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A new approach to soft relations and topology

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Abstract

Let $\mathfrak{R} = (\theta, \mathcal{A})$ be a soft relation over a set $\mathcal{X}$, where $\theta : \mathcal{A} \to \mathcal{P}(\mathcal{X} \times \mathcal{X})$. In this paper, we first associate a mapping $\alpha_\theta : \mathcal{X} \times \mathcal{X} \to \mathcal{P}(\mathcal{A})$ with every soft relation $\mathfrak{R}$ over $\mathcal{X}$. This mapping is then utilized to redefine the concept of soft equivalence relation as developed in [8]. Next, we introduce soft topologies generated by $\mathcal{R}$ and explore several related results within this framework. Specifically, we have characterised soft topologies that are produced by a soft relation and a soft selection. Finally, we applied these concepts to fuzzy frameworks. This revised approach to soft structures brings working with soft set theory into closer alignment with fuzzy set theory.

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1. Introduction

A relation $\mathcal{R}$ defined on a set $\mathcal{X}$ can be characterised as a subset of the Cartesian product $\mathcal{X} \times \mathcal{X}$. In [20], the sets $B_{\mathcal{R}}(m) = \{n \in \mathcal{X} : (m, n) \in \mathrm{R}\}$ and $D_{\mathcal{R}}(m) = \{n \in \mathcal{X} : (n, m) \in R\}$ are defined as the upper and lower contour, respectively. Knoblauch [16] introduced topologies that are induced by $\mathcal{R}$, which are generated using the collection $\{D_{\mathcal{R}}(m), B_{\mathcal{R}}(m)\}_{m \in \mathcal{X}}$ as a subbasis for the open sets. Smithson [20] established a topology on $\mathcal{X}$ using a collection of antisets in $\mathcal{X}$ in relation to $\mathcal{R}$. To induce topologies, various kinds of relations have been used. In [24, 5] an investigation of orderable and preorderable topologies is provided. In the context of fuzzy sets, Fuzzy topologies [6, 13] extend classical topology by allowing for the representation of uncertainty and vagueness inherent in many real-world situations. A fuzzy topology is typically

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generated from fuzzy relations [28], which are mathematical constructs that generalize classical binary relations to accommodate degrees of membership. As a generalization of the related idea in the crisp case, a fuzzy topology derived from a fuzzy relation has been studied by numerous scholars [16, 12, 27, 7, 23, 22, 17]. The idea of a soft set was first presented by Molodtsov [18] as a mathematical tool intended to handle uncertainty without the difficulties of fuzzy set theory. A pair of parameters and a set whose components are described by these parameters are called soft sets. Several soft set operations were presented and described by Maji et al. [15]. The papers [14, 26] examine the use of soft sets. Both fuzzy topology [6] and soft topology [19], which were developed by Shabir and Naz, are extensions of classical topology. There are several applications of soft topology, in both mathematics and real-world situations[2, 10, 9, 1]. The concept of soft relations was first proposed by the authors in [8] as an expansion of fuzzy relations. A soft relation over $\mathcal{X}$ is $\mathfrak{R} = (\theta, \mathcal{A})$ such that $\theta: \mathcal{A} \to P(\mathcal{X} \times \mathcal{X})$. For every parameter $e \in \mathcal{A}$, there exists $\theta(e) \subseteq \mathcal{X} \times \mathcal{X}$. This is because $\mathfrak{R}$ is a family of binary relations on $\mathcal{X}$. The soft relations defined by a subset of the soft product of soft sets serve as the foundation for all research on soft topology produced by soft relations [4, 11, 21, 25]. In this paper, we first associate a mapping $\alpha_\theta: \mathcal{X} \times \mathcal{X} \to \mathcal{P}(\mathcal{A})$ with every soft relation $\mathfrak{R}$ over $\mathcal{X}$. This mapping is then utilized to redefine the concept of soft equivalence relation (**S-eqR**) as developed in [8]. Next, we introduce soft topologies generated by $\mathfrak{R}$ and explore several related results within this framework. In particular, soft topologies generated by a soft relation and a soft selection are characterized. finally, we applied these concepts to fuzzy frameworks.

2. Preliminaries

In this paper, the symbols $\mathcal{X}$ and $\mathcal{P}(\mathcal{X})$ represent a non-empty finite set and the power set of $\mathcal{X}$ respectively, unless otherwise indicated.

Definition 1: ([18]) Consider a collection of parameters $\mathcal{A}$. If $\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is a set-valued mapping, then a pair $(\mathcal{F}, \mathcal{A})$ referred to as a soft set (**S-Set**) over $\mathcal{X}$. Stated differently, a **S-Set** is a parameterised family of subsets of $\mathcal{X}$.

$\mathcal{S}(\mathcal{X})$ is the representation of the collection of all $(\mathcal{F}, \mathcal{A})$ defined over $\mathcal{X}$.

Definition 2: ([18]) Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{A})$ be elements in $\mathcal{S}(\mathcal{X})$, then

1. $(\mathcal{F}, \mathcal{A}) \sqsubseteq (\mathcal{G}, \mathcal{A})$, if $\mathcal{F}(e) \subseteq \mathcal{G}(e)$, $\forall\, e \in \mathcal{A}$.
2. $(\mathcal{F}, \mathcal{A}) = (\mathcal{G}, \mathcal{A})$, if $\mathcal{F}(e) = \mathcal{G}(e)$, $\forall\, e \in \mathcal{A}$.

Definition 3: ([18]) Let $(\mathcal{F}, \mathcal{A}), (\mathcal{G}, \mathcal{A}) \in \mathcal{S}(\mathcal{X})$. Then, the soft sets $(\mathcal{F} \sqcup \mathcal{G}, \mathcal{A})$ and $(\mathcal{F} \sqcap \mathcal{G}, \mathcal{A})$ are defined by

$$(\mathcal{F} \sqcup \mathcal{G})(e) = \mathcal{F}(e) \cup \mathcal{G}(e),$$
$$(\mathcal{F} \sqcap \mathcal{G})(e) = \mathcal{F}(a) \cap \mathcal{G}(e),$$

respectively.

Let $U \in \mathcal{P}(\mathcal{X})$. $U_{\mathcal{X}}$ will represent the constant **S-Set** over $\mathcal{X}$ with value U, that is $U_{\mathcal{X}}(e) = U, \forall e \in \mathcal{A}$.

Definition 4: ([19]) A soft topology **(S-Top)** over $\mathcal{X}$ is defined as a family $T \subseteq \mathcal{S}(\mathcal{X})$ if T satisfies:

(ST1) $\Phi_{\mathcal{X}}, \mathcal{X}_{\mathcal{X}} \in T$,

(ST2) If $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{A})$ are elements in $T \Rightarrow (\mathcal{F} \sqcap \mathcal{G}, \mathcal{A}) \in T$,

(ST3) If $\{(\mathcal{F}_i, \mathcal{A}), i \in I\} \subseteq T$, then $(\coprod_i \mathcal{F}_i, \mathcal{A}) \in T$.

3. Soft relations

A soft relation **(S-Rel)** over $\mathcal{X}$ is a pair $\mathfrak{R} = (\theta, \mathcal{A})$ such that $\theta: \mathcal{A} \to P(\mathcal{X} \times \mathcal{X})$ [8]. That is, for each parameter $e \in \mathcal{A}$, there exists a relation $\theta(e) \subseteq \mathcal{X} \times \mathcal{X}$. We associate with $\mathfrak{R}$ the mapping $\alpha_\theta: \mathcal{X} \times \mathcal{X} \to P(\mathcal{A})$ where $\alpha_\theta(m, n) = \{e \in \mathcal{A}: (m, n) \in \theta(e)\}$. The soft relation $(\theta, \mathcal{A}) \in \mathcal{S}(\mathcal{X} \times \mathcal{X})$ can also be defined by $\theta(e) = \{(m, n) \in \mathcal{X} \times \mathcal{X}: e \in \alpha_\theta(m, n)\}$, if we take $\alpha_\theta: \mathcal{X} \times \mathcal{X} \to P(\mathcal{A})$,

In the example below, we demonstrate that the soft relation $(\theta, \mathcal{A})$ is completely characterized by defining the mapping α_θ.

Example 1: Consider the sets $\mathcal{A} = \{a, b, c\}$ and $\mathcal{X} = \{m, n, w\}$. $\mathfrak{R} = (\theta, \mathcal{A})$ is a **S-Rel** over $\mathcal{X}$ defined by Table 1.

Table 1

Soft relation $(\boldsymbol{\theta}, \boldsymbol{\mathcal{A}})$

$(\theta, \mathcal{A})$	(m,m)	(w,n)	(n,m)	(w,w)	(n,w)	(n,n)	(m,n)	(m,w)	(w,m)
a	1	0	0	1	1	1	0	0	0
b	0	1	0	1	1	0	0	0	0
c	1	1	0	1	1	1	0	0	0

We can read the tabular representation of $\mathfrak{R}$ as follows:

Row 1: The relation $\theta(a) = \{(m, m), (w, w), (n, w), (n, n)\}$ is reflexive, i.e. $(z, z) \in \theta(a)\ \forall x \in \mathcal{X}$.

Row 2: The relation $\theta(b) = \{(w, w), (n, w), (w, n)\}$ is symmetric, i.e. $(y, z) \in \theta(a)$ implies $(z, y) \in \theta(a)\ \forall y, z \in \mathcal{X}$.

Row 3: The relation $\theta(c) = \{(m, m), (w, n), (n, n), (w, w), (n, w)\}$ is transitive, that is (y, z) and (z, x) belong to $\theta(a) \Rightarrow (y, x) \in \theta(a), \forall z, x, y \in \mathcal{X}$.

Now, the associating mapping $\alpha_\theta: \mathcal{X} \times \mathcal{X} \to \mathcal{P}(\mathcal{A})$ is defined as in Table 2:

Table 2

The associating mapping α_θ

α_θ	Parameter: a	Parameter: b	Parameter: c
(m,m)	1	0	1
(n,n)	1	0	1
(w,n)	0	1	1
(w,w)	1	1	1
(n,w)	1	1	1
(n,m)	0	0	0
(m,n)	0	0	0
(m,w)	0	0	0
(w,m)	0	0	0

We can read the tabular representation of the associating mapping α_θ as follows:

$$\alpha_\theta(x,x) = \{a,c\}, \alpha_\theta(y,x) = \emptyset, \alpha_\theta(z,y) = \{b,c\}, \ldots$$

Clearly, Table 2 is obtained by transposing Table 1, which involves swapping its rows and columns, and vice versa.

Lemma 1: Let $(R,\mathcal{A})$, $(\theta,\mathcal{A})$ be **S-Rel**s over $\mathcal{X}$ and $m,n \in \mathcal{X}$. Then,

(1) $\alpha_{R\sqcap\theta}(n,m) = \alpha_R(n,m) \cap \alpha_\theta(n,m)$,

(2) $\alpha_{R\sqcup\theta}(n,m) = \alpha_R(n,m) \cup \alpha_\theta(n,m)$,

(3) $\alpha_{\theta^{-1}}(m,n) = \alpha_\theta(n,m)$, where $\theta^{-1}(e)$ the converse relation of $\theta(e), \forall e \in \mathcal{A}$[8].

Proof: (1) For all $(n,m) \in \mathcal{X} \times \mathcal{X}$, we get

$$\begin{aligned}\alpha_{R\sqcap\theta}(n,m) &= \{e \in \mathcal{A}: (n,m) \in (R \sqcap \theta)(e)\}\\ &= \{e \in \mathcal{A}: (n,m) \in R(e) \sqcap \theta(e)\}\\ &= \{e \in \mathcal{A}: (n,m) \in R(e)\} \cap \{e \in \mathcal{A}: (n,m) \in \theta(e)\}\\ &= \alpha_R(n,m) \cap \alpha_\theta(n,m)\end{aligned}$$

(2) similar to (1).

(3) For all $(n,m) \in \mathcal{X} \times \mathcal{X}$, we have,

$$\begin{aligned}\alpha_{\theta^{-1}}(n,m) &= \{e \in \mathcal{A}: (n,m) \in \theta^{-1}(e)\}\\ &= \{e \in \mathcal{A}: (m,n) \in \theta(e)\}\\ &= \alpha_\theta(m,n).\end{aligned}$$

Recalling from [8], a **S-Rel** $\mathfrak{R}$ over $\mathcal{X}$ is a soft reflexive (a soft symmetric, a soft transitive, an equivalence) relation over $\mathcal{X}$ if the relation $\theta(a) \neq \emptyset$ is reflexive (symmetric, transitive, an equivalence) on $\mathcal{X}$ for all $a \in \mathcal{A}$. Below is a characterization of this definition through the associating mapping α_θ.

Theorem 2: *Let* $\mathfrak{R} = (\theta, \mathcal{A})$ *be a* ***S-Rel*** *over* $\mathcal{X}$. *Then,* $\forall\, z, x, y \in \mathcal{X}$, *the following hold:*

(1) $\mathfrak{R}$ *is soft reflexive* $\Leftarrow \alpha_\theta(x, x) = \mathcal{A}$.

(2) $\mathfrak{R}$ *is soft symmetric* $\Leftarrow \alpha_\theta(x, y) = \alpha_\theta(y, x)$.

(3) $\mathfrak{R}$ *is soft transitive* $\Leftarrow \alpha_\theta(x, y) \supseteq \alpha_\theta(x, z) \cap \alpha_\theta(z, y)$

Proof: $(1)(\Rightarrow)$ Since the relation $\theta(a)$ is reflexive on $\mathcal{X}$ for all $a \in \mathcal{A}$, then $\alpha_\theta(x, x) = \mathcal{A}$. Conversely, if $\alpha_\theta(x, x) = \mathcal{A}$, implies $(x, x) \in \theta(a)$ for all $a \in \mathcal{A}$. Thus $(\theta, \mathcal{A})$ is soft reflexive.

(2) straightforward.

(3) Assume that the relation $\theta(a)$, for all $a \in \mathcal{A}$, is transitive on $\mathcal{X}$ and $(x, w), (w, y) \in \theta(a)$. Then we have $a \in \alpha_\theta(x, w) \cap \alpha_\theta(w, y)$. Sine $\theta(a)$ is a transitive then $(x, y) \in \theta(a)$ and so $a \in \alpha_\theta(x, y)$. That is, $\alpha_\theta(x, w) \cap \alpha_\theta(w, y) \subseteq \alpha_\theta(x, y)$.

$(\Leftarrow)$ is obvious.

Lemma 3: *If* $\mathfrak{R} = (\theta, \mathcal{A})$ *is a* ***S-eqR*** *over* $\mathcal{X}$, *then* $\alpha_\theta(m, n) \cap \alpha_\theta(n, z) = \alpha_\theta(m, z)$, $\forall\, m, z, n \in \mathcal{X}$.

Proof. Suppose $\mathfrak{R}$ is a **S-eqR** over $\mathcal{X}$. By Theorem 2, it is valid that $\alpha_\theta(m, z) \supseteq \alpha_\theta(m, n) \cap \alpha_\theta(n, z)$. Reverse inclusion: let $e \in \alpha_\theta(m, z) \Rightarrow (m, z), (z, z) \in \theta(e)$, since $\theta(e)$ is reflexive. Consequentially, $(m, z) \in \theta(e) \circ \theta(e)$. Hence $\exists\, w \in \mathcal{X}$ where $(m, w), (w, z) \in \theta(e)$. This implies $e \in \alpha_\theta(m, w) \cap \alpha_\theta(w, z)$. Therefore $\alpha_\theta(m, z) = \alpha_\theta(m, w) \cap \alpha_\theta(w, z)$ for all $m, w, z \in \mathcal{X}$.

Based on Theorem 2, we can restate the definition of $S - eqR$s in terms of the corresponding mapping.

Definition 5: A soft relation $\mathfrak{R} = (\theta, \mathcal{A})$ over $\mathcal{X}$ is an $\boldsymbol{S - eqR}$ on $\mathcal{X}$ if the mapping $\alpha_\theta : \mathcal{X} \times \mathcal{X} \to P(\mathcal{A})$ satisfies the following:

(1) $\alpha_\theta(m, m) = \mathcal{A}, \forall m \in \mathcal{X}$;

(2) $\alpha_\theta(m, n) = \alpha_\theta(n, m), \forall n, m \in \mathcal{X}$;

(3) $\alpha_\theta(m, n) \cap \alpha_\theta(n, w) \subseteq \alpha_\theta(m, w), \forall n, w, m \in \mathcal{X}$.

Remark 1: *Let* $(R, \mathcal{A}), (\theta, \mathcal{A})$ *be* ***S-Rel****s. From* [8], the composition of $(\theta, \mathcal{A})$ and $(R, \mathcal{A})$ is the **S-Rel** $(R \circ \theta, \mathcal{A})$ defined by

$$(R \circ \theta)(e) = R(e) \circ \theta(e) = \{(m, n) \in \mathcal{X} \times \mathcal{X} : \exists x \in \mathcal{X}, (m, x) \in R(e), (x, n) \in \theta(e)\}.$$

For every $y, w \in \mathcal{X}$, we obtain

$$\begin{aligned}
\alpha_{\rho\circ\sigma}(y, w) &= \{e \in \mathcal{A} : (x, z) \in (\rho \circ \sigma)(e)\} \\
&= \{e \in \mathcal{A} : (y, w) \in \theta(e) \circ \theta(e)\} \\
&= \{e \in \mathcal{A} : \exists x \in \mathcal{X}, (y, x) \in \rho(a), (x, w) \in \sigma(e)\} \\
&= \cup_{x\in\mathcal{X}} \{e \in \mathcal{A} : (y, x) \in \rho(e)\} \cap \{e \in \mathcal{A} : (x, w) \in \sigma(e)\} \\
&= \cup_{x\in\mathcal{X}} \{\alpha_\rho(y, x) \cap \alpha_\sigma(x, w)\}
\end{aligned}$$

The following conclusion may be simply derived from Remark 1, Definition 5, and Lemma 3.

Proposition 4: Let $\mathfrak{R} = (\theta, \mathcal{A})$ be a $\boldsymbol{S-eqR}$, then

$$\alpha_{\theta\circ\theta}(n,m) = \alpha_{\theta}(n,m), \forall m,n \in \mathcal{X}.$$

Definition 6: Let $\mathfrak{R} = (\theta, \mathcal{A})$ be a $\boldsymbol{S-eqR}$. A soft equivalence class of $m \in \mathcal{X}$ is a **S-Set** $([m], \mathcal{A})$ over $\mathcal{X}$ defined by

$$\alpha_{[m]}(z) = \alpha_{\theta}(m,z)$$

for all $m, z \in \mathcal{X}$.

Proposition 5: Let $\mathfrak{R} = (\theta, \mathcal{A})$ be a $\boldsymbol{S-eqR}$. Then
(1) $([n], \mathcal{A}) = ([m], \mathcal{A}) \Leftrightarrow \alpha_{\theta}(n,m) = \mathcal{A}, \forall n,m \in \mathcal{X}$,
(2) $([m], \mathcal{A}) \sqcap ([n], \mathcal{A}) = \Phi_{\mathcal{X}} \Leftrightarrow \alpha_{\theta}(m,n) = \phi, \forall m,n \in \mathcal{X}$,
(3) $\sqcup_{w\in\mathcal{X}} ([w], \mathcal{A}) = \mathcal{X}_{\mathcal{X}}$.

Proof: (1): If $\alpha_{\theta}(w,y) = \mathcal{A}$ then we obtain

$$\alpha_{[w]}(x) = \alpha_{\theta}(w,x) \supseteq \alpha_{\theta}(w,y) \cap \alpha_{\theta}(y,x) = \alpha_{\theta}(y,x) = \alpha_{[y]}(x), \forall x \in \mathcal{X}.$$

Hence, $[y](e) \subseteq [w](e), \forall e \in \mathcal{A} \Rightarrow ([y], \mathcal{A}) \sqsubseteq ([w], \mathcal{A})$. Similarly, we get $([w], \mathcal{A}) \sqsubseteq ([y], \mathcal{A})$. Therefore, $([w], \mathcal{A}) = ([y], \mathcal{A})$. Conversely, If $([w], \mathcal{A}) = ([y], \mathcal{A})$ then $\alpha_{\theta}(w,y) = \alpha_{[w]}(y) = \alpha_{[y]}(y) = \alpha_{\theta}(y,y) = \mathcal{A}$.

(2): If $\alpha_{\theta}(m,n) = \phi$ then we have

$$\phi = \alpha_{\theta}(m,n) \supseteq \alpha_{\theta}(m,x) \cap \alpha_{\theta}(x,n) = \alpha_{[m]}(x) \cap \alpha_{[n]}(x) \supseteq \phi, \forall x \in \mathcal{X}.$$

By Lemma 1 , $([m], \mathcal{A}) \sqcap ([n], \mathcal{A}) = \Phi_{\mathcal{X}}$. Conversely, Assume that $\forall e \in \mathcal{A}, [m(e) \cap [y](e) = ([n] \sqcap [y])(e) = \phi$. By Lemma 3, we conclude to

$$\phi = \alpha_{[m]ó[n]}(x) = \alpha_{[m]}(x) \cap \alpha_{[n]}(x) = \alpha_{\theta}(m,x) \cap \alpha_{\theta}(n,x) = \alpha_{\theta}(m,n).$$

(3): For every $w \in \mathcal{X}$, we have

$$\alpha_{\sqcup_{y\in\mathcal{X}}[y]}(w) = \cup_{y\in X} \{\alpha_{[y]}(w)\} = \cup_{y\in\mathcal{X}} \{\alpha_{\theta}(y,w)\} = \mathcal{A},$$

because $\alpha_{\theta}(w,w) = \mathcal{A}$ is contained. Therefore, item(3) holds.

4. Soft topology via soft relations

Definition 7: Let $(\theta, \mathcal{A})$ be a soft relation over $\mathcal{X}$. Then, the lower and upper contours of $m \in \mathcal{X}$ are the soft sets $(\theta_m, \mathcal{A})$ and $(\theta^n, \mathcal{A})$, which are defined as

$$\theta_m(e) = \{z \in \mathcal{X}: (z,m) \in \theta(e)\}$$
$$\theta^m(e) = \{z \in \mathcal{X}: (m,z) \in \theta(e)\},$$

respectively.

Easily, $\forall x, w \in \mathcal{X}$,

$$\alpha_{\theta_x}(w) = \alpha_{\theta}(w,x), \quad \alpha_{\theta^x}(w) = \alpha_{\theta}(x,w),$$

Definition 8: The family $\{(\theta_m,\mathcal{A}),(\theta^m,\mathcal{A}): m\in\mathcal{X}\}$ forms a subbasis for a soft topology, which we denote as T_θ.

Example 2: Let $\mathcal{A}=\{a\}$ and $(\theta,\mathcal{A})$ be a **S-Rel** over $\mathcal{X}=\{m,n,z\}$, defined by:

$$\theta(a)=\{(m,n),(m,m),(n,z),(z,m)\}$$

Then, $(\theta_m,\mathcal{A}),(\theta_n,\mathcal{A}),(\theta_z,\mathcal{A}),(\theta^m,\mathcal{A}),(\theta^n,\mathcal{A})$ and $(\theta^z,\mathcal{A})$ are soft sets given by:

$$\theta_m(a)=\{z,m\},\quad \theta_n(a)=\{m\},\quad \theta_z(a)=\{n\},\quad \theta^m(a)=\{n,m\},\quad \theta^n(a)=\{z\},\quad \theta^z(a)=\{m\}.$$

Therefore,

$$T_\theta=\{\Phi,\mathcal{X},\theta_m,\theta_n,\theta_z,\theta^m,\theta^n,\theta^z,\mathcal{G}\},$$

where $\mathcal{G}(e)=\{n,z\}$.

Proposition 6: Let T_1 and T_2 be soft topologies induced by $\{\theta_w\}_{w\in\mathcal{X}}$ and $\{\theta^w\}_{w\in\mathcal{X}}$ respectively. Then $T_1=T_2$, if $\mathfrak{R}=(\theta,\mathcal{A})$ is symmetric.

Proof: For each $m,n\in\mathcal{X},\alpha_\theta(m,n)=\alpha_\theta(n,m)$, Since $\mathfrak{R}$ is symmetric. This implies $\alpha_{\theta_m}(n)=\alpha_{\theta^m}(n)$, and hence $\theta^x(e)=\theta_x(e)$, for each $e\in\mathcal{A}$. Consequently, T_1 and T_2 are identical.

Proposition 7: If $\mathfrak{R}=(\theta,\mathcal{A})$ is a reflexive and transitive **S-Rel** over $\mathcal{X}$, then

(1) If $(\mathcal{G},\mathcal{A})\in T_1$, then $\sqcup_{\alpha_{\mathcal{G}}(x)=\mathcal{A}}\ \theta_x\sqsubseteq\mathcal{G}$.

(2) If $(\mathcal{G},\mathcal{A})\in T_2$, then $\sqcup_{\alpha_{\mathcal{G}}(x)=\mathcal{A}}\ \theta^x\sqsubseteq\mathcal{G}$.

Proof: (1) for every $e\in\mathcal{A}$, let $w\in\bigcup_{\alpha_{\mathcal{G}}(x)=\mathcal{A}}\theta_x(e)\Rightarrow\exists x\in\mathcal{X}$ such that $\alpha_{\mathcal{G}}(x)=\mathcal{A}$, that is $x\in\mathcal{G}(e)$ for each $e\in\mathcal{A}$. Now, since $(\mathcal{G},\mathcal{A})\in T_1$, there exist $\{\theta_{x_i}\}\subseteq T_1$ such that $x\in\cap\,\theta_{x_i}(e)\subseteq\mathcal{G}(e)\Rightarrow x\in\theta_{x_i}(e)\Rightarrow e\in\alpha_{\theta_{x_i}}=\alpha_\theta(x,x_i)$. Since $e\in\alpha_{\theta_x}(w)=\alpha_\theta(w,x)\Rightarrow e\in\alpha_\theta(w,x)\cap\alpha_\theta(x,x_i)\subseteq\alpha_\theta(w,x_i)$. This implies that $e\in\alpha_{\theta_{x_i}}(w)$, for all $i=1,\ldots,n$. Thus $w\in\theta_{x_i}(e)\subseteq\mathcal{G}(e)$. Therefore, item (1) holds.

(2) similarly.

For any soft topology $\mathcal{T}$, can it be generated by a soft relation? The following result provides the answer:

Theorem 8: *Suppose $(\mathcal{X},\mathcal{A},\mathcal{T})$ is a **S-Top**. $\mathcal{T}$ is considered to be induced by a soft relation θ iff it possesses a subbase of the form $\{(\mathcal{F}_w,\mathcal{A}),(\mathcal{G}_w,\mathcal{A})\in\mathcal{S}(\mathcal{X}): w\in\mathcal{X}\}$ such that the condition $\alpha_{\mathcal{F}_z}(w)=\alpha_{\mathcal{G}_w}(z)$ holds for all $w,z\in\mathcal{X}$.*

Proof: $(\Rightarrow)$ Suppose $\mathcal{T}$ is produced by a certain **S-Rel** $(\theta,\mathcal{A})$, then T is generated by $\{(\mathcal{F}_w,\mathcal{A}),(\mathcal{G}_w,\mathcal{A})\in\mathcal{S}(\mathcal{X}): w\in\mathcal{X}\}$ where $\alpha_{\mathcal{F}_w}(y)=\alpha_{\theta_w}(y)=\alpha_\theta(w,y)$ and $\alpha_{\mathcal{G}_w}(y)=\alpha_{\theta^w}(y)=\alpha_\theta(y,w)$ such that $\alpha_{\mathcal{F}_y}(x)=\alpha_{\mathcal{G}_w}(y)$.

($\Leftarrow$) Assume that, $\forall\, m, n \in \mathcal{X}$, the set $\{(\mathcal{F}_m, \mathcal{A}), (\mathcal{G}_m, \mathcal{A}) \in \mathcal{S}(\mathcal{X})\}$ generates $\mathcal{T}$ such that $\alpha_{\mathcal{F}_n}(m) = \alpha_{\mathcal{G}_m}(n)$. To demonstrate that T has a subbase related to a **S-Rel** $\mathfrak{R}$, we define $\mathfrak{R} = (\theta, \mathcal{A})$, $\theta: \mathcal{A} \to \mathcal{P}(\mathcal{X} \times \mathcal{X})$, by $\alpha_\theta(m, n) = \alpha_{F_n}(m) = \alpha_{G_m}(n), \forall n, m \in \mathcal{X}$. Then $\alpha_{\theta_m}(n) = \alpha_{F_m}(n) = \alpha_\theta(n, m$ and $\alpha_{\theta^m}(n) = \alpha_{G_m}(n) = \alpha_\theta(m, n)$ which implies $(\theta_m, \mathcal{A}) = (\mathcal{F}_m, \mathcal{A})$ and $(\theta^m, \mathcal{A}) = (\mathcal{G}_m, \mathcal{A})$. Thus, the family $\{(\theta_m, \mathcal{A}), (\theta^m, \mathcal{A})\}$ functions as a subbasis for $\mathcal{T}$ according to the theorem's postulate. . Thus, $\mathcal{T}$ is induced by the **S-Rel** $\mathfrak{R}$.

Below, we show that a **S-Rel** $\mathfrak{R}$, which generates $\mathcal{T}$, will satisfy some extra features if specific constraints on $\{(\mathcal{F}_w, \mathcal{A}), (\mathcal{G}_w, \mathcal{A}) \in \mathcal{S}(\mathcal{X}): w \in \mathcal{X}\}$ are imposed.

Proposition 9: The soft relation $\mathfrak{R} = (\theta, \mathcal{A})$ where $\alpha_\theta(m, n) = \alpha_{\mathcal{F}_n}(m) = \alpha_{\mathcal{G}_m}(n), \forall m, n \in \mathcal{X}$, satisfies the following:

(1) $\mathfrak{R}$ is soft reflexive $\Leftrightarrow$ $\alpha_{\mathcal{F}_w}(w) = \mathcal{A}, \forall w \in \mathcal{X}$.

(2) $\mathfrak{R}$ is soft symmetric $\Leftrightarrow$ $(\mathcal{F}_w, \mathcal{A}) = (\mathcal{G}_w, \mathcal{A}), \forall w \in \mathcal{X}$.

(3) $\mathfrak{R}$ is soft transitive $\Leftrightarrow$ $\alpha_{\mathcal{F}_z}(w) \supseteq \alpha_{\mathcal{G}_w}(y) \cap \alpha_{\mathcal{F}_z}(y), \forall z, y, w \in \mathcal{X}$.

Proof: Straightforward.

Definition 9: Let $\mathcal{X}$ be a nonempty set. The soft partition of $\mathcal{X}$ is a family $\mathcal{C}$ of **S-Set**s defined over $\mathcal{X}$ such that:

(1) $\forall: (\mathcal{F}, \mathcal{A}) \in \mathcal{C}, \exists\, w \in \mathcal{X}$ with $\alpha_{\mathcal{F}}(w) = \mathcal{A}$.

(2) $\forall\, w \in \mathcal{X}$, only one $(\mathcal{F}, \mathcal{A}) \in \mathcal{C}$ with $\alpha_{\mathcal{F}}(w) = \mathcal{A}$.

(3) If $(\mathcal{F}, \mathcal{A}), (\mathcal{G}, \mathcal{A}) \in \mathcal{C}$, and $\alpha_{\mathcal{F}}(w) = \alpha_{\mathcal{G}}(z) = \mathcal{A}$ for some $w, z \in \mathcal{X}$, then $\alpha_{\mathcal{F}}(z) = \alpha_{\mathcal{G}}(w)$.

Definition 10: Suppose $\mathfrak{R} = (\theta, \mathcal{A})$ is a **S-Rel** over $\mathcal{X}$. Then $\mathfrak{R}$ is called a soft selection if it verifies:

(1) For each $m \in \mathcal{X}, \alpha_\theta(m, m) = \emptyset$.

(2) For every $m, n \in \mathcal{X}, m \neq n$, either $\alpha_\theta(m, n) = \mathcal{A}, \alpha_\theta(n, m) = \emptyset$ or $\alpha_\theta(n, m) = \mathcal{A}, \alpha_\theta(m, n) = \emptyset$

Theorem 10: *Let* $\mathfrak{R}$ *be a soft selection and* T *be a **S-Top** over* $\mathcal{X}$ *generated by* $\mathfrak{R}$. *Then the family* $\{(\mathcal{F}_w, \mathcal{A}), (\mathcal{G}_w, \mathcal{A}) \in \mathcal{S}(\mathcal{X}): w \in \mathcal{X}\}$ *serves as a subbase for* T *and satisfies the conditions:*

(1) $\alpha_{\mathcal{F}_z}(\mathrm{w}) = \alpha_{\mathcal{G}_\mathrm{w}}(\mathrm{z}), \forall \mathrm{w}, \mathrm{z} \in \mathcal{X}$.

(2) $\forall w \in \mathcal{X}$, *the collection* $\mathcal{C} = \{(\mathcal{F}_w, \mathcal{A}), (\mathcal{G}_w, \mathcal{A}), \{w\}_{\mathcal{X}}\}$ *is a soft partition.*

Proof: (1) Let T be a **S-Top** generated by a soft selection. By Theorem 8, item (1) holds.

(2) For each $w \in \mathcal{X}, \alpha_\theta(w, w) = \emptyset$, since $\mathfrak{R}$ is a soft selection. This implies $\alpha_{\mathcal{F}_w}(w) = \alpha_{\mathcal{G}_w}(w) = \alpha_\theta(w, w) = \emptyset$. For $w \neq k$, by Definition 10,

either $\alpha_\theta(w,k) = \mathcal{A}$ and $\alpha_\theta(k,w) = \emptyset$ or $\alpha_\theta(z,w) = \mathcal{A}$ and $\alpha_\theta(w,k) = \emptyset$. Consequently, either $\alpha_{\mathcal{G}_w}(k) = \mathcal{A}$ and $\alpha_{\mathcal{F}_w}(k) = \emptyset$ or $\alpha_{\mathcal{G}_w}(k) = \emptyset$ and $\alpha_{\mathcal{F}_w}(k) = \mathcal{A}$. It remains to prove item(3) of Definition 9. For some $m, n \in \mathcal{X}$, if $\alpha_{\mathcal{F}_m}(m) = \alpha_{\mathcal{G}_x}(n) = \mathcal{A} \Rightarrow \alpha_\theta(m,x) = \alpha_{\mathcal{F}_x}(m) = \mathcal{A}$. So $m \neq x$ and hence $\alpha_\theta(x,m) = \emptyset$, by Definition 10. Similarly, we get $\alpha_\theta(n,x) = \emptyset$, and therefore $\alpha_{\mathcal{F}_m}(n) = \alpha_\theta(n,m) = \alpha_\theta(x,m) = \alpha_{\mathcal{G}_x}(m)$.

5. Application to Fuzzy settings

A fuzzy relation **(F-Rel)** on $\mathcal{X}$ is a mapping $\lambda: \mathcal{X} \times \mathcal{X} \to [0,1]$. We recall from [16] that λ is a $F-eqR$ on X when $\lambda(w, wm, m) = 1, \lambda(m,n) = \lambda(n,m)$ and $\lambda(m,n) \geq \lambda(m,x) \wedge \lambda(x,n), \forall m,x,n \in \mathcal{X}$.

According to [8], the **F-Rel** λ could be written as a **S-Rel** $(\theta_\lambda, [0,1])$ over $\mathcal{X}$ where,

$$\theta_\lambda(t) = \lambda_t = \{(y,w): \lambda(y,w) \geq t\}, \forall t \in [0,1].$$

Theorem 11: [3] *λ is a $F-eqR$ on $\mathcal{X}$ $\Leftrightarrow$ the set λ_t is an **eqR** on $\mathcal{X}$.*

Theorem 12: *A **F-Rel** λ on $\mathcal{X}$ is an $F-eqR$ $\Leftrightarrow (\theta_\lambda, \mathcal{I} = [0,1])$ is an $S-eqR$ over $\mathcal{X}$.*

Proof: $(\Rightarrow)$ From Theorem 11, λ_t is an equivalence relation on $\mathcal{X}, \forall t \in [0,1]$. Hence $(\theta_\lambda, \mathcal{I})$ is a $S-eqR$ over $\mathcal{X}$, by [Proposition 4.3 [8]].

$(\Leftarrow)$ Let (θ_λ, I) be a $S-eqR$ over $\mathcal{X}$. Definition 5 implies that $, \forall m \in \mathcal{X}$

$$\alpha_{\theta_\lambda}(m,m) = \mathcal{I} \Rightarrow (m,m) \in \theta_\lambda(t) = \lambda_t, \forall t \in \mathcal{I},$$

choose $t = 1$ then $\lambda(m,m) = 1 \Rightarrow \lambda$ is reflexive. Let $\lambda(m,z) = t \in \mathcal{I} \Rightarrow (m,z) \in \lambda_t = \theta_\lambda(t) \Rightarrow t \in \alpha_{\theta_\lambda}(m,z) = \alpha_{\theta_\lambda}(z,m)$. Hence $(z,m) \in \theta_\lambda(t) = \lambda_t \Rightarrow \lambda(m,z) = t \leq \lambda(z,m)$ Similarly, we get $\lambda(z,m) \leq \lambda(m,z)$ and therefore, λ is symmetric. To show that λ is transitive and consequently a $\mathbf{F-eqR}$ on $\mathcal{X}$, let $\lambda(m,y) \wedge \lambda(y,n) = t \Rightarrow \lambda(m,y) = t$ or $\lambda(y,n) = t$. Consider the case $\lambda(m,y) = t \leq \lambda(y,n)$, then $m,y),(y,n \in \lambda_t = \theta_\lambda(t) \Rightarrow t \in \alpha_{\theta_\lambda}(m,y) \cap \alpha_{\theta_\lambda}(y,n) \subseteq \alpha_{\theta_\lambda}(m,n)$. Then $(m,n) \in \theta_\lambda(t) \Rightarrow \lambda(y,n) \geq t = \lambda(m,y) \wedge \lambda(y,n)$. That is, λ is transitive.

Definition 11: [6] A pair $(\mathcal{X}, \tau)$ is called a fuzzy (**F-**)topological space, where τ is a family of fuzzy sets on $\mathcal{X}$ that satisfy certain conditions, .

(1) $0_\mathcal{X}, 1_\mathcal{X} \in \tau$ where $0_\mathcal{X}(z) = 0, 1_\mathcal{X}(z) = 1, \forall z \in \mathcal{X}$,

(2) If $\lambda_1, \lambda_2 \in \tau \Rightarrow \lambda_1 \wedge \lambda_2 \in \tau$,

(3) If $\{\lambda_i\} \in \tau \Rightarrow \vee_i \lambda_i \in \tau$,

For any fuzzy topology τ on $\mathcal{X}$, we can associate soft topology over $\mathcal{X}$ in the following manner: Let $\mathcal{I}^0$ =]0,1], and $\mathbf{F}_\tau = \{(F_\delta, \mathcal{I}): \delta \in \tau\}$ be the family of soft sets over $\mathcal{X}$ corresponding to the fuzzy topology τ, where $F_\delta: \mathcal{I} \to \mathcal{P}(\mathcal{X})$,

$$F_\delta(t) = \delta_t = \{z \in \mathcal{X}: \delta(z) \geq t\}$$

Proposition 13: $(\mathcal{X}, \mathbf{F}_\tau, \mathcal{J}^0)$ is a S-Top space.

Proof: Straightforward.

Using a **F-Rel** λ on $\mathcal{X}$, two types of fuzzy sets for every $x \in \mathcal{X}$ are introduced in [16], as follows:

$$\lambda_m(n) = \lambda(n,m), \lambda^m(n) = \lambda(m,n)$$

and the family $\mathcal{C} = \{\lambda_m, \lambda^m\}_{m\in\mathcal{X}}$ induces a **F-Top** t_λ on $\mathcal{X}$.

Theorem 14: $T_{\theta_\lambda} = \mathbf{F}_{\tau_\lambda}$.

Proof: All we need to do is demonstrate that both topologies are generated by the same sets of generators. For $m \in \mathcal{X}$, and from Definition 7, we get

$$\begin{aligned}(\theta_\lambda)_m &= \{n \in \mathcal{X}: (n,m) \in \theta_\lambda(t)\}\\ &= \{n \in \mathcal{X}: (n,m) \in \lambda(t)\}\\ &= \{n \in \mathcal{X}: \lambda(n,m) \geq t\}\\ &= F_{\lambda_m}(t).\end{aligned}$$

and

$$\begin{aligned}(\theta_\lambda)^m &= \{n \in \mathcal{X}: (m,n) \in \theta_\lambda(t)\}\\ &= \{n \in \mathcal{X}: (m,n) \in \lambda(t)\}\\ &= \{n \in \mathcal{X}: \lambda(m,n) \geq t\}\\ &= F_{\lambda^m}(t).\end{aligned}$$

6. Conclusion

In conclusion, this study presents a novel approach to understanding a soft relation $(\theta, \mathcal{A})$ over a set $\mathcal{X}$ by redefining them through the mapping $\alpha_\theta: \mathcal{X} \times \mathcal{X} \to \mathcal{P}(\mathcal{A})$. This redefinition not only clarifies the structure of soft relations but also facilitates the development of **S-eqRs**, building upon the foundational work of Feng [8]. Furthermore, we have introduced the concept of **S-Tops** generated by these **S-Rels**, revealing significant characterizations and properties that enhance our understanding of their behavior. Using these ideas in fuzzy frameworks shows how flexible and important our results are in a wider range of mathematical settings. Future research may expand on these results, exploring additional applications and implications of soft relations and topologies in the theory of rough sets.

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