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Application of α-β-FG-Khan contractions to an initial value problem in b-metric spaces

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Abstract

We establish sufficient criteria for the existence and uniqueness of fixed points for α-β-FG-Khan contractions in b-metric spaces. Examples are provided to illustrate these ideas. Additionally, we apply these theoretical results to solving an initial value problem (IVP).

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1. Introduction

The theory of b-metric spaces, introduced by Bakhtin [6] and later formalized by Czerwik [8, 9], generalizes classical metric spaces by incorporating a scaling coefficient $s \geq 1$ into the triangle inequality. This framework has proven invaluable in areas such as functional analysis, fractal geometry, and computational dynamics. A central idea in these and related structures is the extension of fixed point theory, which serves as a foundation for establishing the existence and uniqueness of solutions in nonlinear analysis. Consequently, its development within b-metric spaces has become a significant and active branch

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of mathematical analysis, driving the exploration of numerous novel contractions (see, e.g., [2]–[5], [7], [10], [15]).

Recent studies have explored generalized contractions to advance fixed point theory. For instance, Hussain, Salimi and Latif [12] presented fixed point results for single and set-valued $\alpha - \eta - \psi -$contractive mappings, while Raturi and Chauhan [16] introduced results in bipolar controlled b-dislocated metric spaces and applied them to fractional differential equations, demonstrating practical applications.

Important developments also include Wardowski [19], who introduced F-contractions, later generalized by Abbas, Ali and Romaguera [1], Wardowski and Dung [20], and Piri and Kumam [21]. Samet, Vetro and Vetro [18] generalized the Banach contraction principle using α -admissible and α-contractive mappings. Building on these ideas, Hussain et al. [13] introduced $\alpha - \beta - FG -$ contractions in b-metric and partially ordered b-metric spaces

In this article, we extend these developments by introducing a new type of contraction, called the $\alpha - \beta - FG -$Khan contraction. We establish criteria for the existence and uniqueness of fixed points for these mappings in b-metric spaces. Furthermore, to demonstrate the applicability of our theoretical results, we apply the proposed framework to solve an initial value problem (IVP), thereby connecting the abstract theory of contractions with practical problems in mathematical analysis.

We are now representing a concept of $\alpha - \psi -$ contractive and α-admissible mapping [11] and b-metric space [8].

Definition 1.1: Consider $\Phi: \Upsilon \to \Upsilon$ is a mapping on a set Υ and $\alpha: \Upsilon \times \Upsilon \to 0, \infty)$ is a function. We say Φ is an α- admissible map, if $\varpi, \upsilon \in \Upsilon$, $\alpha(\varpi, \upsilon) \geq 1$ implies $\alpha(\Phi\varpi, \Phi\upsilon) \geq 1$.

Let Ψ denote all non-decreasing functions $\psi: [0, \infty) \to 0, \infty)$ so that $\sum_{n=1}^{\infty} \Psi^n(\zeta) < \infty$ for every $\zeta < \infty$ in which Ψ^n is Ψ' n-th iteration. According to Salimi, Latif and Hussain [17], their results are as follows:

Theorem 1.2: *Consider (Υ, π) is a complete metric space and Φ is an $\alpha -$admissible mapping. Suppose that $\alpha(\varpi, \upsilon)\pi(\Phi\varpi, \Phi\upsilon) \leq \psi(\pi(\varpi, \upsilon))$, for all $\varpi, \upsilon \in \Upsilon$, where $\psi \in \Psi$. Additionally, assume that there is $\varpi_0 \in \Upsilon$ such that $\alpha(\varpi_0, \Phi\varpi_0) \geq 1$, and either Φ is continuous, or, for any given (ϖ_μ) in Υ with $\alpha(\varpi_\mu, \varpi_{\mu+1}) \geq 1$ for every $\mu \in N \cup \{0\}$ so that $\varpi_\mu \to \varpi$ as $\mu \to \infty$, we have $\alpha(\varpi_\mu, \varpi) \geq 1$ for every $\mu \in N \cup \{0\}$. Then there is a fixed point to Φ.*

Hussain, Kutbi and Salimi [11] and Karap¬nar, Kumam and Salimi, [14] extended this result in the following way:

Definition 1.3: Consider (Υ, π) is a metric space, $\alpha: \Upsilon \times \Upsilon \to 0, \infty)$ and $\Phi: \Upsilon \to \Upsilon$ are mappings. Φ is an α-continuous on (Υ, π) as we say, if, for given $\varpi \in \Upsilon$ and a sequence $(\varpi_\mu), \varpi_\mu$ converges to ϖ and $\alpha(\varpi_\mu, \varpi_{\mu+1}) \geq 1$ for all $\mu \in N$ then $(\Phi\varpi_\mu)$ converges to $(\Phi\varpi)$.

Example 1.4: Let $\Upsilon = [0, \infty)$ and define the metric $\pi(\varpi, \upsilon) = |\varpi - \upsilon|$ on Υ. Define the mapping $\Phi: \Upsilon \to \Upsilon$ as: $\Phi\varpi = \varpi^5$ if $\varpi \in 0,1]$, and $\Phi\varpi = \sin\pi\varpi + 2$ if $\varpi \in (1, \infty)$. Also, define the function $\alpha: \Upsilon \times \Upsilon \to 0, +\infty)$ as: $\alpha(\varpi, \upsilon) = \varpi^2 + \upsilon^2 + 1$, if $\varpi, \upsilon \in 0,1]$, and $\alpha(\varpi, \upsilon) = 0$ otherwise. Although Φ is obviously not continuous, it is α −continuous on (Υ, π).

Definition 1.5: Let $\Phi: \Upsilon \to \Upsilon$ and $\alpha: \Upsilon \times \Upsilon \to 0, +\infty)$ be two functions. We say Φ is a triangular α −admissible mapping, if the following conditions hold:

1. *For all* $\varpi, \upsilon \in \Upsilon$, *if* $\alpha(\varpi, \upsilon) \geq 1$, *then* $\alpha(\Phi\varpi, \Phi\upsilon) \geq 1$,
2. *For all* $\varpi, \upsilon, c \in \Upsilon$, *if* $\alpha(\varpi, c) \geq 1$ *and* $\alpha(c, \upsilon) \geq 1$, *then* $\alpha(\varpi, \upsilon) \geq 1$.

Lemma 1.6: *Consider Φ is a triangular α-admissible mapping. Suppose that there is $\varpi_0 \in \Upsilon$ such that $\alpha(\varpi_0, \Phi\varpi_0) \geq 1$. Define the sequence (ϖ_μ) by $\varpi_\mu = \Phi^\mu \varpi_0$. Then $\alpha(\varpi_\nu, \varpi_\mu) \geq 1$, for all $\nu, \mu \in N$, with $\upsilon < \mu$.*

Definition 1.7: Let Υ be a non empty set. A function $\pi: \Upsilon \times \Upsilon \to R^+$ is called a b-metric space if there exists a constant $s \geq 1$ such that for all $\varpi_1, \varpi_2, \varpi_3 \in \Upsilon$, the following conditions are holds:

i. $\pi(\varpi_1, \varpi_2) = 0$ *if, and only if,* $\varpi_1 = \varpi_2$;
ii. $\pi(\varpi_1, \varpi_2) = \pi(\varpi_2, \varpi_1)$;
iii. $\pi(\varpi_1, \varpi_2) \leq s\ [\pi(\varpi_1, \varpi_3) + \pi(\varpi_3, \varpi_2)]$.

The pair (Υ, π) is then called a b-metric space with parameter $s \geq 1$.

Definition 1.8: Let $(\varpi_\mu) = \{\varpi_\mu,\ \mu \in N\}$ be a sequence in a b-metric space (Υ, π):

i. A sequence (ϖ_μ) converges in Υ if there exists $\varpi \in \Upsilon$ such that $\lim_{\mu\to\infty} \pi(\varpi_\mu, \varpi) = 0$.
ii. A sequence (ϖ_μ) is a Cauchy in Υ if for every $\varepsilon > 0$, there exists $\mu_0 \in N$ such that
$$\pi(\varpi_\mu, \varpi_\nu) < \varepsilon, \forall \nu, \mu > \mu_0.$$
iii. The space Υ is called complete if every Cauchy sequence in Υ converges in Υ.

Note that a b-metric need not be a continuous function.

Lemma 1.9: *Consider (Υ, π) is a b-metric space with $s \geq 1$, and asume that (ϖ_μ) and (υ_μ) are convergent to ϖ and υ, respectively. Then we get*

$$\frac{1}{s^2}\pi(\varpi, \upsilon) \leq \liminf_{\mu\to\infty} \varpi(\varpi_\mu, \upsilon_\mu) \leq \limsup_{\mu\to\infty}\ \pi(\varpi_\mu, \upsilon_\mu) \leq s^2\pi(\varpi, \upsilon).$$

And also, if $\varpi = \upsilon$, then we get $\lim_{n\to\infty}\ \pi(\varpi_\mu, \upsilon_\mu) = 0$. Furthermore, for every $c \in \Upsilon$, we get

$$\frac{1}{s^2}\pi(\varpi, c) \leq \liminf_{\mu\to\infty} \pi(\varpi_\mu, c) \leq \limsup_{\mu\to\infty} \pi(\varpi_\mu, c) \leq s\pi(\varpi, c).$$

2. $\alpha - \beta - FG -$Khan Contractions

In this section, we introduce the concept of an $\alpha - \beta - FG -$ Khan contraction and establish several fixed point theorems in b-metric spaces.

Consider that C_F is the collection of all functions $F: R^+ \to R$, that satisfy the axioms given as follows:

(a_1) F is strictly increasing and continuous;

(a_2) for every $(\zeta_\mu) \subseteq R^+$, $\lim_{\mu\to\infty} \zeta_\mu = 0$ if, and only if, $\lim_{\mu\to\infty} F(\zeta_\mu) = -\infty$.

and $C_{G,\beta}$ is the collection of pairs (G, β), where $G: R^+ \to R$ and $\beta: [0, \infty) \to 0,1)$, satisfying the following axioms:

(a_3) for every $(\zeta_\mu) \subseteq R^+$, $\limsup_{\mu\to\infty} G(\zeta_\mu) \geq 0$ if, and only if, $\limsup_{\mu\to\infty} \zeta_\mu \geq 1$.

(a_4) for every $(\zeta_\mu) \subseteq 0, \infty)$, $\limsup_{\mu\to\infty} \beta(\zeta_\mu) = 1$ implies $\lim_{\mu\to\infty} \zeta_\mu = 0$,

(a_5) for every $(\zeta_\mu) \subseteq R^+$, $\Sigma_{\mu=1}^{\infty} G(\beta(\zeta_\mu)) = -\infty$.

Example 2.1: Let $F(\zeta) = G(\zeta) = ln\zeta$ and $\beta(\zeta) = k \in (0,1)$, then $F \in C_F$ and $(G, \beta) \in C_{G,\beta}$.

Example 2.2; Let $F(\zeta) = \frac{-1}{\sqrt{\zeta}}$, $G(\zeta) = ln\zeta$ and $\beta(\zeta) = \frac{1}{s}e^{-\zeta}$ for $\zeta > 0$ and $\beta(0) = 0$. Then $F \in C_F$ and $(G, \beta) \in C_{G,\beta}$.

Definition 2.3: Consider (Υ, π) is a b-metric space with $s \geq 1$ and $\Phi : \Upsilon \to \Upsilon$ be a mappimg. Suppose that $\alpha: \Upsilon \times \Upsilon \to 0, \infty)$ is a function, we say that Φ is $\alpha - \beta - FG -$Khan contractions if, for all $\varpi, \upsilon \in \Upsilon$ with $\alpha(\varpi, \upsilon) > 1$ and $\pi(\Phi\varpi, \Phi\upsilon) > 0$, we have

$$F(s\lambda\pi(\Phi\varpi, \Phi\upsilon)) \leq F(\kappa_s(\varpi, \upsilon)) + G(\beta(\kappa_s(\varpi, \upsilon))), s\lambda \geq 2, \tag{2.1}$$

where

$$\kappa_s(\varpi, \upsilon) = \pi(\varpi, \upsilon) + \frac{\pi(\Phi\varpi, \varpi)\pi(\varpi, \Phi\upsilon) + \pi(\upsilon, \Phi\upsilon)\pi(\upsilon, \Phi\varpi)}{\pi(\varpi, \Phi\upsilon) + \pi(\upsilon, \Phi\varpi)}.$$

Example 2.4: Let $\Upsilon = [0, \infty)$, and $\pi(\varpi, \upsilon) = (\varpi - \upsilon)^2$. Define $\Phi: \Upsilon \to \Upsilon$ by $\Phi\varpi = \frac{\varpi}{2}$, $\alpha: \Upsilon \times \Upsilon \to 0, \infty), \alpha(\varpi, \upsilon) = 2$ for all $\varpi, \upsilon \in \Upsilon$, $F(\zeta) = -\frac{1}{\zeta}$, $G(\zeta) = \zeta$, $\beta(\zeta) = \frac{1}{2}e^{-\zeta}$, for $\zeta > 0$. Then Φ is a $\alpha - \beta - FG -$ Khan contraction.

Solution: If $\varpi = \upsilon$, this due to $\pi(\Phi\varpi, \Phi\upsilon) = 0$, For $\varpi \neq \upsilon$, let $s = 2$, $\lambda = 1$, $s\lambda \geq 2$, $F(\zeta) = -\frac{1}{\zeta}, G(\zeta) = \zeta$, and $\beta(\zeta) = \frac{1}{2}e^{-\zeta}$, $\pi(\Phi\varpi, \Phi\upsilon) = (\frac{\varpi}{2} - \frac{\upsilon}{2})^2 = \frac{1}{4}(\varpi - \upsilon)^2$,

$$\kappa_s(\varpi,\upsilon)=\pi(\varpi,\upsilon)+\frac{\frac{\varpi^2}{4}(\varpi-\frac{\upsilon}{2})^2+\frac{\upsilon^2}{4}(\upsilon-\frac{\varpi}{2})^2}{(\varpi-\frac{\upsilon}{2})^2+(\upsilon-\frac{\varpi}{2})^2}>\pi(\varpi,\upsilon),$$

then $F(s\lambda\pi(\Phi\varpi,\Phi\upsilon))=\frac{-2}{(\varpi-\upsilon)^2}$, and

$$F(\kappa_s(\varpi,\upsilon))+G(\beta(\kappa_s(\varpi,\upsilon)))=\frac{-1}{\kappa_s(\varpi,\upsilon)}+\frac{1}{2}e^{-\kappa_s(\varpi,\upsilon)}>\frac{-2}{(\varpi-\upsilon)^2}.$$

Therefore equation (2.1) holds for all $\varpi,\upsilon\in\Upsilon$, with $\varpi\neq\upsilon$, then Φ is a $\alpha-\beta-FG$ −Khan contraction.

The following example shows that there exists a $\alpha-\beta-FG$ − Khan contraction which is not an $\alpha-\beta-FG$ contraction.

Example 2.5: Let $\Upsilon=[0,\infty)$, and $\pi(\varpi,\upsilon)=(\varpi-\upsilon)^2$. Define $\Phi:\Upsilon\to\Upsilon$ by $\Phi\varpi=\frac{\varpi}{2}$, $\alpha:\Upsilon\times\Upsilon\to 0,\infty)$, $\alpha(\varpi,\upsilon)=1$ for all $\varpi,\upsilon\in\Upsilon$, $F(\zeta)=-\frac{1}{\zeta}$, $G(\zeta)=\ln\zeta$, $\beta(\zeta)=\frac{1}{1+\zeta}$. Then Φ is a $\alpha-\beta-FG$ −Khan contraction but Φ is not an $\alpha-\beta-FG$ contraction.

Solution: For arbitrary $\varpi,\upsilon\in\Upsilon$, $\pi(\Phi\varpi,\Phi\upsilon)=\frac{1}{4}(\varpi-\upsilon)^2$, and $\kappa_s(\varpi,\upsilon)>\pi(\varpi,\upsilon)$, we get $\frac{-1}{\kappa_s(\varpi,\upsilon)}\geq\frac{-1}{\pi(\varpi,\upsilon)}$

$$\ln\left(\frac{1}{1+\kappa_s(\varpi,\upsilon)}\right)\geq-\ln(1+\kappa_s(\varpi,\upsilon)\geq-\kappa_s(\varpi,\upsilon)$$

let $s=2$, $\lambda=\frac{1}{2}$

$$F(\kappa_s(\varpi,\upsilon))+G(\beta(\kappa_s(\varpi,\upsilon)))=\frac{-1}{\kappa_s(\varpi,\upsilon)}+\ln\left(\frac{1}{1+\kappa_s(\varpi,\upsilon)}\right)$$
$$\geq\frac{-1}{\kappa_s(\varpi,\upsilon)}\geq\frac{-1}{\pi(\varpi,\upsilon)}>\frac{-4}{\pi(\varpi,\upsilon)}=F(s\lambda\pi(\Phi\varpi,\Phi\upsilon))$$

if we take $\varpi=4,\upsilon=0$, Φ is not an $\alpha-\beta-FG$ contraction.

We now present and prove our main theorem:

Theorem 2.6: *Let (Υ,π) be a complete b-metric space with constant $s\geq 1$ and let $\Phi:\Upsilon\to\Upsilon$ be a mapping satisfying the following conditions:*

1. *Φ is a triangular α −admissible;*
2. *Φ is an $\alpha-\beta-FG$ −Khan contraction;*
3. *There exists $\varpi_0\in\Upsilon$ such that $\alpha(\varpi_0,\Phi\varpi_0)\geq 1$;*
4. *Φ is α-continuous.*

Then Φ has a fixed point. Furthermore, if $\alpha(\varpi,\upsilon)\geq 1$ for all $\varpi,\upsilon\in Fix(\Phi)$, then Φ has a unique fixed point.

Proof: Let $\varpi_0\in\Upsilon$ be such that $\alpha(\varpi_0,\Phi\varpi_0)\geq 1$. For $\varpi_0\in\Upsilon$, we define the sequence (ϖ_μ) by $\varpi_\mu=\Phi^\mu\varpi_0=\Phi\varpi_{\mu-1}$. Since Φ is an α -admissible mapping, then $\alpha(\varpi_0,\varpi_1)=\alpha(\varpi_0,\Phi\varpi_0)\geq 1$. We have continued this process

$1 \leq \alpha(\varpi_{\mu-1}, \varpi_\mu)$ for every $\mu \in N$. Further, let there is $\mu_0 \in N$ such that, $\varpi_{\mu_0} = \varpi_{\mu_0+1} = \Phi\varpi_{\mu_0}$, then ϖ_{μ_0} is a fixed point of Φ. We suppose that $\varpi_\mu \neq \varpi_{\mu+1}$, i.e., $\pi(\Phi\varpi_{\mu-1}, \Phi\varpi_\mu) > 0$ for every $\mu \in N \cup \{0\}$. Since, Φ is a $\alpha - \beta - FG$ −Khan contraction, then we obtain,

$$F(\pi(\Phi\varpi_{\mu-1}, \Phi\varpi_\mu)) \leq F(2\pi(\Phi\varpi_{\mu-1}, \Phi\varpi_\mu)) \leq F(\lambda s(\pi(\Phi\varpi_{\mu-1}, \Phi\varpi_\mu)))$$
$$\leq F(\kappa_s(\varpi_{\mu-1}, \varpi_\mu)) + \gamma(\beta(\kappa_s(\varpi_{\mu-1}, \varpi_\mu))),$$

where

$$\kappa_s(\varpi_{\mu-1}, \varpi_\mu) = \pi(\varpi_{\mu-1}, \varpi_\mu) +$$
$$\frac{\pi(\varpi_{\mu-1},\Phi\varpi_{\mu-1})\pi(\varpi_{\mu-1},\Phi\varpi_\mu)+\pi(\varpi_\mu,\Phi\varpi_\mu)\pi(\varpi_\mu,\Phi\varpi_{\mu-1})}{\pi(\varpi_{\mu-1},\Phi\varpi_\mu)+\pi(\varpi_\mu,\Phi\varpi_{\mu-1})}$$
$$= \pi(\varpi_{\mu-1}, \varpi_\mu) + \frac{\pi(\varpi_{\mu-1},\varpi_\mu)\pi(\varpi_{\mu-1},\varpi_{\mu+1})+\pi(\varpi_\mu,\varpi_{\mu+1})\pi(\varpi_\mu,\varpi_\mu)}{\pi(\varpi_{\mu-1},\varpi_{\mu+1})+\pi(\varpi_\mu,\varpi_\mu)}$$
$$= 2\pi(\varpi_{\mu-1}, \varpi_\mu), s \geq 1$$

we get

$$F(\pi(\varpi_{\mu+1}, \varpi_\mu)) \leq F(2\pi(\varpi_{\mu-1}, \varpi_\mu)) + G(\beta(2\pi(\varpi_{\mu-1}, \varpi_\mu))),$$

we are still applying Φ is a $\alpha - \beta - FG$ −Khan contraction, we have

$$F(2\pi(\varpi_{\mu-1}, \varpi_\mu)) \leq F(s(\pi(\Phi\varpi_{\mu-2}, \Phi\varpi_{\mu-1})))$$
$$\leq F(\kappa_s(\varpi_{\mu-2}, \varpi_{\mu-1})) + G(\beta(\kappa_s(\varpi_{\mu-2}, \varpi_{\mu-1})))$$
$$= F(2\pi(\varpi_{\mu-2}, \varpi_{\mu-1})) + G(\beta(2\pi(\varpi_{\mu-2}, \varpi_{\mu-1}))).$$

We are getting on in this way

$$F(\pi(\varpi_{\mu+1}, \varpi_\mu)) \leq F(2\pi(\varpi_0, \varpi_1)) +_1^\mu G(\beta 2(\pi(\varpi_{i-1}, \varpi_i))).$$

Letting the limit as $\mu \to \infty$, we have $\lim F(\pi(\varpi_{\mu+1}, \varpi_\mu)) = -\infty$, this due to $\lim\limits_{\mu\to\infty} \pi(\varpi_{\mu+1}, \varpi_\mu) = 0$, ($F \in C_F$, and $G, \beta \in C_{G,\beta}$). Assume that (ϖ_μ) is not a Cauchy, subsequently there exists $\varepsilon > 0$ and the subsequences $(\varpi_{\mu_\hbar})$ and $(\varpi_{v_\hbar})$ of (ϖ_μ) with $\mu_\hbar > v_\hbar >_\hbar$ such that $\pi(\varpi_{\mu_\hbar}, \varpi_{v_\hbar}) \geq \varepsilon$, and $\mu_\hbar$ is the least number. We obtain $\pi(\varpi_{\mu_{\hbar-1}}, \varpi_{v_\hbar}) < \varepsilon$. By using triangular inequality,

$$\varepsilon \leq \pi(\varpi_{\mu_\hbar}, \varpi_{v_\hbar}) \leq s[\pi(\varpi_{\mu_\hbar}, \varpi_{v_{\hbar+1}}) + \pi(\varpi_{v_{\hbar+1}}, \varpi_{v_\hbar})]$$
$$\frac{\varepsilon}{2s} \leq \lim_{\hbar\to\infty} \sup \pi(\varpi_{\mu_\hbar}, \varpi_{v_{\hbar+1}}).$$

Also $\lim\limits_{\hbar\to\infty} \sup \pi(\varpi_{\mu_{\hbar-1}}, \varpi_{v_\hbar}) \leq \varepsilon$, and $\pi(\varpi_{v_\hbar}, \varpi_{\mu_\hbar}) \leq s(\pi(\varpi_{v_\hbar}, \varpi_{\mu_{\hbar-1}}) + \pi(\varpi_{\mu_{\hbar-1}}, \varpi_{\mu_\hbar}))$,

$$\lim_{\hbar\to\infty} \sup \pi(\varpi_{v_\hbar}, \varpi_{\mu_\hbar}) \leq \varepsilon s.$$

Again we use the triangular inequality, $\pi(\varpi_{v_{\hbar+1}}, \varpi_{\mu_{\hbar-1}}) \leq s(\pi(\varpi_{v_{\hbar+1}}, \varpi_{v_\hbar}) + \pi(\varpi_{v_\hbar}, \varpi_{\mu_{\hbar-1}})$. Taking the upper limit as $\hbar \to \infty$, $\lim\limits_{\hbar\to\infty} \sup \pi(\varpi_{v_{\hbar+1}}, \varpi_{\mu_{\hbar-1}}) \leq \varepsilon s$. So Φ is a triangular α-admissible mapping, we have that $\alpha(\varpi_{v_\hbar}, \varpi_{\mu_\hbar}) \geq 1$, and Φ is a $\alpha - \beta - FG$ −Khan contraction,

$$F(2s(\pi(\varpi_{v_{\hbar+1}},\varpi_{\mu_\hbar}) \leq F(\lambda s(\pi(\Phi\varpi_{v_\hbar},\Phi\varpi_{\mu_{\hbar-1}}) \leq F(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}})) + G(\beta(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}))),$$

$$\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}) = \pi(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}) + \frac{\pi(\varpi_{v_\hbar},\Phi\varpi_{v_\hbar})\pi(\varpi_{v_\hbar},\Phi\varpi_{\mu_{\hbar-1}})+\pi(\varpi_{\mu_{\hbar-1}},\Phi\varpi_{\mu_{\hbar-1}})\pi(\varpi_{\mu_{\hbar-1}},\Phi\varpi_{v_\hbar})}{\pi(\varpi_{v_\hbar},\Phi\varpi_{\mu_{\hbar-1}})+\pi(\varpi_{\mu_{\hbar-1}},\Phi\varpi_{v_\hbar})}.$$

Letting the upper limit as $\hbar \to \infty$, $\lim_{\hbar\to\infty}\sup\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}) = \lim_{\hbar\to\infty}\sup\pi(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}) \leq \varepsilon$, we get

$$F\left(2s\frac{\varepsilon}{2s}\right) \leq F(s\lambda \lim_{\hbar\to\infty}\sup\pi(\varpi_{v_{\hbar+1}},\varpi_{\mu_\hbar}) \leq$$

$$\lim_{\hbar\to\infty}\sup F(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}})) + \lim_{\hbar\to\infty}\sup G(\beta(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}))),$$

$$F(\varepsilon) \leq F(\varepsilon) + \lim_{\hbar\to\infty}\sup G(\beta(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}))),$$

hence $\lim_{\hbar\to\infty}\sup G(\beta(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}))) \geq 0$, and since $\beta(\zeta) < 1$ for all $\zeta \geq 0$, we have $\lim_{\hbar\to\infty}\sup G(\beta(\kappa_s(a_{v_\hbar},a_{\mu_{\hbar-1}}))) = 1$, therefore $\lim_{\hbar\to\infty}\sup G(\beta(\kappa_s(\varpi_{v_\hbar},\varpi_{\mu_{\hbar-1}}))) = 0$, which is impossible. Hence (ϖ_μ) is sequence of Cauchy in Υ. Since (Υ,π) is complete, the sequence (ϖ_μ) converges to some $\varpi \in \Upsilon$, i.e., $\lim_{\mu\to\infty}\pi(\varpi_\mu,\varpi) = 0$. Suppose that $\varpi \neq \Phi\varpi$. By Lemma (1.9), we obtain

$$\frac{1}{s^2}\pi(\varpi,\Phi\varpi) \leq \lim_{\mu\to\infty}\inf\pi(\varpi_\mu,\Phi\varpi_\mu) \leq$$

$$\lim_{\mu\to\infty}\sup\pi(\varpi_\mu,\Phi\varpi_\mu) = \lim_{\mu\to\infty}\sup\pi(\varpi_\mu,\varpi_{\mu+1}) = 0.$$

Hence, we have $\pi(\varpi,\Phi\varpi) = 0$ and so $\Phi\varpi = \varpi$, this ϖ is a fixed point of Φ. Let $\varpi, v \in Fix\Phi$, $\varpi \neq v$ and Φ is a $\alpha - \beta - FG -$ Khan contraction, we get,

$$F(\pi(\Phi\varpi,\Phi v)) \leq F(2\pi(\Phi\varpi,\Phi v)) \leq F(\lambda s(\pi(\Phi\varpi,\Phi v)) \leq$$

$$F(\kappa_s(\varpi,v)) + G(\beta(\kappa_s(\varpi,v))),$$

$$\kappa_s(\varpi,v) = \pi(\varpi,v) + \frac{\pi(\varpi,v)\pi(\varpi,\varpi)+\pi(v,v)\pi(\varpi,v)}{\pi(\varpi,v)+\pi(\varpi,v)} = \pi(\varpi,v),$$

we get $F(\pi(\varpi,v)) = F(\pi(\Phi\varpi,\Phi v)) \leq F(\pi(\varpi,v)) + G(\beta(\pi(\varpi,v)))$, so $G(\beta(\pi(\varpi,v))) \geq 0$, which produces that $\beta(\pi(\varpi,v)) \geq 1$, is a contradiction. Hence $\varpi = v$. Consequently, Φ has a unique fixed point.

Example 2.7; Consider b-metric space $(\Upsilon,\pi), \Upsilon = (-\infty,\infty)$, and $\pi(\varpi,v) = (\varpi - v)^2$. Define $\Phi\colon \Upsilon \to \Upsilon$ by $\Phi\varpi = q\varpi$, $0 < q \leq 2-\sqrt{2}$, $\alpha(\varpi,v) = 2$ for all $\varpi, v \in \Upsilon$, $F(\zeta) = \ln\zeta$, $G(\zeta) = \ln\zeta$, $\beta(\zeta) = q$. For $\varpi \neq v$, we have

$$\ln(2\lambda q^2(\varpi - v)^2) \leq \ln[\pi(\varpi,v) + \frac{\pi(\Phi\varpi,\varpi)\pi(\varpi,\Phi v)+\pi(\Phi v,v)\pi(\Phi\varpi,v)}{\pi(\varpi,\Phi v)+\pi(\Phi\varpi,v)}] + \ln q,$$

its clear that Φ satisfies hypothesis of Theorem 2.6, then Φ has a unique fixed point $\varpi = 0$.

Corollary 2.8: *Let (Υ,π) be a complete b-metric space with $s \geq 1$ and $\Phi: \Upsilon \to \Upsilon$ be a mapping satisfying the following:*

1. Φ *is a triangular α-admissible mapping;*
2. $\tau + F(s\lambda\pi(\Phi\varpi, \Phi\upsilon)) < F(\kappa_s(\varpi,\upsilon))$, *for some $\tau > 0$ and for every $\varpi,\upsilon \in \Upsilon$ with $1 \leq \alpha(\varpi,\upsilon)$ and $\pi(\Phi\varpi,\Phi\upsilon) > 0$, $s\lambda \geq 2$.*
3. *There is $\varpi_0 \in \Upsilon$ such that $\alpha(\varpi_0, \Phi\varpi_0) \geq 1$;*
4. Φ *is α-continuous.*

Then Φ has a fixed point. Furthermore, if $\alpha(\varpi,\upsilon) \geq 1$ for all $\varpi,\upsilon \in Fix(\Phi)$, then Φ has a unique fixed point.

Proof: put $G(\zeta) = \ln\zeta$, $\beta(\zeta) = k \in 0,1)$, and $\tau = -\ln k$ in the Theorem (2.6), we get our result.

Corollary 2.9: *Let (Υ,π) be a complete b-metric space with $s \geq 1$ and $\Phi: \Upsilon \to \Upsilon$ be a mapping satisfying the following:*

1. Φ *is a triangular α −admissible mapping;*
2. $s\lambda\pi(\Phi\varpi, \Phi\upsilon) < \beta(\kappa_s(\varpi,\upsilon))\kappa_s(\varpi,\upsilon)$, *for all $\varpi,\upsilon \in \Upsilon$ with $1 \leq \alpha(\varpi,\upsilon), \beta \in C_{G,\beta}$ and $\pi(\Phi\varpi,\Phi\upsilon) > 0$, $s\lambda \geq 2$.*
3. *There is $\varpi_0 \in \Upsilon$ such that $\alpha(\varpi_0, \Phi\varpi_0) \geq 1$;*
4. Φ *is α-continuous.*

Then Φ has a fixed point. Furthermore, if $\alpha(\varpi,\upsilon) \geq 1$ for all $\varpi,\upsilon \in Fix(\Phi)$, then Φ has a unique fixed point.

Proof: Put $F(\zeta) = G(\zeta) = \ln\zeta$ in the Theorem (2.6), we obtain the result.

3. Application to initial value problem via Green's function

Theorem 3.1: *Let $\Upsilon = C[0,1]$ be a complete b-metric space with $\pi(\varpi,\upsilon) = \sup_{\zeta\in 0,1]} |\varpi(\zeta) - \upsilon(\zeta)|^2$, for all $\varpi,\upsilon \in \Upsilon$. Consider the initial value problem:*

$$\varpi''(\zeta) + p(\zeta)\varpi'(\zeta) + q(\zeta)\varpi(\zeta) = f(\zeta, \varpi(\zeta))$$
$$\varpi(0) = \varpi_0, \varpi'(0) = \varpi_1,$$

where $\zeta \in [0,1]$. Suppose the following conditions hold:

1. $f: [0,1] \times \varpi\,[0,1] \to R$ is continuous, satisfies Lipschitz condition in $\varpi(\zeta)$, i.e.,
$$|f(\zeta, \varpi(\zeta)) - f(\zeta, \upsilon(\zeta))| \leq L|\varpi(\zeta) - \upsilon(\zeta)|,$$
and f non-decreasing in $\varpi(\zeta)$ for each $\zeta \in 0,1]$;
2. $\Phi\varpi \in C[0,1]$ for all $\varpi \in C[0,1]$, where $\Phi\varpi = \varpi_h(\zeta) + \int_0^\zeta g\,(\zeta,\chi)$ $f(\chi, \varpi(\chi))d\chi$, and $g(\zeta,\chi)$ is a Green's function satisfing $g(\zeta,\chi) \geq 0$, for all ζ,χ.

3. $M = \sup \int_0^{\zeta} |g(\zeta,\chi)|\, d\chi$ with $LM \leq \frac{1}{2}$ and $\lambda \geq 1$.

Then there exists a unique solution $\varpi(\zeta) \in C[0,1]$.

Proof: Define a self-map Φ on Υ by

$$\Phi\varpi(\zeta) = \varpi_h(\zeta) + \int_0^{\zeta} g\,(\zeta,\chi) f(\chi,\varpi(\chi)) d\chi, \textit{ for all } \varpi \in C[0,1], \zeta \in [0,1],$$

and $\alpha(\varpi(\zeta),\upsilon(\zeta)) = 2$ if $\varpi(\zeta) \leq \upsilon(\zeta)$, for all ζ and $\alpha(\varpi(\zeta),\upsilon(\zeta)) = 0$, otherwise. Assume that $\alpha(\varpi(\zeta),\upsilon(\zeta)) \geq 1$, implies $\varpi(\zeta) \leq \upsilon(\zeta)$, for all $\zeta \in [0,1]$. We prove Φ satisfy $\alpha(\Phi\varpi(\zeta),\Phi\upsilon(\zeta)) \geq 1$, we require $\Phi\varpi(\zeta) \leq \Phi\upsilon(\zeta)$. It follows from hypothesis, $f(\grave{u},\varpi(\grave{u})) \leq f(\grave{u},\upsilon(\grave{u}))$, for all $\grave{u} \in [0,\zeta]$, and since

$$g(\zeta,\chi) \geq 0, \int_0^{\zeta} g\,(\zeta,\chi) f(\chi,\varpi(\chi)) d\chi \leq \int_0^{\zeta} g(\zeta,\chi)\, f(\chi,\upsilon(\chi)) d\chi.$$

Thus

$$\Phi\varpi(\zeta) = \varpi_h(\zeta) + \int_0^{\zeta} g\,(\zeta,\chi) f(\chi,\varpi(\chi)) d\chi \leq \Phi\upsilon(\zeta),$$

so $\alpha(\Phi\varpi(\zeta),\Phi\upsilon(\zeta)) = 2 \geq 1$. Let $\alpha(\varpi(\zeta),\upsilon(\zeta)) \geq 1$ and $\alpha(\upsilon(\zeta),z(\zeta)) \geq 1$, then $\varpi(\zeta) \leq \upsilon(\zeta)$, and $\upsilon(\zeta) \leq z(\zeta)$ for all ζ, by transitivity of $\leq$, we get $\varpi(\zeta) \leq z(\zeta)$, so $\alpha(\varpi(\zeta),\upsilon(\zeta)) = 2 \geq 1$, hence Φ is a triangular $\alpha-$admissible mapping. Secondly, we prove Φ is a $\alpha - \beta - FG -$ Khan contraction. Let $\beta(\zeta) = \frac{1}{2}$, $F(\zeta) = \ln\zeta$, $G(\zeta) = \ln\zeta$, $s = 2$ and $\lambda \geq 1$.

$$|\Phi\varpi(\zeta) - \Phi\upsilon(\zeta)|^2 = \left|\int_0^{\zeta} g\,(\zeta,\chi)(f(\chi,\varpi(\chi)) - f(\chi,\upsilon(\chi)))d\chi\right|^2 \leq$$

$$\int_0^{\zeta} |g(\zeta,\chi)|^2\, |(f(\chi,\varpi(\chi)) - f(\chi,\upsilon(\chi)))|^2 d\chi$$

$$\leq L^2 \int_0^{\zeta} |g(\zeta,\chi)|^2\, |(\varpi(\chi) - \upsilon(\chi)|^2 d\chi \leq L^2 \|\varpi - \upsilon\|^2 \int_0^{\zeta} |g(\zeta,\chi)|^2\, d\chi$$

$$\sup_{\zeta}\, |\Phi\varpi(\zeta) - \Phi\upsilon(\zeta)|^2 \leq \sup_{\zeta} L^2 \|\varpi - \upsilon\|^2 \int_0^{\zeta} |g(\zeta,\chi)|^2\, d\chi = L^2 M^2 \|\varpi - \upsilon\|^2$$

$$\|\Phi\varpi(\zeta) - \Phi\upsilon(\zeta)\|^2 \leq L^2 M^2 \|\varpi - \upsilon\|^2.$$

Therefore,

$$\pi(\Phi\varpi,\Phi\upsilon) \leq L^2 M^2 \|\varpi - \upsilon\|^2 \leq \frac{1}{4} \|\varpi - \upsilon\|^2 \leq \frac{1}{4} \kappa_s(\varpi,\upsilon). \tag{3.1}$$

Let $F(2\lambda(\pi(\Phi\varpi,\Phi\upsilon))) = \ln 2\lambda(\pi(\Phi\varpi,\Phi\upsilon))$, and $F(\kappa_s(\varpi,\upsilon)) + G(\beta(\kappa_s(\varpi,\upsilon))) = \ln\kappa_s(\varpi,\upsilon) + \ln\frac{1}{2} = \ln\frac{1}{2}\kappa_s(\varpi,\upsilon)$, it satisfied form (3.1),

$$F(2\lambda(\pi(\Phi\varpi,\Phi\upsilon))) \leq F(\kappa_s(\varpi,\upsilon)) + G(\beta(\kappa_s(\varpi,\upsilon))),$$

i.e., $\ln 2\lambda(\pi(\Phi\varpi,\Phi\upsilon)) \leq \ln\frac{1}{2}\kappa_s(\varpi,\upsilon)$, i.e., $\pi(\Phi\varpi,\Phi\upsilon) \leq \frac{1}{4\lambda}\kappa_s(\varpi,\upsilon)$. By Theorem (2.6), Φ has a unique fixed point which solves the IVP.

Example 3.2; Solve $\varpi''(\zeta) + \varpi(\zeta) = 0.5\varpi(\zeta)$, $\varpi(0) = 0, \varpi'(0) = 0$, with Green function $g(\zeta,\grave{u}) = \sin(\zeta - \grave{u})$.

Solution: The solution operator Φ is $\Phi\varpi(\zeta) = 0.5\int_0^\zeta \sin(\zeta-\chi)\varpi(\chi)d\chi$. Define $\alpha(\varpi(\zeta), \upsilon(\zeta)) = 2$ if $\varpi(\zeta) \le \upsilon(\zeta), \forall\zeta$ and $\alpha(\varpi(\zeta), \upsilon(\zeta)) = 0$, otherwise, and $\beta(\zeta) = \frac{1}{2}$, $F(\zeta) = \ln\zeta, G(\zeta) = \ln\zeta$, $s = 2$. Let $f(\zeta, \varpi(\zeta) = 0.5\varpi(\zeta)$ be linear and Lipschitz with $L = \frac{1}{2}$,

$$|\Phi\varpi(\zeta) - \Phi\upsilon(\zeta)|^2 = \left|0.25\int_0^\zeta \sin(\zeta-\chi)(\varpi(\chi)) - \upsilon(\chi))d\chi\right|^2$$

$$\int_0^\zeta |\sin(\zeta-\chi)|^2 \, |\varpi(\chi)) - \upsilon(\chi)|^2 d\chi$$

$$\pi(\Phi\varpi, \Phi\upsilon) \le 0.25\sup\left[\int_0^\zeta |\varpi(\chi) - \upsilon(\chi)|\right]^2 d\chi \le 0.25\pi(\varpi, \upsilon).$$

For $\lambda = 1$, $F(s\lambda(\pi(\Phi\varpi, \Phi\upsilon))) = \ln(2(\pi(\Phi\varpi, \Phi\upsilon)) \le \ln(0.5\pi(\varpi, \upsilon))$, and

$$F(\kappa_s(\varpi, \upsilon)) + G(\beta(\kappa_s(\varpi, \upsilon))) = ln\kappa_s(\varpi, \upsilon) + ln\frac{1}{2},$$

and

$$\kappa_s(\varpi, \upsilon) = \pi(\varpi, \upsilon) + \frac{\pi(\Phi\varpi,\varpi)\pi(\varpi,\Phi\upsilon)+\pi(\Phi\upsilon,\upsilon)\pi(\Phi\varpi,\upsilon)}{\pi(\varpi,\Phi\upsilon)+\pi(\Phi\varpi,\upsilon)} \ge \pi(\varpi, \upsilon),$$

$$ln\kappa_s(\varpi, \upsilon) + ln\frac{1}{2} \ge ln0.5\pi(\varpi, \upsilon) \ge ln(2(\pi(\Phi\varpi, \Phi\upsilon)) = F(s\lambda(\pi(\Phi\varpi, \Phi\upsilon))).$$

The operator Φ satisfies the Φ is a $\alpha - \beta - FG$ −Khan contraction. By Theorem (2.6), Φ has a unique fixed point which solves the IVP.

Conclusion

We constructed a new fixed point theorem for $\alpha - \beta - FG -$ Khan contractions in b-metric spaces to expand the theoretical foundations of nonlinear operator theory and we applied these theoretical results to solving initial value problems (IVPs). The depth and significance of $\alpha - \beta - FG$ −Khan contractions may be further explored in future studies by investigating their extensions to fractional calculus and stochastic differential equations.

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