,t_3]=t_4, [t_2,t_3]=[t_1,t_4]=t_5, [t_2,t_4]=[t_1,t_5]=t_6, [t_1,t_2]=t_3$ |
| $L_{6,14}$ | $[t_1,t_3]=t_4, [t_2,t_3]=[t_1,t_4]=t_5, -[t_3,t_4]=[t_2,t_5]=t_6, [t_1,t_2]=t_3$ |
| $L_{6,13}$ | $[t_2,t_4]=[t_1,t_3]=t_5, [t_3,t_4]=[t_1,t_5]=t_6, [t_1,t_2]=t_3$ |
| $L_{6,12}$ | $[t_1,t_3]=t_4, [t_2,t_5]=[t_1,t_4]=t_6, [t_1,t_2]=t_3$ |
| $L_{6,11}$ | $[t_1,t_3]=t_4, [t_1,t_2]=t_3, [t_2,t_3]=[t_1,t_4]=[t_2,t_5]=t_6$ |
| $L_{6,10}$ | $[t_4,t_5]=[t_1,t_3]=t_6, [t_1,t_2]=t_3$ |
| $L_{5,9}$ | $[t_1,t_3]=t_4, [t_1,t_2]=t_3, [t_2,t_3]=t_5$ |
| $L_{5,8}$ | $[t_1,t_3]=t_5, [t_1,t_2]=t_4$ |
| $L_{5,7}$ | $[t_1,t_3]=t_4, [t_1,t_2]=t_3, [t_1,t_4]=t_5$ |
| $L_{5,6}$ | $[t_1,t_3]=t_4, [t_1,t_2]=t_3, [t_2,t_3]=[t_1,t_4]=t_5$ |
| $L_{5,5}$ | $[t_2,t_4]=[t_1,t_3]=t_5, [t_1,t_2]=t_3$ |
| $L_{4,3}$ | $[t_1,t_3]=t_4, [t_1,t_2]=t_3$ |

**Theorem 1.1:** (see [16]). *Assume that $L$ and $K$ are two Lie algebras having finite dimension. Then*

$$\dim \mathcal{M}(L \oplus K) = \dim \mathcal{M}(L) + \dim (L/L^2) \dim (K/K^2) + \dim \mathcal{M}(K).$$

**Theorem 1.2:** (see [8]). *Suppose that $L$ is a $d$-dimensional NLA satisfying $dim L^2 = 1$. Then there exists some $m \geq 1$ such that $L \cong H(m) \oplus A(d-2m-1)$. Moreover,*

$$t(L) = \begin{cases} d, & m \geq 2, \\ d-2, & m = 1. \end{cases}$$

Now, we state another result obtained by Niroomand and Russo, which is essential.

**Theorem 1.3:** (see [16]). *Assume that $L$ is a nonabelian NLA with dimension $d$ and that $dim\, L^2 = n \geq 1$. Then $dim\, \mathcal{M}(L) \leq \frac{1}{2}(d-n-1)(d+n-2)+1$.*

**Table 2**
**The structure of 7-dimensional indecomposable NLAs with $\dim L^2 = 3$.**

| Name | Nonzero multiplication |
|---|---|
| $37A$ | $[t_2,t_4]=t_7, [t_2,t_3]=t_6, [t_1,t_2]=t_5$ |
| $137A$ | $[t_3,t_4]=t_6, [t_1,t_2]=t_5, [t_1,t_5]=[t_3,t_6]=t_7$ |
| $147A$ | $[t_1,t_3]=t_5, [t_2,t_5]=[t_1,t_6]=[t_3,t_4]=t_7, [t_1,t_2]=t_4$ |
| $257A$ | $[t_2,t_4]=[t_1,t_3]=t_6, [t_1,t_5]=t_7, [t_1,t_2]=t_3$ |
| $1357A$ | $[t_2,t_3]=[t_1,t_4]=t_5, [t_2,t_6]=[t_1,t_5]=-[t_3,t_4]=t_7, [t_1,t_2]=t_4$ |
| $1457A$ | $[t_1,t_3]=t_4, [t_5,t_6]=[t_1,t_4]=t_7, [t_1,t_2]=t_3$ |
| $37B$ | $[t_2,t_3]=t_6, [t_1,t_2]=t_5, [t_3,t_4]=t_7$ |
| $137B$ | $[t_3,t_4]=t_6, [t_3,t_6]=[t_1,t_5]=[t_2,t_4]=t_7, [t_1,t_2]=t_5$ |
| $147B$ | $[t_1,t_3]=t_5, [t_2,t_6]=[t_1,t_4]=[t_3,t_5]=t_7, [t_1,t_2]=t_4$ |
| $257B$ | $[t_1,t_3]=t_6, [t_2,t_5]=[t_1,t_4]=t_7, [t_1,t_2]=t_3$ |
| $1357B$ | $[t_2,t_3]=[t_1,t_4]=t_5, [t_3,t_6]=[t_1,t_5]=-[t_3,t_4]=t_7, [t_1,t_2]=t_4$ |
| $1457B$ | $[t_1,t_3]=t_4, [t_2,t_3]=[t_1,t_4]=[t_5,t_6]=t_7, [t_1,t_2]=t_3$ |
| $37C$ | $[t_2,t_3]=t_6, [t_3,t_4]=[t_1,t_2]=t_5, [t_2,t_4]=t_7$ |
| $137C$ | $[t_1,t_4]=[t_2,t_3]=t_6, [t_3,t_5]=-t_7, [t_1,t_6]=t_7, [t_1,t_2]=t_5$ |
| $257C$ | $[t_2,t_4]=[t_1,t_3]=t_6, [t_2,t_5]=t_7, [t_1,t_2]=t_3$ |
| $1357C$ | $[t_2,t_3]=[t_1,t_4]=t_5, [t_2,t_4]=[t_1,t_5]=[t_3,t_6]=-[t_3,t_4]=t_7, [t_1,t_2]=t_4$ |
| $37D$ | $[t_1,t_3]=t_6, [t_3,t_4]=[t_1,t_2]=t_5, [t_2,t_4]=t_7$ |
| $137D$ | $[t_2,t_3]=[t_1,t_4]=t_6, [t_2,t_4]=[t_1,t_6]=-[t_3,t_5]=t_7, [t_1,t_2]=t_5$ |
| $257D$ | $[t_2,t_4]=[t_1,t_3]=t_6, [t_1,t_2]=t_3, [t_2,t_5]=[t_1,t_4]=t_7$ |
| $257E$ | $[t_4,t_5]=[t_1,t_3]=t_6, [t_1,t_2]=t_3, [t_2,t_4]=t_7$ |
| $257F$ | $[t_4,t_5]=[t_2,t_3]=t_6, [t_2,t_4]=t_7, [t_1,t_2]=t_3$ |
| $257G$ | $[t_4,t_5]=[t_1,t_3]=t_6, [t_1,t_5]=[t_2,t_4]=t_7, [t_1,t_2]=t_3$ |
| $257H$ | $[t_2,t_4]=[t_1,t_3]=t_6, [t_4,t_5]=t_7, [t_1,t_2]=t_3$ |
| $257I$ | $[t_1,t_2]=t_3, [t_1,t_3]=[t_1,t_4]=t_6, [t_1,t_5]=[t_2,t_3]=t_7$ |
| $257J$ | $[t_2,t_4]=[t_1,t_3]=t_6, [t_2,t_3]=[t_1,t_5]=t_7, [t_1,t_2]=t_3$ |
| $257K$ | $[t_1,t_3]=t_6, [t_2,t_3]=[t_4,t_5]=t_7, [t_1,t_2]=t_3$ |
| $257L$ | $[t_2,t_4]=[t_1,t_3]=t_6, [t_2,t_3]=[t_4,t_5]=t_7, [t_1,t_2]=t_3$ |

Now, we state a characterization of nonabelian NLAs from [15, 9, 20, 21] in the following theorem.

**Theorem 1.4:** (see [9, 15, 20, 21]). *Assume that $L$ is a nonabelian NLA with dimension $n$. Then the following properties are satisfied:*

(i) $s(L) = 0$ if and only if $L \cong H(1) \oplus A(n-3)$.

(ii) $s(L) = 1$ if and only if $L \cong L_{5,8}$.

(iii $s(L) = 2$ if and only if $L \cong L_{4,3}$, $L_{5,8} \oplus A(1)$ or $H(r) \oplus A(n-2r-1)$ for some $r \geq 2$.

(iv) $s(L) = 3$ if and only if $L \cong L_{4,3} \oplus A(1)$, $L_{5,5}$, $L_{6,22}(\varepsilon)$, $L_{6,26}$ or $L_{5,8} \oplus A(2)$.

(v) $s(L) = 4$ if and only if $L \cong L_{5,8} \oplus A(3)$, $L_{4,3} \oplus A(2)$, $L_{5,5} \oplus A(1)$, $L_{5,6}$, $L_{5,7}$, $L_{5,9}$, $L_{6,22}(\varepsilon) \oplus A(1)$, or $37A$.

(vi) $s(L) = 5$ if and only if $L \cong 37B$, $L_{6,23}$, $37C$, $L_{6,10}$, $37D$,$L_{6,27}$, $L_{6,25}$, $L_{6,26} \oplus A(1)$, $L_{5,5} \oplus A(2)$, $L_{6,22}(\varepsilon) \oplus A(2)$, $L_{4,3} \oplus A(3)$, $L_{5,8} \oplus A(4)$.

For the definition of algebras used in above theorems, see Tables 1 and 2.

**Lemma 1.5:** *The dimension of $\mathcal{M}(L)$ and $t(L)$ for nonabelian NLAs up to dimension six over an arbitrary field are as Lie algebras presented in Table 3.*

***Proof:*** By [5, Lemmas 1, 2 and Theorem 5], the result follows.

**Table 3**
**Nonabelian NLAs up to dimension six**

| **Lie algebras** | $\dim M(L)$ | $t(L)$ |
|---|---|---|
| $H(1)$ | 2 | 1 |
| $L_{4,3}$ | 2 | 4 |
| $L_{6,14}$, $L_{6,16}$ | 2 | 13 |
| $L_{5,6}$, $L_{5,9}$, $L_{5,7}$ | 3 | 7 |
| $L_{6,15}$, $L_{6,17}$, $L_{6,18}$ | 3 | 12 |
| $H(1) \oplus A(1)$ | 4 | 2 |
| $L_{4,3} \oplus A(1)$, $L_{5,5}$ | 4 | 6 |
| $L_{6,13}$, $L_{6,28}$, $L_{6,21}(\epsilon)$ | 4 | 11 |
| $H(2)$ | 5 | 5 |
| $L_{5,7} \oplus A(1)$, $L_{5,6} \oplus A(1)$, $L_{5,9} \oplus A(1)$ $L_{6,24}(\epsilon)$, $L_{6,12}$, $L_{6,11}$, $L_{6,20}$, $L_{6,19}(\epsilon)$ | 5 | 10 |
| $L_{5,8}$ | 6 | 4 |
| $L_{6,10}, L_{6,23}, L_{6,25}, L_{6,27}$ | 6 | 9 |
| $H(1) \oplus A(2)$ | 7 | 3 |
| $L_{4,3} \oplus A(2)$, $L_{5,5} \oplus A(1)$ | 7 | 8 |
| $L_{6,26}, L_{6,22}(\epsilon)$ | 8 | 7 |
| $H(2) \oplus A(1)$, $L_{5,8} \oplus A(1)$ | 9 | 6 |
| $H(1) \oplus A(3)$ | 11 | 4 |

**Lemma 1.6:**

(i) *The* $7$*-dimensional NLAs with* $dim\ L^2 = 2$ *are isomorphic to* $L_{4,3} \oplus A(3),\ L_{5,5} \oplus A(2),\ L_{5,8} \oplus A(2),\ L_{6,10} \oplus A(1),\ L_{6,22}(\varepsilon) \oplus A(1),$ $27A,\ 27B$, or $157$.

(ii) *The* $7$*-dimensional NLAs with* $dimL^2 = 3$ *are isomorphic to* $L_{6,i} \oplus A(1),\ i = 11, 12, 13, 20, 23, 25, 26,\ L_{5,j} \oplus A(2),\ j = 6, 7, 9$ $L_{6,k}(\varepsilon) \oplus A(1),\ k = 19, 24,$ *or one of the algebras defined in Table 2*.

***Proof:*** Based on the classification of NLAs with dimension seven the proof follows from [11].

We note that the definition of algebras of the above lemma are defined in Tables 1, 2, and 4.

**Table 4**
**The structure of 7-dimensional indecomposable NLAs with** $\dim \boldsymbol{L^2} = \mathbf{2}$**.**

| Name | Nonzero Lie multiplication |
|---|---|
| $27A$ | $[t_1, t_4] = [t_3, t_5] = t_7, [t_1, t_2] = t_6$ |
| $27B$ | $[t_3, t_4] = [t_1, t_2] = t_6, [t_2, t_3] = [t_1, t_5] = t_7$ |
| $157$ | $[t_5, t_6] = [t_2, t_4] = [t_1, t_3] = t_7, [t_1, t_2] = t_3$ |

## 2. Main results

Assume that $L$ is an arbitrary $n$-dimensional NLA. Then two nonnegative integers $t(L)$ and $s(L)$ exist such that $s(L) = \frac{1}{2}(n^2 - 3n) - \dim \mathcal{M}(L) + 2$ and $t(L) = \frac{1}{2}(n^2 - n) - \dim \mathcal{M}(L)$. Thus

$$t(L) = n + s(L) - 2. \tag{1}$$

Now, we are ready to classify NLAs $L$ satisfying $t(L) = 13$. It follows from relation (1) that $15 = n + s(L)$, which implies that $n \leq 15$. In this regard, we discuss the classification according to $n$.

The structure of algebras needed in this section is shown in Tables 1, 2, 4, and 5.

First, we prove a theorem for $n = 15$.

**Lemma 2.1:** *The only* $15$*-dimensional NLA* $L$ *with* $t(L) = 13$ *is* $H(1) \oplus A(12)$.

***Proof:*** Since $s(L) = 0$, it follows from Theorem 1.4 (i) that $L = H(1) \oplus A(12)$.

**Lemma 2.2:** *For* $n = 14,\ 12,\ 11,$ *or* $10$*, there exist no* $n$*-dimensional NLAs* $L$ *with* $t(L) = 13$.

***Proof:*** By relation (1), NLAs $L$ of dimension $14$, $12$, $11$, and $10$ with $t(L) = 13$ are algebras with $s(L) = 1$, $3$, $4$, and $5$, respectively. According to Theorem 4.1 (ii, iv, v, vi), the proof is complete.

Now, for $n = 13$, we have the following result.

**Lemma 2.3:** *The only* $13$-*dimensional NLAs* $L$ *with* $t(L) = 13$ *are isomorphic to* $H(i) \oplus A(12 - 2i)$, *for all* $i = 2, \dots, 6$.

***Proof:*** Since $s(L) = 2$, the proof follows from Theorem 1.4 (iii).

Above theorem for $n = 9$ is stated as follows.

**Lemma 2.4:** *The only* $9$-*dimensional NLA* $L$ *with* $t(L) = 13$ *is* $L_{6,22}(\varepsilon) \oplus A(3)$.

***Proof:*** In this case, we have $s(L) = 6$. The proof is complete from [17, Theorem 2.10].

Now, we prove a theorem for $n = 8$.

**Lemma 2.5:** *The only* $8$-*dimensional NLAs* $L$ *with* $t(L) = 13$ *are isomorphic to*

$$L_{6,i} \oplus A(2), \text{ for all } i = 10,26,$$
$$M \oplus A(1), \text{ for all } M = 27A, 157, 37B, 37C, 37D,$$
$$H(1) \oplus H(2),\ S_1,\ S_2, \text{ or } S_3.$$

***Proof:*** In this case, $s(L) = 7$. It follows from [17, Theorem 3.8] that $L$ is isomorphic to

$$L_{6,i} \oplus A(2), \text{ for } i = 10,26,$$
$$M \oplus A(1),\ M = 27A, 157, 37B, 37C, 37D,$$
$$H(1) \oplus H(2),\ S_1,\ S_2, \text{ or } S_3.$$

Note that the undefined Lie algebras of the above lemma are defined in Table 5.

**Table 5**
**The structure of 8-dimensional indecomposable NLAs with $\dim L^2 = 2$ and $s(L) = 7$.**

| Name | Nonzero Lie multiplication |
|---|---|
| $S_1$ | $[t_3, t_5] = [t_2, t_7] = [t_1, t_4] = t_8, [t_1, t_2] = t_6$ |
| $S_2$ | $[t_7, t_8] = [t_6, t_7] = [t_1, t_3] = t_5, [t_1, t_2] = t_4$ |
| $S_3$ | $[t_3, t_4] = [t_7, t_8] = [t_1, t_2] = t_5, [t_1, t_3] = t_6$ |

For $n = 7$, we characterize NLAs by assuming different cases of $\dim L^2$.

**Lemma 2.6:** *There is no* $7$-*dimensional NLAs* $L$ *with* $t(L) = 13$ *when* $\dim L^2 = 5,2,1$.

***Proof:*** (i) Let $\dim L^2 = 5$. Then by Theorem 1.3, $\dim \mathcal{M}(L) \le 6$; thus $t(L) \ge 15$.

(ii) By Lemma 1.6, the 7-dimensional NLAs with $\dim L^2 = 2$ are isomorphic to $L_{6,10} \oplus A(1)$, $27A$, $27B$, $157$, $L_{4,3} \oplus A(3)$, $L_{5,5} \oplus A(2)$, $L_{5,8} \oplus A(2)$, or $L_{6,22}(\varepsilon) \oplus A(1)$. Lemmas 1.5 and 1.1, imply that $\dim \mathcal{M}(L_{6,22}(\varepsilon) \oplus A(1)) = 12$, that $\dim \mathcal{M}(L_{4,3} \oplus A(3)) = \dim \mathcal{M}(L_{5,5} \oplus A(2)) = 11$, that $\dim \mathcal{M}(L_{5,8} \oplus A(2)) = 13$, and that $\dim \mathcal{M}(L_{6,10} \oplus A(1)) = 10$. By applying the method used by Batten, Moneyhun, and Stitzinger in [2], $\dim \mathcal{M}(27B) = 9$ and $\dim \mathcal{M}(27A) = \dim \mathcal{M}(157) = 10$. Thus in this cases, $t(L) \neq 13$, a contradiction.

(iii) If $\dim L^2 = 1$, then By Theorem 1.2, $L$ is isomorphic to $H(2) \oplus A(2)$, $H(1) \oplus A(4)$, or $H(3)$. Thus $t(L) = 5,\ 7$, and $7$, respectively, a contradiction.

Now, we characterize the above Lie algebras for $\dim L^2 = 3$.

**Lemma 2.7:** The 7-dimensional NLAs $L$ with $t(L) = 13$ when $\dim L^2 = 3$ are isomorphic to

$$L_{6,i} \oplus A(1), \text{ for all } i = 11,12,20,$$
$$L_{5,j} \oplus A(2), \text{ for all } j = 6,7,9,$$
$$L_{6,k}(\varepsilon) \oplus A(1), \text{ for all } k = 19,24,$$

$L_{4,3} \oplus H(1)$, $147A$, $147B$, $257B$, $257D$, $257E$, $257G$, $257H$, $257I$, or $257J$.

***Proof:*** The 7-dimensional NLAs $L$ with $\dim L^2 = 3$ over an arbitrary field are collected in Lemma 1.6. By the definition of $t(L)$, the only 7-dimensional NLAs $L$ with $\dim \mathcal{M}(L) = 8$ are acceptable. The proof follows from Table 6.

**Table 6**
**7-dimensional NLAs with $\dim L^2 = 3$.**

| Nonabelian NLAs | $\dim \mathcal{M}(L)$ |
|---|---|
| $37A$ | 12 |
| $37B$, $37C$, $37D$, $L_{6,26} \oplus A(1)$ | 11 |
| $257A$, $257C$, $257F$, $L_{6,23} \oplus A(1)$, $L_{6,25} \oplus A(1)$ | 9 |
| $L_{5,i} \oplus A(2)$, $i = 6,7,9$ $L_{6,j} \oplus A(1)$, $j = 11,12,20$ $L_{6,12} \oplus A(1)$, $L_{6,k}(\varepsilon) \oplus A(1)$, $k = 19,24$, $L_{4,3} \oplus H(1)$, $147A$, $147B$, $257B$, $257D$, $257E$, $257G$, $257H$, $257I$, $257J$ | 8 |
| $L_{6,13} \oplus A(1)$, $137A$, $1357A$, $137B$, $137C$, $137D$ | 7 |
| $257K$, $257L$, $1357B$, $1357C$, $1457A$, $1457B$ | 6 |

**Lemma 2.8:** *Assume that $I$ is a 1-dimensional central ideal of $L$. Then there is no 7-dimensional NLAs $L$ with $t(L) = 13$ and $\dim L^2 = 4$ when $L/I$ is isomorphic to*

$$L_{5,i} \oplus A(1), \text{ for } i = 6,7,9,$$

$L_{6,j}$, for all $j = 11,12,12,20$,
$L_{6,19}(\varepsilon)$, or $L_{6,24}(\varepsilon)$.

***Proof:*** Let $L$ be a 7-dimensional NLA satisfying $t(L) = 13$ and $\dim L^2 = 4$. Thus $\dim \mathcal{M}(L) = 8$. If $L/I$ is isomorphic to $L_{5,i} \oplus A(1)$, for $i = 6,7,9$, $L_{6,j}$, for all $j = 11,12,12,20$, $L_{6,19}(\varepsilon)$, or $L_{6,24}(\varepsilon)$ when $I \subseteq Z(L)$, then it follows from Lemma 1.5, $\dim\mathcal{M}(L/I) \leq 5$. In these cases we have a contradiction by using of [16, Corollary 2.3].

**Table 7**
**The structure of 7-dimensional NLAs with $\dim L^2 = 4$ and $L/I$ isomorphic to $L_{6,23}$, $L_{6,25}$, or $L_{6,26}$ when $I \subseteq Z(L)$.**

| Name | Nonzero multiplication |
|---|---|
| $247A$ | $[t_1,t_3]=t_5, [t_1,t_5]=t_7, [t_1,t_4]=t_6, [t_1,t_2]=t_4$ |
| $357A$ | $[t_1,t_3]=t_5, [t_2,t_4]=t_6, [t_1,t_4]=t_7, [t_1,t_2]=t_3$ |
| $2457A$ | $[t_1,t_3]=t_4, [t_1,t_5]=t_7, [t_1,t_4]=t_6, [t_1,t_2]=t_3$ |
| $2357A$ | $[t_1,t_4]=t_5, [t_2,t_3]=t_5+t_6, -[t_3,t_4]=[t_1,t_5]=t_7, [t_1,t_2]=t_4$ |
| $247B$ | $[t_1,t_3]=t_5, [t_3,t_5]=t_7, [t_1,t_4]=t_6, [t_1,t_2]=t_4$ |
| $357B$ | $[t_1,t_3]=t_5, [t_2,t_3]=t_6, [t_1,t_4]=t_7, [t_1,t_2]=t_3$ |
| $2357B$ | $[t_2,t_3]=[t_1,t_4]=t_5, [t_1,t_3]=t_6, -[t_3,t_4]=[t_1,t_5]=t_7, [t_1,t_2]=t_4$ |
| $2457B$ | $[t_1,t_3]=t_4, [t_2,t_5]=t_6, [t_1,t_4]=t_7, [t_1,t_2]=t_3$ |
| $247C$ | $[t_1,t_3]=t_5, [t_3,t_5]=[t_1,t_4]=t_6, [t_1,t_5]=t_7, [t_1,t_2]=t_4$ |
| $357C$ | $[t_2,t_4]=[t_1,t_3]=t_5, [t_2,t_3]=t_6, [t_1,t_4]=t_7, [t_1,t_2]=t_3$ |
| $2457C$ | $[t_1,t_3]=t_4, [t_2,t_5]=[t_1,t_4]=t_6, [t_1,t_5]=t_7, [t_1,t_2]=t_3$ |
| $247D$ | $[t_1,t_3]=t_5, [t_1,t_4]=t_6, [t_3,t_4]=[t_2,t_5]=t_7, [t_1,t_2]=t_4$ |
| $2457D$ | $[t_1,t_3]=t_4, [t_2,t_3]=[t_1,t_4]=[t_2,t_5]=t_6, [t_1,t_5]=t_7, [t_1,t_2]=t_3$ |
| $1357D$ | $[t_3,t_4]=[t_1,t_6]=[t_2,t_5]=t_7, [t_2,t_4]=t_6, [t_2,t_3]=t_5, [t_1,t_2]=t_3$ |
| $1357E$ | $[t_4,t_6]=[t_2,t_5]=t_7, [t_2,t_4]=t_6, [t_2,t_3]=t_5, [t_1,t_2]=t_3$ |
| $2457F$ | $[t_1,t_3]=t_4, [t_2,t_3]=[t_1,t_4]=t_6, [t_1,t_5]=t_7, [t_1,t_2]=t_3$ |
| $1357F$ | $[t_2,t_3]=t_5, [t_1,t_2]=t_3, [t_2,t_4]=t_6, [t_2,t_5]=[t_1,t_3]=-[t_4,t_6]=t_7$ |
| $1357G$ | $[t_1,t_6]=[t_2,t_5]=t_7, [t_1,t_4]=t_6, [t_2,t_3]=t_5, [t_1,t_2]=t_3$ |
| $1357H$ | $[t_2,t_5]=[t_1,t_6]=[t_2,t_6]=t_7, [t_1,t_4]=t_6, [t_2,t_3]=t_5, [t_1,t_2]=t_3$ |
| $247I$ | $[t_1,t_3]=t_5, [t_3,t_5]=t_7, [t_3,t_4]=[t_2,t_5]=t_6, [t_1,t_2]=t_4$ |
| $2457I$ | $[t_1,t_3]=t_4, [t_2,t_3]=[t_1,t_4]=t_6, [t_2,t_5]=t_7, [t_1,t_2]=t_3$ |
| $1357I$ | $[t_2,t_5]=[t_4,t_6]=t_7, [t_1,t_4]=t_6, [t_2,t_3]=t_5, [t_1,t_2]=t_3$ |
| $1357J$ | $[t_4,t_6]=[t_1,t_3]=[t_2,t_5]=t_7, [t_1,t_2]=t_3, [t_1,t_4]=t_6, [t_2,t_3]=t_5$ |
| $247L$ | $[t_1,t_3]=t_5, [t_1,t_5]=t_7, [t_1,t_4]=[t_2,t_3]=t_6, [t_1,t_2]=t_4$ |
| $1357L$ | $[t_1,t_3]=[t_2,t_4]=t_5, [t_1,t_4]=t_6, [t_1,t_2]=t_3,$ $[t_2,t_3]=[t_1,t_5]=t_7, [t_2,t_6]=[t_3,t_4]=\frac{1}{2}t_7$ |
| $247M$ | $[t_1,t_3]=t_5, [t_3,t_5]=t_7, [t_1,t_4]=[t_2,t_3]=t_6, [t_1,t_2]=t_4$ |
| $1357M$ | $[t_1,t_3]=[t_2,t_4]=t_5, [t_1,t_5]=t_7, [t_1,t_2]=t_3, [t_1,t_4]=t_6,$ $[t_2,t_6]=\lambda t_7, [t_3,t_4]=(1-\lambda)t_7$ |
| $1357M$ | $[t_2,t_4]=[t_1,t_3]=t_5, [t_1,t_5]=t_7, [t_1,t_4]=t_6, [t_2,t_6]=\lambda t_7, [t_3,t_4]=(1-\lambda)t_7, [t_1,t_2]=t_3$ |
| $1357N$ | $[t_2,t_4]=[t_1,t_3]=t_5, [t_3,t_4]=[t_1,t_5]=[t_4,t_6]=t_7, [t_1,t_4]=t_6, [t_2,t_3]=\lambda t_7, [t_1,t_2]=t_3$ |
| $247O$ | $[t_1,t_3]=t_5, [t_2,t_3]=[t_1,t_5]=t_7, [t_1,t_4]=[t_3,t_5]=t_6, [t_1,t_2]=t_4$ |
| $247Q$ | $[t_1,t_3]=t_5, [t_2,t_5]=[t_3,t_4]=t_7, [t_1,t_4]=[t_2,t_3]=t_6, [t_1,t_2]=t_4$ |

For $\dim L^2 = 4$, we need the structure and dimension of $\mathcal{M}(L)$ presented in Tables 7 and 8.

**Lemma 2.9:** *Let $L$ be a 7-dimensional NLA satisfying $\dim L^2 = 4$, let $I$ be a 1-dimensional central ideal of $L$, and let $L/I$ be isomorphic to $L_{6,26}$, $L_{6,23}$, or $L_{6,25}$. Then the structure and the dimension of $\mathcal{M}(L)$ are presented in Tables 7 and 8.*

***Proof:*** Based on the classification of 7-dimensional NLAs with $\dim L^2 = 4$ of [11], the algebras are obtained. For the calculation of $\dim \mathcal{M}(L)$, we apply the method used by Hardy and Stitzinger [16]. For computing $\dim \mathcal{M}(247C)$, we need the structure of this algebra as follows: Start with

$$\begin{array}{lll}
[t_1,t_2] = s_1 + t_4, & [t_2,t_4] = s_8, & [t_3,t_7] = s_{15}, \\
[t_1,t_3] = s_2 + t_5, & [t_2,t_5] = s_9, & [t_4,t_5] = s_{16}, \\
[t_1,t_4] = s_3 + t_6, & [t_2,t_6] = s_{10}, & [t_4,t_6] = s_{17}, \\
[t_1,t_5] = s_4 + t_7, & [t_2,t_7] = s_{11}, & [t_4,t_7] = s_{18} \\
[t_1,t_6] = s_5, & [t_3,t_4] = s_{12}, & [t_5,t_6] = s_{19}, \\
[t_1,t_7] = s_6, & [t_3,t_5] = s_{13} + t_6, & [t_5,t_7] = s_{20}, \\
[t_2,t_8] = s_7, & [t_3,t_6] = s_{14}, & [t_6,t_7] = s_{21},
\end{array}$$

where $s_1, \ldots, s_{21}$ generate $\mathcal{M}(247C)$. It follows from the the Jacobi identity on all possible triples that $t'_4 = t_4 + s_1, t'_5 = t_5 + s_2$, $t'_6 = t_6 + s_3$, and $t'_7 = t_7 + s_4$. A change of variables implies that $s_1 = s_2 = s_3 = s_4 = 0$. Thus $\mathcal{M}(247C) = \langle s_5, s_6, s_7, s_8, s_9, s_{11}, s_{13} \rangle$.

Finally, for $n \leq 6$, we state the following result.

**Lemma 2.10:** The only $n$-dimensional NLAs $L$ with $t(L) = 13$ when $n \leq 6$ are $L_{6,14}$ and $L_{6,16}$.

***Proof:*** By using Lemma 1.5 and Table 3, the proof is clear.

Consequently, the following main theorem characterizes NLAs with $t(L) = 13$. We recall that Throughout this article, we assume that all fields which are under assumption have the characteristics other than two.

**Table 8**

**7-dimensional NLAs with $\dim L^2 = 4$ and $L/I$ be isomorphic to $L_{6,23}$, $L_{6,25}$ or $L_{6,26}$ when $I$ is a 1-dimensional central ideal of $L$.**

| Nonabelian NLAs | $\dim \mathcal{M}(L)$ |
|---|---|
| $357A$, $2357B$, $1357L$, $1357M(\lambda = -1,2)$, $1357N(\lambda \neq 0)$ | 8 |
| $357B$, $357C$, $247C$, $247D$, $247I$, $247L$, $247O$, $247Q$, $2457A$, $2457B$, $2457C$, $2457D$, $2457F$, $2457I$, $1357D$, $1357H$, $1357M(\lambda \neq -1,2)$, $1357N(\lambda = 0)$ | 7 |
| $247A$, $247B$, $247M$, $2357A$, $1357G$ | 6 |
| $1357E$, $1357F$, $1357I$, $1357J$ | 5 |

**Theorem 2.11:** *Suppose that* $L$ *is an NLA. Then* $t(L) = 13$ *if and only if* $L$ *is isomorphic to one of the following algebras:*

$L_{4,3} \oplus H(1)$, $H(1) \oplus H(2)$, $H(1) \oplus A(12)$,

$H(i) \oplus A(12-2i)$, for all $i = 2, \dots, 6$,

$M \oplus A(1)$, for all $M = 157, 27A, 37B, 37C, 37D$,

$L_{6,i} \oplus A(1)$ for all $i = 11,12,20$,

$L_{6,j}(\varepsilon) \oplus A(1)$, for all $j = 19,24$,

$L_{6,k} \oplus A(2)$ for all $k = 10,26$,

$L_{5,l} \oplus A(2)$, for all $l = 6,7,9$,

$L_{6,22}(\varepsilon) \oplus A(3)$,

$S_1$, $S_2$, $S_3$, $L_{6,14}$, $L_{6,16}$, $147A$, $357A$, $147B$, $257B$, $2357B$, $257D$, $257E$, $257G$, $257H$, $257I$, $257J$, $1357L$, $1357M(\lambda = -1,2)$, $1357N(\lambda \neq 0)$.

***Proof:*** Let $L$ be an $n$-dimensional NLA satisfying $t(L) = 13$. By relation (1), we have $15 = n + s(L)$. According to Lemmas 2.1, 2.3, and 2.4, the only 15-, 13-, and 9-dimensional NLAs with $t(L) = 13$ are $H(1) \oplus A(12)$, $H(i) \oplus A(12-2i)$, for all $i = 2, \dots, 6$, and $L_{6,22}(\varepsilon) \oplus A(3)$.

According to Lemma 2.2, for all $n = 14$, $12$, $11$, or $10$, there is no $n$-dimensional NLAs $L$ with $t(L) = 13$.

By Lemma 2.5, the only 8-dimensional NLAs $L$ satisfying $t(L) = 13$ are $L_{6,10} \oplus A(2)$, $L_{6,26} \oplus A(2)$, $H(1) \oplus H(2)$, $S_1$, $S_2$, $S_3$, and $M \oplus A(1)$, for all $M = 157, 27A, 37B, 37C, 37D$.

By Lemma 2.10, the only $n$-dimensional NLAs $L$ with $t(L) = 13$ when $n \leq 6$ are $L_{6,14}$ and $L_{6,16}$.

By Lemma 2.6, there is no 7-dimensional NLAs $L$ satisfying $t(L) = 13$ when $\dim L^2 = 5,2$ or $1$.

It follows from Lemma 2.7 that the 7-dimensional NLAs $L$ with $t(L) = 13$ when $\dim L^2 = 3$ are

$L_{6,i} \oplus A(1)$ for all $i = 11,12,20$, $L_{6,j}(\varepsilon) \oplus A(1)$, for all $j = 19,24$,

$L_{5,k} \oplus A(2)$, for all $k = 6,7,9$, $L_{4,3} \oplus H(1)$, $147A$, $147B$, $257B$, $257D$, $257E$, $257G$, $257H$, $257I$, or $257J$.

By Lemma 2.8, when $I$ is a 1-dimensional central ideal of $L$, there is no 7-dimensional NLAs $L$ with $t(L) = 13$ and $\dim L^2 = 4$ when $L/I$ is isomorphic to one of the following algebras:

$L_{5,i} \oplus A(1)$, for all $i = 6,7,9,$, $L_{6,j}$, for all $j = 11,12,13,20$, $L_{6,19}(\varepsilon)$, or $L_{6,24}(\varepsilon)$.

By Lemma 2.9, the only 7-dimensional NLAs with $t(L) = 13$ are $357A$, $2357B$, $1357L$, $1357M(\lambda = -1,2)$, or $1357N(\lambda \neq 0)$. This completes the proof.

**References**

[1] Peggy Batten, Multipliers and covers of Lie algebras, Ph.D. dissertation, Dept. Math., North Carolina State Univ., Raleigh, NC, USA (1993).

[2] Peggy Batten, Kay Moneyhun, and Ernest Stitzinger, "On characterizing nilpotent Lie algebras by their multipliers," *Communications in Algebra*, vol. 24, pp. 4319–4330 (1996).

[3] Lindsey Bosko, "On Schur multipliers of Lie algebras and groups of maximal class," *International Journal of Algebra and Computation*, vol. 20, pp. 807–821 (2010).

[4] Serena Cicalò, Willem A. de Graaf, and Csaba Schneider, "Six-dimensional nilpotent Lie algebras," *Linear Algebra and Its Applications*, vol. 436, pp. 163–189 (2012).

[5] Hamid Darabi, Mehdi Eshrati, and Homayoon Arabyani, "Characterizing nilpotent n-Lie algebras by their multiplier," *Results in Mathematics*, vol. 76, no. 1, p. 23 (2021).

[6] Hamid Darabi, Mehdi Eshrati, and Babak Jabbar Nezhad, "On the multiplier of filiform Filippov algebras," *Results in Mathematics*, vol. 76, no. 4, p. 190 (2021).

[7] Hamid Darabi and Farshid Saeedi, "On the Schur multiplier of n-Lie algebras," *Journal of Lie Theory*, vol. 27, pp. 271–281 (2017).

[8] Hamid Darabi, Farshid Saeedi, and Mehdi Eshrati, "A characterization of finite dimensional nilpotent Filippov algebras," *Journal of Geometry and Physics*, vol. 101, pp. 100–107 (2016).

[9] Hamid Darabi, Farshid Saeedi, and Mehdi Eshrati, "Capable n-Lie algebras and the classification of nilpotent n-Lie algebras with s(A)=3," *Journal of Geometry and Physics*, vol. 110, pp. 25–29 (2016).

[10] Hamid Darabi and Mostafa Sajedi, "Characterizing nilpotent n-Lie algebras by their multiplier, t(A) = 11, 12," in Proc. 14th Iranian Int. Group Theory Conf., pp. 189–194, Feb. 3–4 (2022).

[11] Ming-Peng Gong, Classification of nilpotent Lie algebras of dimension 7, Ph.D. dissertation, Dept. Pure Math., Univ. of Waterloo, Waterloo, ON, Canada (1998).

[12] Peter Hardy, "On characterizing nilpotent Lie algebras by their multipliers III," *Communications in Algebra*, vol. 33, pp. 4205–4210 (2005).

[13] Peter Hardy and Ernest Stitzinger, "On characterizing nilpotent Lie algebras by their multipliers t(L) = 3, 4, 5, 6," *Communications in Algebra*, vol. 26, pp. 3527–3539 (1998).

[14] Saudamini Nayak, "Multipliers of nilpotent Lie superalgebras," *Communications in Algebra*, vol. 47, no. 2, pp. 689–705 (2019).

[15] Peyman Niroomand, "On dimension of the Schur multiplier of nilpotent Lie algebras," *Central European Journal of Mathematics*, vol. 9, pp. 57–65 (2011).

[16] Peyman Niroomand and Francesco Russo, "A note on the Schur multiplier of a nilpotent Lie algebra," *Communications in Algebra*, vol. 39, pp. 1293–1297 (2011).

[17] Peyman Niroomand and Afsaneh Shamsaki, "On characterizing nilpotent Lie algebra by their multiplier, s(L) = 6, 7," *Results in Mathematics*, vol. 77, no. 2, p. 82 (2022).

[18] Hesam Safa, "The Schur multiplier of an n-Lie superalgebra," *Communications in Algebra*, vol. 50, no. 7, pp. 2983–2996 (2022).

[19] Issai Schur, "Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen," *Journal für die reine und angewandte Mathematik* (Crelle's Journal), vol. 127, pp. 20–50 (1904).

[20] Afsaneh Shamsaki and Peyman Niroomand, "On characterizing nilpotent Lie algebras by their multiplier, s(L) = 4," *Rendiconti del Circolo Matematico di Palermo*, vol. 69, pp. 259–272 (2020).

[21] Afsaneh Shamsaki and Peyman Niroomand, "Characterizing finite dimensional nilpotent Lie algebras by the dimension of their Schur multipliers, s(L) = 5," *Mathematical Reports*, vol. 25, no. 2, pp. 301–312 (2023).

[22] Hasan M. S. Shlaka and Falah S. Kareem, "Abelian non Jordan-Lie inner ideals of the orthogonal finite dimensional Lie algebras," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 25, no. 5, pp. 1547–1561 (2022).

*Received June, 2024*