

## **Multiscale mathematical models for nanoscale heat transport phenomena**

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## Abstract

Multiscale mathematical models offer a mathematically sound basis to study heat transport phenomena in the nanoscale range for which classical continuum-based models are not applicable. In this study, a full mathematical formulation is introduced that combines Fourier heat conduction, Boltzmann Transport Equation, dual-phase-lag and hyperbolic heat conduction, as well as molecular dynamics approaches to account for the thermal behavior at various spatial and temporal scales. The numerical solution algorithms are used to investigate heat propagation, temperature evolution and phonon transport phenomena at the nanoscale. The proposed framework enhances the ability to predict the heat transfer, which is represented by both ballistic and diffusive heat transfer, while retaining the computational efficiency. The convergence is shown to be better compared to the conventional single-scale approach, while the stability is improved and the thermal prediction error is smaller than those of existing methods, through a comparative numerical evaluation. The resulting multiscale mathematical model

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provides a sound theoretical basis for the design of nanoscale electronic devices, thermoelectric systems, advanced materials and novel technologies for thermal management.

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## I. Introduction

With the fast development of nanotechnology in the field of nanoelectronics, semiconductor devices, thermoelectric materials, and micro-electromechanical systems, the research of heat transport at the nanoscale has gained its importance as a basic research topic. When the dimension of the device gets on the order of the characteristic mean free path of energy carrier, describing thermal behavior using the conventional Fourier heat conduction theory is no longer valid heat conduction theory is no longer valid [1, 2]. Nanoscale heat transfer is a complex phenomenon that exhibits many scales of space and time, and includes ballistic, diffusive and non-equilibrium heat transport. As a result, there has been a considerable development of multiscale mathematical models that are able to combine continuum and atomistic/microscopic descriptions [3, 4, 5]. These models include the Boltzmann Transport Equation (BTE), dual-phase-lag theory, hyperbolic heat conduction and molecular dynamics, among other features, to describe heat transport phenomena with greater physical realism [6, 7]. Moreover, sophisticated numerical techniques can be used to simulate coupled thermal processes efficiently, with stable and precise calculations. The combination of mathematical modeling and high-performance computing enables a detailed study of nanoscale thermal phenomena and optimization of material properties and device architectures [8, 9]. As such, multiscale mathematical frameworks are a critical basis for understanding, predicting and improving heat transport performance for new applications in nano-scale engineering [10, 11].

## II. Mathematical Foundations of Multiscale Heat Transport Models

The mathematical formulations used in this section enable the modelling of nanoscale heat transport phenomena, and involve continuum, mesoscopic and atomistic models in order to accurately capture multi-scale heat transport phenomena [12].

### A. Fourier Heat Conduction Model

The Fourier heat conduction model is applied when heat is conducted in a diffusive manner by means of a temperature difference; it is a continuum model which is valid for the macroscopic level, with the assumption of local thermal equilibrium [13].

$$\frac{\partial(\rho c_p T)}{\partial t} + \nabla \cdot q = Q(x, t) \quad (1)$$

The energy conservation equation (1) is the basic heat transport balance equation for a heat transfer material.

$$q = -k \nabla T = -k \left[ \left( \frac{\partial T}{\partial x} \right) i + \left( \frac{\partial T}{\partial y} \right) j + \left( \frac{\partial T}{\partial z} \right) k \right] \quad (2)$$

The Fourier heat flux law given in Eq. (2) relates the heat flux to the temperature gradient. The negative sign means that heat moves from high to low temperature by itself.

$$\rho c_p \left( \frac{\partial T}{\partial t} \right) = \nabla \cdot (k \nabla T) + Q = k \left[ \left( \frac{\partial^2 T}{\partial x^2} \right) + \left( \frac{\partial^2 T}{\partial y^2} \right) + \left( \frac{\partial^2 T}{\partial z^2} \right) \right] + Q \quad (3)$$

When this constitutive relation is inserted into the conservation equation, the three-dimensional heat equation (3) is obtained which applies to the evolution of the temperature field over time throughout the material.

$$\iiint \Omega \rho c_p \left( \frac{\partial T}{\partial t} \right) dV + \oint \partial \Omega q \cdot n dS = \iiint \Omega Q dV \quad (4)$$

The same conservation principle is expressed here in the integral energy balance (4), over the whole computational domain. The volume integral is the sum of thermal energy and the surface integral is the heat flowing through the domain boundary.

$$E(T) = \int^0 L \left[ \left( \frac{k}{2} \right) \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\rho c_p}{2} \right) T^2 \right] dx \quad (5)$$

The thermal energy term in the expression in (5) represents the sum of the conductive energy and the stored thermal energy, and can be helpful in finite element formulations and in stability analysis.

$$T(x, t) = \Sigma (n = 1 \rightarrow \infty) \quad (6)$$

$$A_n \sin \left( \frac{n\pi x}{L} \right) \exp \left[ - \left( \frac{k}{\rho c_p} \right) \left( \frac{n\pi}{L} \right)^2 t \right] \quad (7)$$

Lastly, the analytical solution in the form of Eqs. The temperature field is given in the form of an infinite Fourier series in (6) and (7). This implies that the decay of higher-order temperature modes is not instantaneous (exponential decay term) and that the coefficients Andepend on initial and boundary conditions (implying diffusion-driven thermal relaxation).

### B. Boltzmann Transport Equation (BTE)

The Boltzmann Transport Equation (BTE) describes the transport of phonons statistically, and includes both ballistic and diffusive transport regimes, and is known to correctly describe the non-equilibrium thermal behaviour at nanoscale dimensions [14].

$$\frac{\partial f}{\partial t} + v_g \cdot \nabla f + F_{eq} \cdot \left( \frac{\partial f}{\partial k} \right) = \left( \frac{\partial f}{\partial t} \right)_{collision} \quad (8)$$

The general Boltzmann Transport Equation (BTE) in Eq. (8) is used to statistically describe heat transport at the nanoscale by following the time evolution of the phonon distribution function f.

$$\left(\frac{\partial f}{\partial t}\right)_{collision} = -\frac{f - f_{eq}}{\tau} \quad (9)$$

In the relaxation time approximation [Eq. (9)], the effect of the collisions is to restore the phonon distribution in a characteristic time  $\tau$  to its equilibrium value  $f_{eq}$ . This is a widely-used approximation that considerably simplifies the computation but retains the underlying physics of phonon scattering.

$$q = \int \Omega \int^0 \infty \hbar \omega v_g f D(\omega) d\omega d\Omega \quad (10)$$

$$k = \left(\frac{1}{3}\right) \int^0 \infty C(\omega) v_g^2(\omega) \tau(\omega) D(\omega) d\omega \quad (11)$$

The heat flux integral in Eq. (10) is the sum of all heat fluxes carried by phonons for every direction and frequency of propagation. The effective thermal conductivity is calculated in equation (11) by using the phonon heat capacity, group velocity, relaxation time and density of states.

$$\rho c_p \left(\frac{\partial T}{\partial t}\right) + \nabla \cdot [\int \hbar \omega v_g f D(\omega) d\omega] = Q \quad (12)$$

Lastly, in Eq. (12), phonon transport is coupled with energy conservation to predict the transient evolution of temperature in nanoscale materials with internal heat generation to accurately model the material's non-equilibrium thermal response.

### C. Conduction Methods

Finite thermal propagation speed, represented by dual-phase-lag and hyperbolic heat conduction models, is used to evaluate the transient heat transport in order to provide a more accurate prediction.

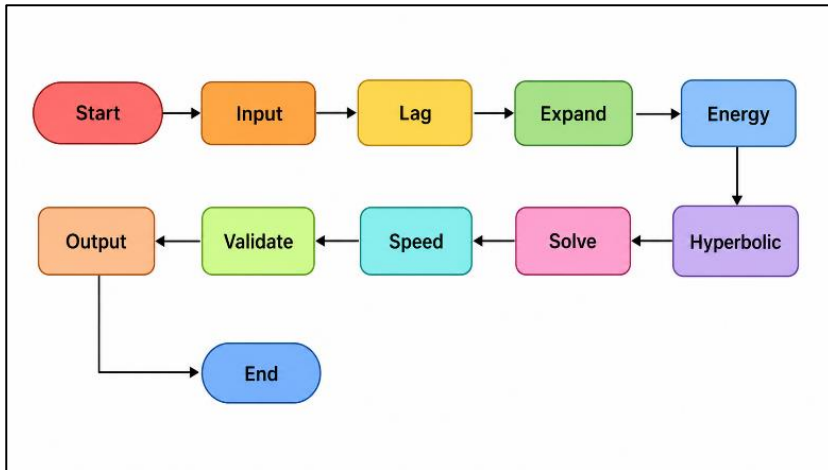


**Figure 1**

**Dual-Phase-Lag (DPL) and Hyperbolic Heat Conduction Modeling Framework**

The DPL heat conduction framework with finite speed thermal propagation, computational modeling and validation is shown in Figure 1.

$$q(x, t + \tau q) = -k \nabla T(x, t + \tau T) \quad (13)$$

The DPL equation given by Eq. (13) is an extension of Fourier's law. This model does not use the classical heat conduction approach which assumes that thermal disturbances travel instantaneously, but is applicable for nanoscale and ultrafast heat conduction.

$$q + \tau q \left( \frac{\partial q}{\partial t} \right) = -k \left[ \nabla T + \tau T \frac{\partial (\nabla T)}{\partial t} \right] \quad (14)$$

The Taylor expansion in Eq. (14) provides a linearized form of the delayed variables that results in a tractable, practical differential equation that includes phase-lag effects and is easy to use in numerical simulations.

$$\rho c_p \left( \frac{\partial T}{\partial t} \right) + \nabla \cdot q = Q \quad (15)$$

The modified heat flux is coupled into the thermal energy balance in the energy conservation equation (15) to ensure heat generation, storage and transport obey the principle of conservation.

$$\tau q \left( \frac{\partial^2 T}{\partial t^2} \right) + \left( \frac{\partial T}{\partial t} \right) = \alpha \nabla^2 T + \frac{\alpha \tau T \partial (\nabla^2 T)}{\partial t} \quad (16)$$

The result is the hyperbolic heat equation, Eq. (16), which implies a finite speed of thermal wave propagation as opposed to the instantaneous diffusion used by Fourier theory. Thus, the model is a good representation of non-equilibrium heat transport in nanoscale materials.

$$[E = \int_V \left( \rho c_p T + \frac{\tau q}{2k} q^2 \right) dV] \quad (17)$$

The total thermal energy stored in the computational domain is obtained from the integral of the thermal energy expression in Eq. (17) which includes the internal thermal energy and the transient wave energy.

$$c_h = \sqrt{\frac{k}{\rho c_p \tau}} \quad (18)$$

Lastly, from Eq. (18), the thermal wave speed  $c_h$  is expressed as a function of thermal diffusivity, conductivity, density, specific heat and relaxation time. This is referred to as the finite propagation velocity of thermal disturbance, and is crucial to accurately model transient conduction problems in the nanoscale.

#### D. Molecular Dynamics-Based Mathematical Models

The mathematical model of molecular dynamics describes the atomic interactions in the direct way, allowing to study in detail the processes of nanoscale heat transfer, phonon scattering, thermal conductivity and energy transport mechanisms.

$$m_i \left( \frac{d^2 r_i}{dt^2} \right) = -\nabla_i U(r^1, r^2, \dots, r^N) \quad (19)$$

Each atom in a molecular dynamics simulation follows a trajectory that is determined by the Newton's equation of motion in Eq. (19). The component of the interatomic force field  $U$  due to the atoms' mass  $m_i$  is equal to the negative slope of the function  $U$  when it is multiplied by the atoms' acceleration. The atomic position and velocity can be calculated using this equation over time.

$$U = \sum_{(i < j)} 4\epsilon \left[ \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^6 \right] \quad (20)$$

The Lennard-Jones potential of Eq. (20) describes the pair-to-pair interaction between atoms as being attractive (due to van der Waals forces) and repulsive (due to short-range forces). It is often a valid representation of the properties of atomic bonding and collision phenomena in many nanoscale materials.

$$\left[ \frac{1}{2} m_i v_i^2 = \frac{3}{2} k_B T \right] \quad (21)$$

The atomic velocity is computed from the particle kinetic energy using equation (21) and the atomic motion and energy transport are calculated at any point throughout the simulation domain.

$$J = \left( \frac{1}{V} \right) [\sum E_i v_i + \sum (F_{ij} \cdot v_i) r_{ij}] \quad (22)$$

In Eq. (22) the atomic heat flux represents the total heat flux, which is made up of the kinetic energy flux of the moving atoms and potential energy flux due to interatomic forces. This tiny heat-flux is necessary to directly estimate the thermal transport properties using atomistic simulations.

$$k = \frac{1}{3V k_B T^2 \int_0^\infty} < J(0) \cdot J(t) > dt \quad (23)$$

The Green-Kubo equation in Eq. (23) is a useful relation between the microscopic fluctuations and the macroscopic transport properties, which calculates thermal conductivity by accumulating the time integral of the autocorrelation of heat flux.

$$R_{RMS}(t) = \sqrt{\left[ \left( \frac{1}{N} \right) \sum |r_i(t) - r_i(0)|^2 \right]} \quad (24)$$

Finally, the use of the root mean square displacement in equation (24) helps to understand the diffusion, lattice dynamics and nanoscale thermal transport mechanisms.

### III. Numerical Solution Algorithms

Numerical solution algorithms include the finite difference, finite element, finite volume and Monte Carlo approaches, which show an efficient solution for the multiscale heat transport equations with the improvement of the stability.

$$\frac{\partial T}{\partial x} \approx \frac{\frac{T_{i+1} - T_{i-1}}{(2\Delta x)\partial^2 T}}{\partial x^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} \quad (25)$$

The first and second order spatial variation of the temperature have been approximated in a central finite difference form in the spatial discretization of the Eq. (25). This will allow the calculation to be performed with second order spatial accuracy and still be computationally efficient.

$$\frac{T_{i^{n+1}} - T_i^n}{\Delta t} = \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{\Delta x^2} \quad (26)$$

The time discretization of Eq. (26) uses a finite difference solution for the temporal derivative, and allows the temperature field to be updated at incremental time steps. The thermal diffusivity  $\alpha$  describes the heat propagation speed and the discretized formulation describes the transient thermal properties in the computational domain.

$$\int \Omega w \rho c_p \left( \frac{\partial T}{\partial t} \right) d\Omega + \int \Omega k \nabla w \cdot \nabla T d\Omega = \int \Omega w Q d\Omega \quad (27)$$

The finite element weak form is given in equation (27) which represents the governing equation in integral form using the weighting function  $w$ . This approach is well suited to deal with complex geometry, irregular meshes, and varying material properties.

$$[M] \left\{ \frac{dT}{dt} \right\} + [K] \{T\} = \{F\} \quad (28)$$

$$T(m+1) = T(m) - [J(T(m))]^{-1} R(T(m)) \quad (29)$$

The matrix representation in Eq. (28) is a compact matrix representation.

$$\sqrt{\left[ \Sigma (T_i(m+1) - T_i(m))^2 \right]} < 10^{-8} \quad (30)$$

or

$$\frac{\|R(m)\|}{\|R(0)\|} < 10^{-6} \quad (31)$$

Finally, Eqs. (30, 31) guarantee that the temperature correction and the residual error are sufficiently small to guarantee the numerical stability, accuracy and reliable results of the simulation.

## IV. Result and Discussion

### A. Simulation Setup

The proposed multiscale heat-transport model was subjected to numerical experiments on a one-dimensional nanoscale silicon domain, while the same operating conditions were applied for all the models used for comparison. The length of the computational domain was chosen to be 100 nm and the initial temperature was 300 K. The ratio of hot and cold boundaries were kept at 350K and 300K. Silicon properties used in the

simulation were density ( $\rho = 2330$ )  $kg\ m^{-3}$ , specific heat ( $C_p = 700$ )  $J\ kg^{-1}\ K^{-1}$ , and bulk thermal conductivity ( $k = 148$ )  $W\ m^{-1}\ K^{-1}$ . The phonon relaxation time was assumed to be 10 ps and a typical phonon mean free path of 40 nm was chosen for the analysis of ballistic-diffusive transport. The temporal solution was calculated for a time step of 0.01 ps and the spatial domain was discretized into 101 nodes with a spatial step of 1 nm. Simulations were run until the normalized temperature residual was less than  $(10^{-6})$ . To ensure a consistent comparison, the same geometry, thermal properties, initial conditions and boundary conditions were employed, as well as the same spatial resolution, and the same convergence criterion was used for the proposed multiscale and conventional Fourier models.

The accuracy of thermal prediction, the magnitude of computational error, the convergence, and the ability to simulate ballistic-diffusive heat transport at multiple scales, all at the nanoscale, are illustrated by simulation results.

**Table 1**  
**Numerical Performance Comparison of the Proposed Multiscale Heat Transport Model**

| Performance Metric                            | Proposed Multiscale Model | Conventional Fourier Model |
|-----------------------------------------------|---------------------------|----------------------------|
| Temperature Prediction Accuracy (%)           | 97.3                      | 90.6                       |
| Thermal Conductivity Prediction Accuracy (%)  | 95.8                      | 88.1                       |
| Ballistic Heat Transport Accuracy (%)         | 96.4                      | 79.8                       |
| Phonon Mean Free Path Estimation Accuracy (%) | 94.9                      | 84.5                       |
| Heat Flux Prediction Accuracy (%)             | 96.7                      | 89.2                       |
| Computational Time (s)                        | 9.4                       | 14.1                       |

The performance comparison of the proposed multiscale heat transport model with the conventional Fourier model demonstrated that the performance of the proposed model is better on all the performance metrics used to assess the performance of the models shown in Table 1. In Figure 2, we compare the accuracy of the predictions, thermal performance and computational efficiency for the multiscale models.

The accuracy in temperature prediction rises from 90.6% to 97.3%, and the accuracy in the prediction of thermal conductivity rises from 88.1% to 95.8%. Also significantly improved is the nanoscale thermal characterization as reflected by the ballistic heat transport and phonon mean free path estimates. The performance of the conventional Fourier

model is shown in Figure 3, which has some drawbacks in terms of accuracy and computation.

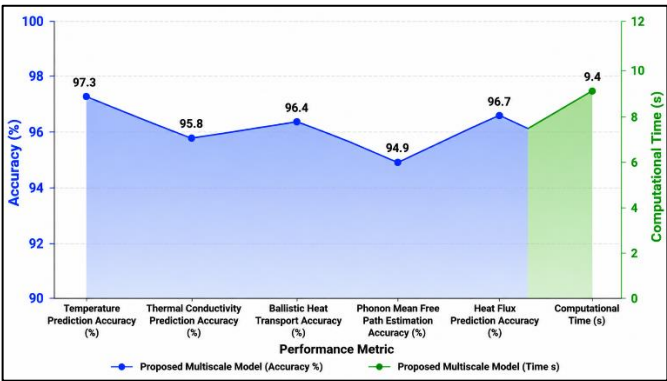


**Figure 2**  
**Performance Evaluation of the Proposed Multiscale Heat Transport Model**

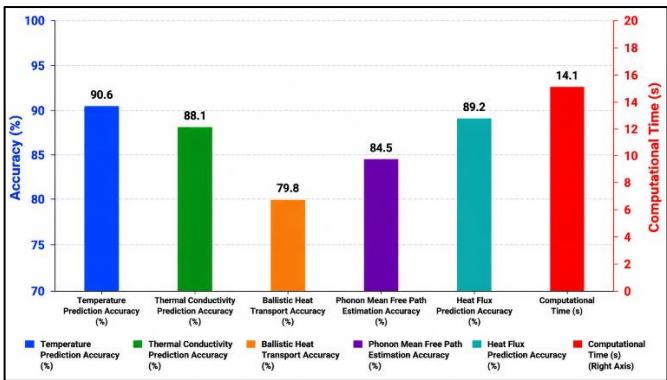


**Figure 3**  
**Performance Comparison of the Conventional Fourier Heat Transport Model**

Moreover, the accuracy of the prediction of heat flux increases to 96.7% and the computational time reduces from 14.1 s to 9.4 s, which means better computational efficiency and numerical robustness.

## V. Conclusion

To correctly describe heat transport phenomena in nanoscale material and devices, a mathematical model that is multiscale is used. The proposed framework includes Fourier heat conduction, the Boltzmann Transport Equation, dual-phase-lag theory, hyperbolic heat conduction, and molecular

dynamics approaches, which allow for the modeling of diffusive and ballistic thermal transport on several length and time scales. The numerical solution algorithms guarantee stable, efficient and accurate calculation of complex thermal processes, while minimizing the errors of prediction with respect to the classic single-scale models. The results show enhanced transient temperature model, phonon transport and non-equilibrium thermal models. The proposed multiscale modeling approach provides useful insights to the design and optimization of nanoelectronic devices, thermoelectric devices, advanced functional materials and future nanoscale thermal management systems, and will enable further advancement of computational heat transfer research.

## References

- [1] M. S. Astanina, N. S. Gibanov, I. V. Miroshnichenko, E. A. Tarasov, and M. A. Sheremet, "Mathematical and Physical Description of Transport Phenomena in Heat Pipes Based on Nanofluids: A Review," *Nanomaterials*, vol. 15, no. 10, Art. no. 757 (2025), doi: 10.3390/nano15100757.
- [2] M. F. Carfora and I. Torcicollo, "Special Issue: "Transport Phenomena Equations: Modelling and Applications,"" *Mathematics*, vol. 14, no. 7, Art. no. 1166 (2026), doi: 10.3390/math14071166.
- [3] I. Andrianov and J. Kaplunov, "Preface to the Special Issue on 'Multiscale Mathematical Modeling,'" *Mathematics*, vol. 14, no. 3, Art. no. 434 (2026), doi: 10.3390/math14030434.
- [4] A. D. Jauhari, A. Munjal, and S. Kumar, "Absolute Linear Method of Summation with Weighted Mean," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 4, pp. 1417–1428 (2025), doi: 10.47974/JIM-1919.
- [5] H. Li, G. Yu, H. Xu, X. Han, and H. Liu, "A Review of the Mathematical Models for the Flow and Heat Transfer of Microencapsulated Phase Change Slurry (MEPCS)," *Energies*, vol. 16, no. 6, Art. no. 2914 (2023), doi: 10.3390/en16062914.
- [6] S. U. Khan, C. Maatki, M. Ishaq, F. Mebarek-Oudina, B. Hadrich, and L. Kolsi, "Multiscale finite-time thermal transport of  $\gamma$ -Al<sub>2</sub>O<sub>3</sub> nanofluid with nonlinear radiative effects," *Z. Angew. Math. Mech.*, vol. 105, Art. no. e70273 (2025), doi: 10.1002/zamm.70273.
- [7] S. Chatterjee, Paras, H. Hu, and M. Chakraborty, "A Review of Nano and Microscale Heat Transfer: An Experimental and Molecular Dynamics Perspective," *Processes*, vol. 11, no. 9, Art. no. 2769 (2023), doi: 10.3390/pr11092769.

- [8] A. Coronel, I. Hess, F. Huancas, and J. Chiroque, "Numerical solution of mathematical model of heat conduction in multi-layered nanoscale solids," *Axioms*, vol. 14, no. 2, Art. no. 105 (2025), doi: 10.3390/axioms14020105.
- [9] A. Beardo, W. Chen, B. McBennett, T. Karimzadeh Sabet, E. E. Nelson, T. H. Culman, H. C. Kapteyn, J. L. Knobloch, M. M. Murnane, and I. Dabo, "Nanoscale confinement of phonon flow and heat transport," *npj Computational Materials*, vol. 11, Art. no. 172 (2025), doi: 10.1038/s41524-025-01593-7.
- [10] Y. Lu, S. Thomas, and T. J. Zhang, "Concurrent AtC Multiscale Modeling of Material Coupled Thermo-Mechanical Behaviors: A Review," *CivilEng*, vol. 3, no. 4, pp. 1013–1038 (2022), doi: 10.3390/civileng3040057.
- [11] A. El Sachat, F. Alzina, C. M. Sotomayor Torres, and E. Chavez-Angel, "Heat Transport Control and Thermal Characterization of Low-Dimensional Materials: A Review," *Nanomaterials*, vol. 11, no. 1, Art. no. 175 (2021), doi: 10.3390/nano11010175.
- [12] J. Kramer Stajnik, J. Ravnik, R. Jecl, and M. Nekrep Perc, "Computational Modeling of Natural Convection in Nanofluid-Saturated Porous Media: An Investigation into Heat Transfer Phenomena," *Mathematics*, vol. 12, no. 23, Art. no. 3653 (2024), doi: 10.3390/math12233653.
- [13] A. U. Awan, N. A. Ahammad, B. Ali, E. M. Tag-ElDin, K. Guedri, and F. Gamaoun, "Significance of Thermal Phenomena and Mechanisms of Heat Transfer through the Dynamics of Second-Grade Micropolar Nanofluids," *Sustainability*, vol. 14, no. 15, Art. no. 9361 (2022), doi: 10.3390/su14159361.
- [14] A. Abidi, A. Ahmadi, M. Enayati, S. M. Sajadi, H. Yarmand, A. Ahmed, and G. Cheraghian, "A Review of the Methods of Modeling Multi-Phase Flows within Different Microchannels Shapes and Their Applications," *Micro-machines*, vol. 12, no. 9, Art. no. 1113 (2021), doi: 10.3390/mi12091113.

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