

A novel modification of the fractional conformable triple Sumudu transformation to investigate nonlinear fractional PDEs

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Abstract

In this paper, we introduce a novel modification of the fractional conformable triple Sumudu transformation and subsequently establish fundamental theorems that pave the way for various practical applications. Our theoretical discoveries contribute significantly to the advancement of this field by creating innovative avenues for tackling nonlinear partial fractional differential equations, particularly within the framework of the Adomian Sumudu decomposition method.

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1. Introduction

The FPDEs (fractional partial differential equations) have been used for many aspects of the modeling of feasible problems, related to sciences and engineering applications, such as dynamics of fluids, mathematical biology, control theory, optics, image processing, Quantum Mechanics, etc. ([6, 16, 17, 18, 21, 26, 32, 33]). There are many definitions of fractional derivatives and integrals, such as Riesz, Weyl, Riemann-Liouville (RL), Caputo (C), Hadamard (H), and others, which have been explored in the literature. These definitions bring forth peculiar properties of fractional derivatives, such as the non-universality of the chain rule and specific rules governing multiplication and division, all of which have significant implications for engineering physics.

Khalil *et al.* [24] introduced a novel derivative known as the conformable fractional derivative ($\mathcal{C}_{\text{FD}}$) in 2014. This derivative exhibits adherence to the classical properties commonly associated with well-known derivatives. For any given function $q : (0, \infty) \rightarrow \mathbb{R}$ and a selected order $\varpi \in (0, 1]$, the $\mathcal{C}_{\text{FD}}$ is precisely defined as form:

$$\mathcal{C}_{\text{FD}}^{\varpi}(q(y)) = \lim_{v \rightarrow 0} \frac{1}{v} [q(y + vy^{1-\varpi}) - q(y)], \quad y > 0. \quad (1.1)$$

In recent years, numerous researchers have delved into novel facets of analytics and numerical methods that offer precise and approximate solutions. One notable development in this field is the Sumudu transform (ST), which emerged as an extension of the classical Fourier transform. This transformative concept was first introduced by Watugala in the early 1990s [36]. Since then, it has found widespread application in addressing a multitude of real-world problems, particularly within the realm of engineering sciences.

What makes the ST particularly valuable is its inherent ability to preserve both unit and scale, making it an exceptionally efficient tool for tackling a wide array of physical problems. The ST is formally defined over a set of functions:

$$\psi = \{q(y) : \exists K, \tau_1, \tau_2 > 0, |q(y)| < Ke^{\frac{|y|}{\tau_j}}, \text{ if } y \in (-1)^j \times \mathbb{R}^+\}, \quad (1.2)$$

by,

$$S[q(y)] = \int_0^\infty q(vy)e^{-y} dy, \quad v \in (-\tau_1, \tau_2). \quad (1.3)$$

In the context of the current literature, it is worth noting that Belgacem, F.B.M conducted an extensive analysis of the ST in [7]. This work delved into a deeper understanding of the ST and explored its various applications. For more comprehensive insights into the properties of the ST and its derivatives, along with additional applications, we recommend referring to sources [4, 8, 9]. Furthermore, in references [5, 15, 20, 29, 30, 31], researchers have established the existence and uniqueness of some fundamental properties associated with the ST, particularly in the context of FODE (Fractional Ordinary Differential Equations) and FPDEs involving singularities. These studies also explored the connections and relationships between the ST and other integral transforms, such as the Laplace transform (LT). These references provide valuable resources for anyone seeking a deeper understanding of the ST and its applications in various mathematical and scientific contexts.

In [22], the ST found application in solving the one-dimensional neutron transport equation. Notably, a strong relationship between the double Sumudu and double LT was elucidated in [28]. Furthermore, the ST was extended to distributions, with its properties explored based on $\mathcal{C}_{FD}$ s in references [2, 35]. In [3], authored by Alfaqih, Bakcerler, and Misirli, delved into the application and results of the fractional conformable double Kiliçman, Eltayeb, and Agarwal applied this transform to tackle systems of differential equations in [27]. Bhanotar and Belgacem [12] and Bhanotar and Kaabar [13] introduced a novel method for solving nonlinear FPDEs in 3-space in a conformable sense, employing the triple LT and making comparisons with the ST in references [12, 13]. Recent advancements in the conformable fractional ST were outlined in references [1, 10, 11, 14, 19, 23, 25, 34], demonstrating its utility in solving nonlinear stochastic FDEs, transient solutions of Maxwell's equations, and convergence criteria for the double ST.

To address nonlinear FPDEs, this paper proposes the use of the Sumudu Decomposition Method, combined with the $\mathcal{C}_{FD}$ triple ST ($\mathcal{C}_{FD}$ -3ST). This approach leverages the advantages of the ST over the LT and offers accurate solutions for nonlinear FPDEs in three-dimensional space.

The paper is structured as follows: Section 2 introduces the basic concepts of Fractional Calculus. Section .1 investigates the main results and theorems related to the $\mathcal{C}_{FD}$ -3ST. In Section 3.2, an iterative method is discussed, employing the Adomian Decomposition Method (ADM) to address general nonhomogeneous FPDEs. Section 4 presents a numerical experiment using the proposed method to validate the obtained results. Finally, in Section 5, the paper concludes with a summary of the research work.

2. Basic concepts

We present some basic definitions and concepts for fractional calculus theory, which are used later in this paper.

Definition 2.1: The fractional RL integral operator of order $\varpi > 0$ for $y > 0$ is expressed as:

$$\mathcal{J}^{\varpi} q(y) = \begin{cases} \int_0^y \frac{(y-\tilde{x})^{\varpi-1}}{\Gamma(\varpi)} q(\tilde{x}) d\tilde{x}, & \varpi > 0, \\ q(y) & \varpi = 0, \end{cases} \quad (2.1)$$

and the RL-derivative of the n th order is given by:

$$\mathcal{D}_y^{\varpi} q(y) = \frac{d^n}{dy^n} (\mathcal{J}^{(n-\varpi)} q(y)) = \frac{d^n}{dy^n} \left(\int_0^y \frac{(y-\tilde{x})^{n-\varpi-1}}{\Gamma(n-\varpi)} q(\tilde{x}) d\tilde{x} \right). \quad (2.2)$$

While the RL derivative is useful in modeling real-world phenomena with FDEs, there is a drawback associated with its use. Therefore, M. Caputo proposed a new fractional derivative operator in his dissertation on viscoelasticity theory in [23], which is as follows.

Definition 2.2: The fractional C-derivative of a function $q(y)$ of order $\varpi > 0$ with $y > 0$ is defined as follows:

$$\mathcal{J}^{(n-\varpi)} \frac{d^n}{dy} q(y) = \int_0^y \frac{(y-\tilde{x})^{n-\varpi-1}}{\Gamma(n-\varpi)} q^{(n)}(\tilde{x}) d\tilde{x}. \quad (2.3)$$

In this expression, $n - 1 < \varpi \leq n$, and the notation $\mathcal{J}^{(n-\varpi)}$ represents the fractional C-derivative of order $n - \varpi$, while $\frac{d^n}{dy}$ denotes the n th traditional derivative of the function $q(y)$. This formula provides a way to calculate the fractional C-derivative of a function $q(y)$ in terms of its traditional derivatives and an integral involving the Gamma function.

Definition 2.3: Let $q: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function. The fractional partial derivatives of the function $q(y_1, y_2, y_3)$ with respect to y_i of order $\varpi_i \in (0, 1]$, $i = 1, 2, 3$ respectively, are expressed as follows, for $y_1, y_2, y_3 > 0$:

$$\begin{aligned} \partial_{y_1}^{\varpi_1} q &= \frac{\partial^{\varpi_1}}{\partial y_1^{\varpi_1}} q(y_1, y_2, y_3) = \lim_{v \rightarrow 0} \frac{1}{v} [q(y_1 + v y_1^{1-\varpi_1}, y_2, y_3) - q(y_1, y_2, y_3)], \\ \partial_{y_2}^{\varpi_2} q &= \frac{\partial^{\varpi_2}}{\partial y_2^{\varpi_2}} q(y_1, y_2, y_3) = \lim_{v \rightarrow 0} \frac{1}{v} [q(y_1, y_2 + v y_2^{1-\varpi_2}, y_3) - q(y_1, y_2, y_3)], \\ \partial_{y_3}^{\varpi_3} q &= \frac{\partial^{\varpi_3}}{\partial y_3^{\varpi_3}} q(y_1, y_2, y_3) = \lim_{v \rightarrow 0} \frac{1}{v} [q(y_1, y_2, y_3 + v y_3^{1-\varpi_3}) - q(y_1, y_2, y_3)]. \end{aligned}$$

In these expressions, $\partial_{y_i}^{\varpi_i}$ represent the fractional partial derivatives of order ϖ_i , $i = 1, 2, 3$. These derivatives provide a generalization of the traditional concept of partial derivative to accommodate fractional orders, making them suitable for analyzing functions defined in three dimensions on the positive real numbers.

3. Main results

We present the following theorem and some properties of the fractional C-derivative, which elucidate the relationship between the Conformable Partial Fractional Derivatives ($\mathcal{C}_{\text{PFD}}$) and traditional partial derivatives.

Theorem 3.1: For fractional orders $\varpi_i \in (0, 1]$, $i = 1, 2, 3$ and a function $q(y_1, y_2, y_3)$ that is differentiable at a point for $y_1, y_2, y_3 > 0$, the following relationship hold: $\mathcal{C}_{\text{PFD}}$ with Respect to y_i :

$$\partial_{y_i}^{\varpi_i} q = \frac{\partial^{\varpi_i}}{\partial y_i^{\varpi_i}} q(y_1, y_2, y_3) = y_i^{1-\varpi_i} \frac{\partial q(y_1, y_2, y_3)}{\partial y_i} = y_i^{1-\varpi_i} \partial_{y_i} q, \quad i = 1, 2, 3. \quad (3.1)$$

These relationships demonstrate that the $\mathcal{C}_{\text{PFD}}$ with respect to y_i can be expressed as the product of a power of the respective variable ($y_i^{1-\varpi_i}$) and the traditional partial derivative of $q(y_1, y_2, y_3)$ with respect to y_i , $i = 1, 2, 3$ respectively.

Proof: By the definition of $\mathcal{C}_{\text{PFD}}$, for $i = 1$, we have

$$\begin{aligned} \frac{\partial^{\varpi_1}}{\partial y_1^{\varpi_1}} q(y_1, y_2, y_3) &= \lim_{v \rightarrow 0} \frac{1}{v} [q(y_1 + v y_1^{1-\varpi_1}, y_2, y_3) - q(y_1, y_2, y_3)], \\ &= \lim_{\tilde{x} \rightarrow 0} \frac{1}{\tilde{x} y_1^{(\varpi_1-1)}} [q(y_1 + \tilde{x}, y_2, y_3) - q(y_1, y_2, y_3)] \\ &= y_1^{1-\varpi_1} \lim_{\tilde{x} \rightarrow 0} \frac{1}{\tilde{x}} [q(y_1 + \tilde{x}, y_1, y_3) - q(y_1, y_2, y_3)], \end{aligned}$$

where $\dot{x}_1 = \nu y_1^{1-\varpi_1}$. By using the same maner, we can see that

$$\begin{aligned}\frac{\partial^{\varpi_2}}{\partial y_2^{\varpi_2}} q(y_1, y_2, y_3) &= \lim_{\nu \rightarrow 0} \frac{1}{\nu} [q(y_1, y_2 + \nu y_2^{1-\varpi_2}, y_3) - q(y_1, y_2, y_3)], \\ &= \lim_{\dot{x}_2 \rightarrow 0} \frac{1}{\dot{x}_2 y_2^{(\varpi_2-1)}} [q(y_1, y_2 + \dot{x}_2, y_3) - q(y_1, y_2, y_3)] \\ &= y_2^{1-\varpi_2} \lim_{\dot{x}_2 \rightarrow 0} \frac{1}{\dot{x}_2} [q(y_1, y_2 + \dot{x}_2, y_3) - q(y_1, y_2, y_3)], \\ \frac{\partial^{\varpi_3}}{\partial y_3^{\varpi_3}} q(y_1, y_2, y_3) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\nu} [q(y_1, y_2, y_3 + \nu y_3^{1-\varpi_3}) - q(y_1, y_2, y_3)] \\ &= \lim_{\dot{x}_3 \rightarrow 0} \frac{1}{\dot{x}_3 y_3^{(\varpi_3-1)}} [q(y_1, y_2, y_3 + \dot{x}_3) - q(y_1, y_2, y_3)] \\ &= y_3^{1-\varpi_3} \lim_{\dot{x}_3 \rightarrow 0} \frac{1}{\dot{x}_3} [q(y_1, y_2, y_3 + \dot{x}_3) - q(y_1, y_2, y_3)],\end{aligned}$$

where $\dot{x}_2 = \nu y_2^{1-\varpi_2}$, $\dot{x}_3 = \nu y_3^{1-\varpi_3}$. This completes the proof.

The following properties showcase the versatility and applicability of fractional differentiation in various contexts, including linear combinations, monomials, and trigonometric expressions. They are valuable tools for solving differential equations involving fractional orders.

Proposition 3.2: The following properties hold for $a, b \in \mathbb{R}$, $\varpi_i \in (0, 1]$, $l_i \in \mathbb{N}$, $i = 1, 2, 3$:

P1) **Linearity of Differentiation:** The fractional derivative of a linear combination of two functions $q(y_1, y_2, y_3)$ and $q'(y_1, y_2, y_3)$ with respect to y_1 is equal to the corresponding linear combination of their fractional derivatives with the same order ϖ_1 . This property is a fundamental property of differentiation.

$$\frac{\partial^{\varpi_1}}{\partial y_1^{\varpi_1}} (aq + bq') = a \frac{\partial^{\varpi_1}}{\partial y_1^{\varpi_1}} q + b \frac{\partial^{\varpi_1}}{\partial y_1^{\varpi_1}} q'.$$

P2) **Derivative of a Monomial:** The derivative of a monomial $y_1^{l_1} y_2^{l_2} y_3^{l_3}$ with respect to y_1 , y_2 , and y_3 for fractional orders ϖ_1 , ϖ_2 , and ϖ_3 . It demonstrates how the exponents of the variables change after differentiation:

$$\frac{\partial^{\varpi_1 + \varpi_2 + \varpi_3}}{\partial y_1^{\varpi_1} \partial y_2^{\varpi_2} \partial y_3^{\varpi_3}} (y_1^{l_1} y_2^{l_2} y_3^{l_3}) = l_1 l_2 l_3 y_1^{(l_1 - \varpi_1)} y_2^{(l_2 - \varpi_2)} y_3^{(l_3 - \varpi_3)}.$$

P3) **Derivative of a Monomial with respect to y_1 :** The derivative of a monomial $y_1^{l_1} y_3^{l_3}$ with respect to y_1 for a fractional order ϖ_1 . It shows how the exponent of y_1 changes after differentiation:

$$\frac{\partial^{\varpi_1}}{\partial y_1^{\varpi_1}} (y_1^{l_1} y_3^{l_3}) = l_1 y_1^{(l_1 - \varpi_1)} y_3^{l_3}.$$

P4) **Derivative of a Trigonometric Expression:** The derivative of the given trigonometric expression with respect to y_1 for a fractional order ϖ_1 . It illustrates how fractional differentiation affects trigonometric functions:

$$\frac{\partial^{\overline{\omega}_1}}{\partial y_1^{\overline{\omega}_1}} \left(\sin \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right) \tan \left(\frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3} \right) \right) = \cos \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right) \tan \left(\frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3} \right).$$

3.1 New results and theorems

We present results, theorems and properties provide a foundation for understanding the behavior of the $\mathcal{C}_{\text{FD}}$ -3ST and how it operates on various types of functions and their derivatives. These transforms have applications in solving FDEs and analyzing complex systems in science and engineering.

Definition 3.3: Over the set of functions,

$$A_1 = \left\{ q \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right) \mid \exists K, \tau_1, \tau_2 > 0, \left| q \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right) \right| < K \exp \left(\frac{1}{\tau_j} \left| \frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right| \right), y_1^{\overline{\omega}_1} \in (-1)^j \times [0, \infty) \right\}.$$

The $\mathcal{C}_{\text{FD}}$ -ST of function $q: [0, \infty) \rightarrow \mathbb{R}$ of the exponential of order is defined by the integral as follows,

$$G_{\overline{\omega}_1}(u) = S_{y_1}^{\overline{\omega}_1} \left(q \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right) \right) = \int_0^\infty e^{-y_1^{\overline{\omega}_1} \overline{\omega}_1} q \left(u \frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} \right) y_1^{\overline{\omega}_1-1} dy_1, \quad y_1 > 0.$$

Definition 3.4: For a given function $q(y_1^{\overline{\omega}_1} \overline{\omega}_1, y_2^{\overline{\omega}_2} \overline{\omega}_2)$, we define the double $\mathcal{C}_{\text{FD}}$ -3ST over the set,

$$A_2 = \{ q(y_1, y_2) : \exists K, \tau_1, \tau_2 > 0, |q(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2})| < K \exp \left(\frac{1}{\tau_j^2} \left| \frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} + \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2} \right| \right) \},$$

if $j = 1, 2$ and $(y_1^{\overline{\omega}_1}, y_2^{\overline{\omega}_2}) \in \mathbb{R}_+^2$. And given by the integral,

$$\begin{aligned} G_{(\overline{\omega}_1, \overline{\omega}_2)}(u_1, u_2) &= S_{(y_1, y_2)}^{(\overline{\omega}_1, \overline{\omega}_2)} \left(q \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2} \right) : (u_1, u_2) \right) \\ &= \int_0^\infty \int_0^\infty \exp \left(- \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} + \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2} \right) \right) q \left(u_1 \frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, u_2 \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2} \right) y_1^{\overline{\omega}_1-1} y_2^{\overline{\omega}_2-1} dy_1 dy_2, \end{aligned}$$

where $y_1^{\overline{\omega}_1}, y_2^{\overline{\omega}_2} > 0$, and u_1, u_2 are the complex variables.

Definition 3.5: For $\overline{\omega}_i \in (0, 1]$, $i = 1, 2, 3$, the $\mathcal{C}_{\text{FD}}$ -ST of $q(y_1^{\overline{\omega}_1} \overline{\omega}_1, y_2^{\overline{\omega}_2} \overline{\omega}_2, y_3^{\overline{\omega}_3} \overline{\omega}_3)$ is defined as follows:

$$\begin{aligned} G_{\overline{\omega}_1}(u_1) &= S_{y_1}^{\overline{\omega}_1} (q(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}) : u_1) = \int_0^\infty e^{-y_1^{\overline{\omega}_1} \overline{\omega}_1} q(u_1 \frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}) y_1^{\overline{\omega}_1-1} dy_1, \\ G_{\overline{\omega}_2}(u_2) &= S_{y_2}^{\overline{\omega}_2} (q(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}) : u_2) = \int_0^\infty e^{-y_2^{\overline{\omega}_2} \overline{\omega}_2} q(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, u_2 \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}) y_2^{\overline{\omega}_2-1} dy_2, \\ G_{\overline{\omega}_3}(u_3) &= S_{y_3}^{\overline{\omega}_3} (q(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}) : u_3) = \int_0^\infty e^{-y_3^{\overline{\omega}_3} \overline{\omega}_3} q(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, u_3 \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}) y_3^{\overline{\omega}_3-1} dy_3. \end{aligned}$$

Definition 3.6: For a given real-valued piecewise continuous function $q(y_1^{\overline{\omega}_1} \overline{\omega}_1, y_2^{\overline{\omega}_2} \overline{\omega}_2, y_3^{\overline{\omega}_3} \overline{\omega}_3)$, the triple $\mathcal{C}_{\text{F}}$ ST is defined as follows:

$$\begin{aligned} G_{(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3)}(u, v, w) &= S_{(y_1, y_2, y_3)}^{(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3)} \left(q \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3} \right) : (u, v, w) \right) \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \exp \left(- \left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1} + \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2} + \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3} \right) \right) \end{aligned}$$

$$\cdot q\left(u \frac{y_1^{\overline{w}_1}}{\overline{w}_1}, v \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, w \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right) y_1^{\overline{w}_1-1} y_2^{\overline{w}_2-1} y_3^{\overline{w}_3-1} dy_1 dy_2 dy_3.$$

The inverse triple ST is defined by

$$\begin{aligned} q(y_1, y_2, y_3) &= S^{-1}\left(G_{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}(u, v, w)\right) \\ &= \frac{1}{2\pi i} \int_{\overline{w}_1-i\infty}^{\overline{w}_1+i\infty} e^{y_1^{\overline{w}_1} \overline{w}_1} \left[\frac{1}{2\pi i} \int_{\overline{w}_2-i\infty}^{\overline{w}_2+i\infty} e^{y_2^{\overline{w}_2} \overline{w}_2} \right. \\ &\quad \left. \times \left[\frac{1}{2\pi i} \int_{\overline{w}_3-i\infty}^{\overline{w}_3+i\infty} e^{y_3^{\overline{w}_3} \overline{w}_3} G_{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}(u, v, w) dw \right] dv \right] du. \end{aligned}$$

Theorem 3.7: The $\mathcal{C}_{FD}$ -3ST for functions defined over the variables y_i with fractional orders $\overline{w}_i \in (0, 1]$, $i = 1, 2, 3$. Let

$$\begin{aligned} S_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(q\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}, \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right)\right) &= G_q(u, v, w), \\ S_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(q'\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}, \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right)\right) &= G_{q'}(u, v, w), \end{aligned}$$

and k , l , and m are constants. Then the following properties can be determined:

- **Linearity property:** The $\mathcal{C}_{FD}$ -3ST is a linear operation. For constants k and l and functions q and q' , the $\mathcal{C}_{FD}$ -3ST of $kq + lq'$ is equal to k times the $\mathcal{C}_{FD}$ -3ST of q plus l times the $\mathcal{C}_F$ ST of q' , which can be expressed as:

$$\begin{aligned} S_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(kq\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}, \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right) + lq'\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}, \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right)\right) \\ = kS_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(q\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}, \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right)\right) + lS_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(q'\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}, \frac{y_2^{\overline{w}_2}}{\overline{w}_2}, \frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right)\right) \\ = kG_q(u, v, w) + lG_{q'}(u, v, w). \end{aligned}$$

- **Constant Property:** The $\mathcal{C}_{FD}$ -3ST of a constant k with respect to the variables u , v , and w is equal to the constant itself:

$$S_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}(k; (p, q, s)) = k.$$

- **Product Property:** The $\mathcal{C}_{FD}$ -3ST of a product of functions involving powers of y_i , $i = 1, 2, 3$ is related to the gamma function:

$$\begin{aligned} S_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(\left(\frac{y_1^{\overline{w}_1}}{\overline{w}_1}\right)^p \left(\frac{y_2^{\overline{w}_2}}{\overline{w}_2}\right)^q \left(\frac{y_3^{\overline{w}_3}}{\overline{w}_3}\right)^r; (k, l, m)\right) \\ = \Gamma(p+1)\Gamma(q+1)\Gamma(r+1)u^p v^q w^r, \end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function, and $\Gamma(n+1) = n!$ for $n = 0, 1, 2, 3, \dots$.

- **Exponential Property:** The $\mathcal{C}_{FD}$ -3ST of an exponential function with linear arguments results in a rational expression:

$$S_{(y_1, y_2, y_3)}^{(\overline{w}_1, \overline{w}_2, \overline{w}_3)}\left(e^{(ly_1^{\overline{w}_1} \overline{w}_1 + my_2^{\overline{w}_2} \overline{w}_2 + ny_3^{\overline{w}_3} \overline{w}_3)}\right) = \frac{1}{(1-lu)(1-mv)(1-nw)}.$$

- **Multiplication Property:** The $\mathcal{C}_{FD}$ -3ST of the product of two functions is equal to the product of their individual $\mathcal{C}_{FD}$ -3STs:

$$S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(q \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) q' \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right) = G_q(u, v, w) G_{q'}(u, v, w).$$

These properties are useful in various mathematical and engineering applications involving the $\mathcal{C}_{FD}$ -3ST, allowing for simplifications and manipulations of transformed functions. Let us illustrate as follows:

$$\begin{aligned} & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\sin \left(\frac{ly_1^{\varpi_1}}{\varpi_1} \right) \sin \left(\frac{my_2^{\varpi_2}}{\varpi_2} \right) \sin \left(\frac{ny_3^{\varpi_3}}{\varpi_3} \right) \right) \\ &= S_{y_1}^{\varpi_1} \left(\sin \left(\frac{ly_1^{\varpi_1}}{\varpi_1} \right) \right) S_{y_2}^{\varpi_2} \left(\sin \left(\frac{my_2^{\varpi_2}}{\varpi_2} \right) \right) S_{y_3}^{\varpi_3} \left(\sin \left(\frac{ny_3^{\varpi_3}}{\varpi_3} \right) \right) \\ &= \frac{lu}{(1+l^2u^2)} \frac{mv}{(1+m^2v^2)} \frac{nw}{(1+n^2w^2)}, \end{aligned}$$

and

$$S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\cos \left(\frac{ly_1^{\varpi_1}}{\varpi_1} \right) \cos \left(\frac{my_2^{\varpi_2}}{\varpi_2} \right) \cos \left(\frac{ny_3^{\varpi_3}}{\varpi_3} \right) \right) = \frac{1}{(1+l^2u^2)} \frac{1}{(1+m^2v^2)} \frac{1}{(1+n^2w^2)}.$$

Theorem 3.8: For $0 < \varpi_i \leq 1$, $i = 1, 2, 3$. The $\mathcal{C}_{FD}$ -3ST applied to the set of derivatives

$$\frac{\partial^{\varpi_1} q}{\partial y_1^{\varpi_1}}, \frac{\partial^{\varpi_2} q}{\partial y_2^{\varpi_2}}, \frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}}, \frac{\partial^2 \varpi_1 q}{\partial y_1^2 \varpi_1}, \frac{\partial^2 \varpi_2 q}{\partial y_2^2 \varpi_2}, \frac{\partial^2 \varpi_3 q}{\partial y_3^2 \varpi_3}, \frac{\partial^3 \varpi_1 q}{\partial y_1^3 \varpi_1},$$

can be represented as follows:

$$\begin{aligned} & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^{\varpi_1} q}{\partial y_1^{\varpi_1}} \right) = u^{-1} [G(u, v, w) - G(0, v, w)]; \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^{\varpi_2} q}{\partial y_2^{\varpi_2}} \right) = v^{-1} [G(u, v, w) - G(u, 0, w)]; \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}} \right) = w^{-1} [G(u, v, w) - G(u, v, 0)]; \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^2 \varpi_1 q}{\partial y_1^2 \varpi_1} \right) = u^{-2} G(u, v, w) - u^{-2} G(0, v, w) - u^{-1} G_{y_1}(0, v, w); \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^2 \varpi_2 q}{\partial y_2^2 \varpi_2} \right) = v^{-2} G(u, v, w) - v^{-2} G(u, 0, w) - v^{-1} G_{y_2}(u, 0, w); \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^2 \varpi_3 q}{\partial y_3^2 \varpi_3} \right) = w^{-2} G(u, v, w) - w^{-2} G(u, v, 0) - w^{-1} G_{y_3}(u, v, 0); \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^3 \varpi_1 q}{\partial y_1^3 \varpi_1} \right) = u^{-3} G(u, v, w) - u^{-3} G(0, v, w) - u^{-2} G_{y_1}(0, v, w) \\ & \quad - u^{-1} G_{y_1 y_1}(0, v, w); \\ & S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^3 \varpi_2 q}{\partial y_2^3 \varpi_2} \right) = v^{-3} G(u, v, w) - v^{-3} G(u, 0, w) - v^{-2} G_{y_2}(u, 0, w) \\ & \quad - v^{-1} G_{y_2 y_2}(u, 0, w); \end{aligned}$$

$$S_{(y_1, y_2, y_3)}^{(\varpi_1, \varpi_2, \varpi_3)} \left(\frac{\partial^3 \varpi_3 q}{\partial y_3^3} \right) = w^{-3} G(u, v, w) - w^{-3} G(u, v, 0) - w^{-2} G_{y_3}(u, v, 0) \\ - w^{-1} G_{y_3 y_3}(u, v, 0).$$

3.2 New iterative method

In this section, we propose an iterative method that combines the ADM with conformable fractional calculus techniques to solve the nonlinear term of a given PDE in a conformable sense. The ADM is a powerful analytical method for solving nonlinear PDEs, and in this context, it is applied to handle the nonlinear terms of the PDE while conformable fractional calculus techniques are used to account for fractional derivatives and integrals. The use of such an iterative method can be beneficial in solving PDEs with complex nonlinear terms that are difficult to handle using traditional techniques. The combination of ADM and conformable fractional calculus allows for the systematic decomposition and solution of nonlinear terms, making it a useful approach in mathematical modeling and analysis of various real-world phenomena.

Consider the following nonlinear FPDE:

$$\frac{\partial^{\varpi_1 n}}{\partial y_1^{\varpi_1 n}} \left(q \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right) + Lq \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) + Nq \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \\ = F \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right), \quad (3.2)$$

where $n \in \mathbb{N}$ and $\varpi_i \in (0, 1]$, $i = 1, 2, 3$ with the initial conditions,

$$\frac{\partial^{\varpi_1 n-1}}{\partial y_1^{\varpi_1 n-1}} \left(q \left(0, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right) = G_{\varpi_1 n-1} \left(0, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right), \quad (3.3)$$

where L and N address the linear and nonlinear partial fractional operators, and $F(\cdot, \cdot, \cdot)$ is the source term. The following are the steps to determine the analytic solution of Eq. (3.2) using the proposed iterative Adomian Sumudu transform.

Step 1: Applying the $\mathcal{C}_{FD}$ -3ST to Eq. (3.2), we have:

$$S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(\frac{\partial^{\varpi_1 n}}{\partial y_1^{\varpi_1 n}} \left(q \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right) \right) \\ + S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} (Lq) + S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} (Nq) = S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(F \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right).$$

Therefore,

$$u^{-n} [G(u, v, w) - \sum_{k=0}^{n-1} u^k S_{y_2, y_3}^{\varpi_2, \varpi_3} \left(q \left(0, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right)] + S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} (Lq) + \\ S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} (Nq) \\ = S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(F \left(\frac{y_1^{\varpi_1}}{\varpi_1}, \frac{y_2^{\varpi_2}}{\varpi_2}, \frac{y_3^{\varpi_3}}{\varpi_3} \right) \right) \quad (3.4)$$

Step 2: Apply the inverse $\mathcal{C}_{\text{FD}}$ -3ST, and Solving for $q(\cdot, \cdot, \cdot)$, we get:

$$q\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right) = F\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right) - S_{u,v,w}^{-1}\{u^n[S_{y_1,y_2,y_3}^{\overline{\omega}_1,\overline{\omega}_2,\overline{\omega}_3}(Lq) + S_{y_1,y_2,y_3}^{\overline{\omega}_1,\overline{\omega}_2,\overline{\omega}_3}(Nq)]\}, \quad (3.5)$$

where $F(\cdot, \cdot, \cdot)$ can be obtained from the source term and prescribed initial conditions.

Step 3: Apply the Adomian Sumudu transform method. Let the general solution of Eq. (3.2) be an infinite series:

$$q\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right) = \sum_{n=0}^{\infty} q_n\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right), \quad (3.6)$$

and the nonlinear term can be represented as Adomian Polynomials:

$$Nq\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right) = \sum_{n=0}^{\infty} A_n, \quad (3.7)$$

where A_n is the Adomian Polynomial of q_i , $i = 1, 2, \dots, n$ and it can be obtained by the following formula:

$$A_n = \frac{1}{n!} \left[\frac{\partial^n}{\partial \omega^n} (N \sum_{i=1}^{\infty} \omega^i q_i) \right]_{\omega=0}. \quad (3.8)$$

Using Eqs. (3.6), (3.7) and (3.8) in Eq. (3.5), we obtain,

$$\begin{aligned} & \sum_{n=0}^{\infty} q_n\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right) \\ &= F(y_1, y_2, y_3) - S_{u,v,w}^{-1}\{u^n[S_{y_1,y_2,y_3}^{\overline{\omega}_1,\overline{\omega}_2,\overline{\omega}_3}(L \sum_{n=0}^{\infty} q_n(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3})) \\ & \quad + N \sum_{n=0}^{\infty} A_n]\}. \end{aligned}$$

Step 4: Comparing both sides, we obtain recursive relations:

$$q_n\left(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}\right) = \begin{cases} F(\frac{y_1^{\overline{\omega}_1}}{\overline{\omega}_1}, \frac{y_2^{\overline{\omega}_2}}{\overline{\omega}_2}, \frac{y_3^{\overline{\omega}_3}}{\overline{\omega}_3}), & n = 0, \\ -S_{u,v,w}^{-1}\{u^{n-1}[S_{y_1,y_2,y_3}^{\overline{\omega}_1,\overline{\omega}_2,\overline{\omega}_3}(Lq_{n-1}) + NA_{n-1}]\}, & n \geq 1, \end{cases}$$

Step 5: The iterative solution to the given problem can be approximated as the dynamic analytical solution (3.6).

4. Applications

In this section, we demonstrate the application of the $\mathcal{C}_{\text{FD}}$ -3ST in conjunction with the ADM to solve nonlinear FPDEs. This approach appears to be a novel and powerful technique for addressing nonlinear FPDEs, combining the capabilities of the $\mathcal{C}_{\text{FD}}$ -3ST to handle fractional derivatives and integrals with the systematic decomposition and solution procedure of ADM.

Example 4.1: Consider a dynamic nonhomogeneous nonlinear FPDEs with initial conditions:

$$\frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}} + 6u \frac{\partial^{\varpi_1 + \varpi_2} q}{\partial y_2^{\varpi_2} \partial y_1^{\varpi_1}} + \frac{\partial^2 \varpi_1 q}{\partial y_1^2} = 0, \quad (4.1)$$

under the initial condition $q(0, y_2, y_3) = y_2 y_3$, and where $\varpi_i \in (0, 1]$, $y_i \in \mathbb{R}^+$, $i = 1, 2, 3$. By employing the $\mathcal{C}_{\text{FD}}$ -3ST on both sides of Eq. (4.1) and using obtained results we get,

$$w^{-1}[G(u, v, w) - G(0, v, w)] = -S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(6q \frac{\partial^{\varpi_1 + \varpi_2} q}{\partial y_2^{\varpi_2} \partial y_1^{\varpi_1}} + \frac{\partial^2 \varpi_1 q}{\partial y_1^2} \right). \quad (4.2)$$

With the help of the initial condition and applying the inverse $\mathcal{C}_{\text{FD}}$ -3ST to Eq. (4.2), we obtain,

$$G(y_1, y_2, y_3) = S_{u, v, w}^{-1}(vw) - S_{u, v, w}^{-1} \left[w S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(6q \frac{\partial^{\varpi_1 + \varpi_2} q}{\partial y_2^{\varpi_2} \partial y_1^{\varpi_1}} + \frac{\partial^2 \varpi_1 q}{\partial y_1^2} \right) \right]. \quad (4.3)$$

By applying the proposed method, it leads to the generalized recursive relation:

$$q_n(y_1, y_2, y_3) = \begin{cases} y_2 y_3, & n = 0, \\ -S_{u, v, w}^{-1} \left[w S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(6A_{n-1} + \frac{\partial^2 \varpi_1 q_{n-1}}{\partial y_1^2} \right) \right], & n \geq 1, \end{cases} \quad (4.4)$$

where A_n is the Adomian polynomial, which is given by

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \omega^n} F(\sum_{i=0}^{\infty} \omega^i q_i) |_{\omega=0}. \quad (4.5)$$

Let the nonlinear term be expressed as:

$$F(q_n(y_1, y_2, y_3)) = q_n \frac{\partial^{\varpi_1 + \varpi_2} q_n}{\partial y_2^{\varpi_2} \partial y_1^{\varpi_1}}. \quad (4.6)$$

Therefore Eq. (4.4) leads to,

$$\begin{aligned} q_1 &= -[6y_2^{2-\varpi_2} y_3^{2-\varpi_3} + (1 - \varpi_1) y_2^{1-2\varpi_2} y_3] y_1, \\ q_2 &= (18[1 + (2 - \varpi_2)(2 - \varpi_3)] y_2^{3-2\varpi_2} y_3^{3-2\varpi_3} \\ &\quad + 3(3 - 2\varpi_2)(2 - 2\varpi_3) y_2^{2-3\varpi_2} y_3^{2-\varpi_3} \\ &\quad + \frac{1}{2}(1 - \varpi_2)(1 - 2\varpi_2)(1 - 3\varpi_2) y_2^{1-4\varpi_2} y_3) y_1^2. \end{aligned}$$

The proposed solution can be obtained in series form:

$$\begin{aligned} q(y_1, y_2, y_3) &= y_2 y_3 - [6y_2^{2-\varpi_2} y_3^{2-\varpi_3} + (1 - \varpi_1) y_2^{1-2\varpi_2} y_3] y_1 \\ &\quad + (18[1 + (2 - \varpi_2)(2 - \varpi_3)] y_2^{3-2\varpi_2} y_3^{3-2\varpi_3} \\ &\quad + 3(3 - 2\varpi_2)(2 - 2\varpi_3) y_2^{2-3\varpi_2} y_3^{2-\varpi_3} \\ &\quad + \frac{1}{2}(1 - \varpi_2)(1 - 2\varpi_2)(1 - 3\varpi_2) y_2^{1-4\varpi_2} y_3) y_1^2 + \dots \end{aligned} \quad (4.7)$$

To get the exact solution for an integer order PDE, let us take $\varpi_i = 1$ ($i = 1, 2, 3$), then Eq. (4.7) reduces to,

$$\begin{aligned} q(y_1, y_2, y_3) &= y_2 y_3 - 6y_1 y_2 y_3 + 36y_1^2 y_2 y_3 - 216y_1^3 y_2 y_3 \\ &\quad + \dots = y_2 y_3 \sum_{n=0}^{\infty} (-1)^n (6y_1)^n. \end{aligned}$$

In the next example, we consider new type of FPDE.

Example 4.2: Consider the following dynamic nonhomogeneous nonlinear FPDE:

$$\frac{\partial^{\varpi_2} q}{\partial y_2^{\varpi_2}} \frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}} - \frac{\partial^{2\varpi_1} q}{\partial y_1^{2\varpi_1}} = q(y_1, y_2, y_3), \quad \varpi_i \in (0, 1], y_i \in \mathbb{R}^+ \quad (i = 1, 2, 3), \quad (4.8)$$

via the initial conditions:

$$q(0, y_2, y_3) = y_2 y_3, \quad q_{y_1}(0, y_2, y_3) = -1. \quad (4.9)$$

Applying the $\mathcal{C}_{\text{FD}}\text{-3ST}$ to both sides of Eq. (4.8), we have:

$$S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(\frac{\partial^{2\varpi_1} q}{\partial y_1^{2\varpi_1}} \right) = S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(\frac{\partial^{\varpi_2} q}{\partial y_2^{\varpi_2}} \frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}} - q(y_1, y_2, y_3) \right). \quad (4.10)$$

Using the obtained result, we get:

$$\begin{aligned} G(u, v, w) &= G(0, v, w) + u G_{y_1}(0, v, w) + u^2 S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(\frac{\partial^{\varpi_2} q}{\partial y_2^{\varpi_2}} \frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}} - q(y_1, y_2, y_3) \right) \\ q(y_1, y_2, y_3) &= S_{(u, v, w)}^{-1}(vw - u) + S_{(u, v, w)}^{-1} \left(u^2 S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} \left(\frac{\partial^{\varpi_2} q}{\partial y_2^{\varpi_2}} \frac{\partial^{\varpi_3} q}{\partial y_3^{\varpi_3}} - q(y_1, y_2, y_3) \right) \right). \end{aligned} \quad (4.11)$$

Applying the proposed method, we get the recursive formula:

$$q_n = \begin{cases} y_2 y_3 - y_1, & n = 0, \\ S_{(u, v, w)}^{-1} \left(u^2 S_{y_1, y_2, y_3}^{\varpi_1, \varpi_2, \varpi_3} (A_n - q_n) \right), & n \geq 1, \end{cases} \quad (4.12)$$

where A_n is the Adomian polynomial, which is as follows:

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \omega^n} F(\sum_{i=0}^{\infty} \omega^i q_i) \Big|_{\omega=0}. \quad (4.13)$$

Let the nonlinear term be represented as:

$$F(q_n) = \frac{\partial^{\varpi_2} q_n}{\partial y_2^{\varpi_2}} \frac{\partial^{\varpi_3} q_n}{\partial y_3^{\varpi_3}}. \quad (4.14)$$

These Eqs. (4.12), (4.13), (4.14) gives:

$$\begin{aligned} q_1 &= \frac{y_1^2}{2!} y_2^{2-\varpi_2} y_3^{2-\varpi_3} - \frac{y_1^2}{2!} y_2 y_3 + \frac{y_1^3}{3!}, \\ q_2 &= (4 - \varpi_3 - \varpi_2) \frac{y_1^4}{4!} y_2^{3-2\varpi_2} y_3^{3-2\varpi_3} - \frac{y_1^4}{8} y_2^{2-\varpi_2} y_3^{2-\varpi_3} + \frac{y_1^4}{4!} y_2 y_3 - \frac{y_1^5}{5!}. \end{aligned}$$

Therefore, the analytical approximate series solution is:

$$\begin{aligned} q(y_1, y_2, y_3) &= (y_2 y_3 - y_1) + \left(\frac{y_1^2}{2} y_2^{2-\varpi_2} y_3^{2-\varpi_3} - \frac{y_1^2}{2} y_2 y_3 + \frac{y_1^3}{3!} \right) \\ &\quad + ((4 - \varpi_3 - \varpi_2) \frac{y_1^4}{4!} y_2^{3-2\varpi_2} y_3^{3-2\varpi_3} \\ &\quad - \frac{y_1^4}{8} y_2^{2-\varpi_2} y_3^{2-\varpi_3} + \frac{y_1^4}{4!} y_2 y_3 - \frac{y_1^5}{5!}) + \dots, \end{aligned} \quad (4.15)$$

where $\varpi_i \in (0, 1]$, $y_i \in \mathbb{R}^+$, $i = 1, 2, 3$.

5. Conclusion

This study is dedicated to the exploration and application of the $\mathcal{C}_{\text{FD}}\text{-3ST}$ in conjunction with the ADM. The primary focus of this work involves: Establishing and defining the $\mathcal{C}_{\text{FD}}\text{-3ST}$. Investigating and presenting novel

results and theorems related to fractional conformable transformations. Applying the newly introduced $\mathcal{C}_{FD}$ -3ST in combination with the ADM to achieve consistency in solving nonlinear dynamic models, encompassing both homogeneous and non-homogeneous three-dimensional PDEs. The effectiveness and applicability of this innovative methodology are demonstrated through numerical experiments, validating its proficiency in handling complex nonlinear problems. Furthermore, it is worth noting that this method is versatile and can also be effectively applied to linear FPDEs, extending its utility to a broad spectrum of mathematical models. By integrating the $\mathcal{C}_{FD}$ -3ST with the ADM, this work contributes to the advancement of fractional calculus techniques, providing a powerful tool for tackling intricate mathematical problems across various scientific disciplines.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

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Authors' contributions

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References

- [1] Z. Ahmed, M. I. Idrees, F. B. M. Belgacem, and Z. Perveen, "On the convergence of double Sumudu transform," *Journal of Nonlinear Sciences and Applications*, vol. 13, no. 3, pp. 154-162 (2020).
- [2] Z. Al-Zhour, F. Alrawajeh, N. Al-Mutairi, and R. Alkhasawneh, "New results on the conformable fractional Sumudu transform: Theories and applications," *International Journal of Analysis and Applications*, vol. 17, no. 6, pp. 1019-1033 (2019).
- [3] S. Alfaqeh, G. Bakcerler, and E. Misirli, "Conformable double Sumudu transform with applications," *Journal of Applied and Computational Mechanics*, vol. 7, no. 2, pp. 578-586 (2021).

- [4] M. A. Asiru, "Further properties of the Sumudu transform and its applications," *International Journal of Mathematical Education in Science and Technology*, vol. 33, no. 3, pp. 441-449 (2002).
- [5] A. Atangana and A. Kilicman, "A novel integral operator transform and its application to some FODE and FPDE with some kind of singularities," *Mathematical Problems in Engineering*, vol. 2013, pp. 7 (2013).
- [6] D. Avc, B. B. Iskender Eroglu, and N. Ozdemir, "Conformable heat equation on a radial symmetric plate," *Thermal Science*, vol. 21, no. 2, pp. 819-826 (2017).
- [7] F. B. M. Belgacem, "Introducing and analyzing deeper Sumudu properties," *Nonlinear Studies*, vol. 13, no. 1, pp. 23-41 (2006).
- [8] F. B. M. Belgacem and A. A. Karaballi, "Sumudu transform fundamental properties investigations and applications," *International Journal of Stochastic Analysis*, vol. 2006, pp. 24 (2006).
- [9] F. B. M. Belgacem and R. Silambarasan, "Theory of natural transform," *Mathematics in Engineering, Science and Aerospace*, vol. 3, no. 1, pp. 99-124 (2012).
- [10] F. B. M. Belgacem and S. Sivasundaram, "New developments in computational techniques and transform theory applications to nonlinear fractional and stochastic differential equations and systems," *Nonlinear Studies*, vol. 22, no. 4, pp. 561-563 (2015).
- [11] S. A. Bhanota, "Exploration of novel analytical solutions of boundary layer equation via the modified sumudu transform," *Examples and Counterexample*, vol. 5, pp. 100140 (2024).
- [12] S. A. Bhanotar and F. B. M Belgacem, "Theory and applications of distinctive conformable triple Laplace and Sumudu transforms decomposition methods," *Journal of Partial Differential Equations*, vol. 35, no. 1, pp. 49-77 (2022).
- [13] S. A. Bhanotar and M. K. A. Kaabar, "Analytical solutions for the nonlinear partial differential equations using the conformable triple Laplace transform decomposition method," *International Journal of Differential Equations*, vol. 2021, pp. 18 (2021).
- [14] S. A. Bhanotar and S. Sivasundaram, "Exploration of novel iterative solution of nonlinear partial fractional differential equations with mixed derivatives via Adomian multi Sumudu transform," *Nonlinear Studies*, vol. 30, no. 3, pp. 921-936 (2023).

- [15] V. G. Gupta, B. Sharma, and A. Kiliçman, "A note on fractional Sumudu transform," *Journal of Applied Mathematics*, vol. 2010, pp. 9 (2010).
- [16] H. A. Hammad, R. A. Rashwan, A. Nafea, M. E. Samei, and M. De la Sen, "Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations," *Symmetry*, vol. 14, pp. 2579 (2022).
- [17] H. A. Hammad, R. A. Rashwan, A. Nafea, M. E. Samei, and S. Noeiaghdam, "Stability analysis for a tripled system of fractional pantograph differential equations with nonlocal conditions," *Journal of Vibration and Control*, vol. 30, no. 3-4, pp. 632-647 (2024).
- [18] Mohamed Houas, M. E. Samei, S. S. Santra, and J. Alzabut, "On Duffing-type oscillator differential equation on the transition to chaos with fractional q-derivatives," *Journal of Inequality and Applications*, vol. 2024, pp. 12 (2024).
- [19] M. G. Hussain and F. B. M. Belgacem, "Transient solutions of Maxwell's equations based on Sumudu transform," *Progress in Electromagnetics Research*, vol. 74, pp. 273-289 (2007).
- [20] Q. Hussein and H. Altaie, "Solutions of nonlinear fractional integro-differential equations by combined sumudu transform with Adomian decomposition approach," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 7, pp. 1665-1670 (2023).
- [21] Sajad Iqbal, F. M. González, M. K. A. Kaabar, and M. E. Samei, "A novel Elzaki transform homotopy perturbation method for solving time-fractional non-linear partial differential equations," *Boundary Value Problems*, vol. 2022, pp. 91 (2022).
- [22] A. Kadem, "Solving the one-dimensional neutron transport equation using Chebyshev polynomials and the Sumudu transform," *Analele Universitatii din Oradea. Fascicola Matematica*, vol. XII, pp. 153-171 (2005).
- [23] Q. D. Katatbeh and F. B. M. Belgacem, "Applications of the Sumudu transform to fractional differential equations," *Nonlinear Studies*, vol. 18, no. 1, pp. 99-112 (2011).
- [24] R. Khalil, M. A. Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65-70 (2014).
- [25] T. Kherraz, M. Benbachir, M. Lakrib, M. E. Samei, M. K. A. Kaabar, and S. A. Bhanotar, "Existence and uniqueness results for fractional boundary value problems with multiple orders of fractional deriva-

- tives and integrals," *Chaos, Solitons & Fractals*, vol. 166, pp. 113007 (2023).
- [26] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam (2006).
 - [27] A. Kiliçman, H. Eltayeb, and R. P. Agarwal, "On Sumudu transform and system of differential equations," *Abstract and Applied Analysis*, vol. 2010, pp. 11 pages (2010).
 - [28] A. Kilicman and H. Eltayeb, "A note on integral transforms and partial differential equations," *Applied Mathematical Sciences*, vol. 4, no. 3, pp. 109-118 (2010).
 - [29] A. Kilicman and H. Eltayeb, "On the applications of Laplace and Sumudu transforms," *Journal of the Franklin Institute*, vol. 347, no. 5, pp. 848-862 (2010).
 - [30] H. Z. Mjthap and S. N. Al-Azzawi, "Mixing Sumudu transform and Adomain decomposition method for solving Riccati equation of variable fractional order," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 8, pp. 1559-1563 (2019).
 - [31] H. Z. Mjthap and S. N. Al-Azzawi, "Some properties of Sumudu transformation," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 8, pp. 1543-1547(2019).
 - [32] I. Podlubny, *Fractional differential equations*, Academic Press, New York (1993).
 - [33] Shahram Rezapour, Hakimeh Mohammadi, and M. E. Samei, "SEIR epidemic model for COVID-19 transmission by Caputo derivative of fractional order," *Advances in Difference Equations*, vol. 2020, pp. 490 (2020).
 - [34] J. Singh, D. Kumar, and A. Kiliçman, "Homotopy perturbation method for fractional gas dynamics equation using Sumudu transform," *Abstract and Applied Analysis*, vol. 2013, pp. 8 (2013).
 - [35] H. Thabet and S. Kendre, "Analytical solutions for conformable space time-fractional partial differential equations via fractional differential transform," *Chaos Solitons & Fractals*, vol. 109, pp. 238-245 (2018).
 - [36] G. K. Watugala, "Sumudu transform: A new integral transform to solve differential equations and control engineering problems," *International Journal of Mathematical Education in Science and Technology*, vol. 24, no. 1, pp. 35-43 (1993).