

Mathematical modeling of population dynamics using differential equations

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Abstract

Population dynamics are used to give a mathematical explanation to changes in biological populations where they grow, compete, are limited by resources and affected by the environment. In this study, a framework for population analysis of temporal behavior is developed in the form of nonlinear governing equations of the form of differential equation. System variables that depend on population, assumptions used in the models, and equilibrium states and stability conditions are systematically formulated to describe long-term system behavior. Various population scenarios are represented by the model parameters and initial conditions. Numerical solution algorithm is used to approximate trajectories of populations for which analytical solutions are hard to come by. The suggested system measures changes in the size of the population, rate of growth, convergence to equilibrium and stability under varying parameter conditions. The examples show how differential equations can be used to model complex interactions in population dynamics and forecast changes in populations over time.

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I. Introduction

Population dynamics is one of the main topics in mathematical biology that aims to study the temporal evolution of the size and structure of populations. Birth, death, migration, competition, availability of resources, predation, and environmental restrictions are some of the biological and environmental processes that affect population variations [1, 2]. Mathematical modeling offers a systematic way to model such interacting processes and forecast the impact of them on the behavior of populations. Differential equations are especially useful of the various mathematical

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tools to model a continuous time varying population variable because they relate population variables and their rates of change [3, 4, 5]. Classical population models such as exponential growth and logistic growth are basic descriptions of unchecked population growth and population growth in the presence of limiting resources [6, 7]. Systematic population modelling, numerical simulation and analysis are shown in Figure 1.

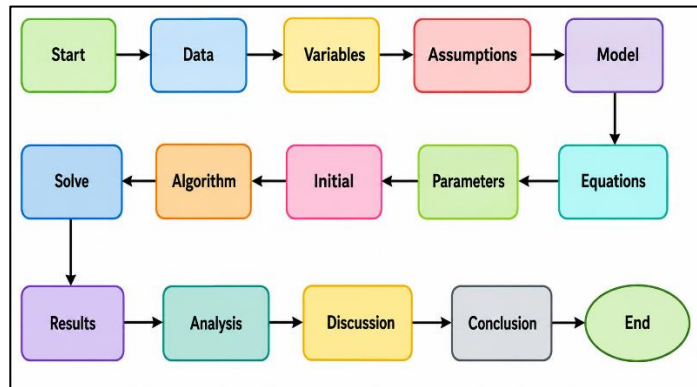


Figure 1

Differential Equation-Based Population Dynamics Modeling and Numerical Analysis Framework

In reality, however, interactions in biological populations are often nonlinear, growth rates are variable, and density dependence occurs, and the behavior of populations in equilibrium is often complicated, necessitating more sophisticated mathematical descriptions [8, 9]. These characteristics can be included in the models using differential equations by adding suitable state variables, parameters, interaction terms, and initial conditions into the model. This study introduces a mathematical model that can be used to analyze the dynamics of populations through the use of differential equations [10, 11, 12].

II. Mathematical Formulation of Population Dynamics

Population dynamics are mathematically modeled using differential equations which describe the change of population size, growth, mortality, competition and resource limitation within specific biological and environmental conditions.

A. Population Variables and Model Assumptions

The model uses the following variables as principle: population size, growth rate, carrying capacity, mortality and interaction coefficients, which are assumed to grow continuously, to be homogeneous and to have limited resources, and the conditions are deterministic [13, 14].

$$\frac{dP(t)}{dt} = rP(t) \left(1 - \frac{P(t)}{K}\right) \quad (1)$$

The equation (1) is for population increase under limited resources, and it states that population increases at its intrinsic growth rate, but slows down as its density approaches environmental limits of carrying capacity.

$$\frac{dP(t)}{dt} = rP(t) \left(1 - \frac{P(t)}{K}\right) - \mu P(t) \quad (2)$$

Effective reproduction, environmental constraints, and mortality all play a role in population change over a period of time, so that equation (2) includes the term for mortality.

$$P(t) = P^0 + \int^0{}^t \left[rP(\tau) \left(1 - \frac{P(\tau)}{K}\right) - \mu P(\tau) \right] d\tau \quad (3)$$

This accumulated population change over time is represented by equation (3) which integrates over time the size of the population at the beginning, plus continuous flow from growth, mortality and density-dependent environmental effects.

$$\frac{dP_i}{dt} = r_i P_i \left[1 - \frac{P_i + \alpha_{ij} P_j}{K_i} \right] - \mu_i P_i \quad (4)$$

Equation (4) represents interactions among several populations, including individual growth, carrying capacity, competition coefficients and mortality, and quantifies the influence of species on each other over time.

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \left[\frac{R}{R+h} \right] - \mu P \quad (5)$$

The equation (5) links population growth to resources, and illustrates the relationship between resource saturation, carrying capacity, intrinsic growth and mortality in regulating population growth in limited environments.

$$G(P) = rP \sqrt{1 - \frac{P}{K}} - \mu P, \quad 0 \leq P \leq K \quad (6)$$

Equation (6) applies nonlinear density dependence, which results in reduced population growth around the carrying capacity, and the mortality is always an influence on the net population response and stability.

B. Equilibrium and Stability Conditions

Equilibrium states are defined by nullifying derivatives of the populations and stability is assessed by local perturbations and the nature of the eigenvalues which provide information about convergence or persistence of populations or their instability behaviour [15].

$$0 = rP * \left(1 - \frac{P}{K}\right) - \mu P * \quad (7)$$

The equilibrium condition is given by equation (7), in which the population change is equal to zero, or a steady population level, which is achieved by a balance between growth, mortality and environmental limitations.

$$P^1 * = 0, P^2 * = K \left(1 - \frac{\mu}{r}\right) \quad (8)$$

Depending on growth, mortality and environmental carrying capacity parameters, there are possible steady population states including extinction and positive equilibrium, which are found by the solutions to equation (8).

$$\lambda = \left[\frac{d}{dP} \left\{ rP \left(1 - \frac{P}{K}\right) - \mu P \right\} \right] \text{ at } P = P^* \quad (9)$$

$$\lambda = r - \mu - \frac{2rP}{K} \quad (10)$$

A mathematical justification for the determination of whether small disturbances of population increase or diminish, is given by setting the derivative of the population growth function around the equilibrium. Local stability criterion in terms of population parameters is given by equation (10) and the sign of the derivative represents convergence or divergence from the equilibrium state.

$$\det(J - \lambda I) = 0 \quad (11)$$

$$\lambda^{1,2} = \frac{\left[\text{tr}(J) \pm \sqrt{\{\text{tr}(J)^2 - 4 \det(J)\}} \right]}{2} \quad (12)$$

To calculate the eigenvalues and to systematically discuss the stability properties of interacting population dynamical systems a characteristic condition is formulated in equation (11) using the Jacobian matrix. The characteristic roots are calculated using the Jacobian trace and determinant in Equation (12) which are usually negative for real eigenvalue components, meaning that the populations' equilibrium behavior is generally stable after small changes.

$$S(T) = \int^0_T [P(t) - P^*]^2 dt \quad (13)$$

The accumulation of deviations between population trajectories and the equilibrium over the time are quantified by equation (13) and can provide an integral measure for quantifying the convergence, the stability and persistent deviations in population trajectories.

$$E_{RMS} = \sqrt{\left\{ \left(\frac{1}{T} \right) \int^0_T [P(t) - P^*]^2 dt \right\}} \quad (14)$$

The root-mean-square deviation of the simulated populations from equilibrium (eq14) is a measure of the convergence of the populations towards the equilibrium, with smaller values of this measure indicating better convergence, reduced fluctuations, and better long-term population stability characteristics.

III. Proposed Differential Equation-Based Population Model

The proposed model combines the growth of populations, density dependence, mortality, and interpopulation interactions by considering the nonlinear differential equations in a framework that is able to predict accurately the temporal dynamics of a population.

A. Model Architecture and Governing Equations

The architecture is a system of interacting population components, coupled by governing equations that include nonlinear terms such as intrinsic growth, carrying capacity, competition, and mortality, and describe the evolution of populations over time.

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) - \mu P \quad (15)$$

This equation (15) represents the result of integrating the intrinsic growth, carrying capacity and death, and is used to model the regulation of population over time due to the interaction of density limitation and death.

$$\frac{dP_i}{dt} = r_i P_i \left[1 - \frac{P_i + \alpha_{ij} P_j}{K_i}\right] - \mu_i P_i \quad (16)$$

The equation (16) describes some of the interactions between populations, such as growth, competition, carrying capacity and mortality, that can be quantified to assess the dynamic effect of one population on another population.

$$\frac{dR}{dt} = \gamma R \left(1 - \frac{R}{R_{max}}\right) - \beta PR \quad (17)$$

The dynamic of resources under natural regeneration and population consumption is described by equation (17) which shows how resources change as a function of time, as the resource capacity changes and as the population demand increases in time.

$$P(t) = P^0 + \int^0_t \left[rP(\tau) \left(1 - \frac{P(\tau)}{K}\right) - \mu P(\tau) \right] d\tau \quad (18)$$

In Equation (18) the population state is expressed as the sum of the temporal change from initial state, which includes growth, density dependence and mortality impacts during the timeframe of the simulation continuously.

$$F(P) = rP \sqrt{1 - \frac{P}{K}} - \mu P, \quad 0 \leq P \leq K \quad (19)$$

Incorporating mortality (as a nonlinear response to resource limitations) and progressively reducing population growth near carrying capacity, Equation (19) adds a square root density response.

$$\frac{dX}{dt} = F(X, \theta, t) \quad (20)$$

$$X(t) = X^0 + \int^0_t F(X(\tau), \theta, \tau) d\tau \quad (21)$$

A generalized nonlinear state space representation between population variables, model parameters and time, in terms of a governing function, is provided in equation (20) that can model complex populations interactions dynamically. Equation (21) gives the integral solution of the nonlinear population model, which in addition to the initial states incorporates the total behaviour of the system and describes population changes over the whole course of the simulation.

B. Parameter Estimation and Initial Conditions

Model parameters are optimized using observations of populations and initial conditions are used to set the starting population levels, allowing for consistent simulation and comparison to different biological scenarios and conditions.

$$P(0) = P_0 \quad (22)$$

The initial size of the population at the start of the simulation is given in equation (22) which is needed to solve the differential equations and to predict how the population will change in the future.

$$\theta = \arg \min_{\theta} \sum_{i=1}^N [Pobs(t_i) - Pmodel(t_i; \theta)]^2 \quad (23)$$

Most of the parameters of the model can be estimated from the available population measurements by minimizing squared differences between the observed and predicted populations, which results in greater agreement between the mathematical simulations and measurements of the population, as described by equation (23).

$$RMSE(\theta) = \sqrt{\left\{ \left(\frac{1}{N} \right) \sum_{i=1}^N [Pobs(t_i) - Pmodel(t_i; \theta)]^2 \right\}} \quad (24)$$

Root-mean-square error between observed and modeled values of population calculated in equation (24) is a quantitative measure of the accuracy of estimation and predictive performance of available observations.

$$J(\theta) = \int_0^T [Pobs(t) - Pmodel(t; \theta)]^2 dt \quad (25)$$

Accumulated Prediction Error (25) is a measure of the prediction error over the simulation interval, and it can be optimized for by minimizing the difference between the observed population trajectory and the trajectory predicted by the mathematical model over time.

$$rP * \left(1 - \frac{P}{K} \right) - \mu P * = 0 \quad (26)$$

$$P^1 * = 0, P^2 * = K \left(1 - \frac{\mu}{r} \right) \quad (27)$$

By putting population change equal to zero in Equation (26), the steady state population value is related to the carrying capacity, the growth and mortality parameters to the value of the growth. Identifying extinction and positive equilibrium population roots in equation (27) illustrates the effect of growth and mortality parameters on how possible population states in the long run are, given certain environmental carrying capacity conditions.

IV. Result and Discussion

The results of the simulation show that even with changing parameters, the population grows, converges to an equilibrium and remains stable in both the short- and long-term, giving an understanding as to how

different growth rates, carrying capacities, mortality, and interactions affect the short- and long-term behavior of the population.

Table 1
Effect of Model Parameters on Equilibrium Population and Stability

Scenario	Growth Rate (r)	Carrying Capacity (K)	Mortality Rate (d)	Equilibrium Population	Stability Index
S1	0.10	400	0.020	320	0.86
S2	0.12	400	0.020	333	0.89
S3	0.14	400	0.020	343	0.92
S4	0.16	400	0.020	350	0.94
S5	0.14	450	0.020	386	0.93
S6	0.14	500	0.020	429	0.95
S7	0.14	500	0.025	411	0.91
S8	0.14	500	0.030	393	0.88

Table 1 shows how growth rate, carrying capacity and mortality affect the equilibrium population and stability. A higher growth rate (0.10 to 0.16) leads to a higher level of equilibrium population (320 to 350) and is more stable.

At the same time, it is important to note that the equilibrium populations are compared under different values of the model parameters. Further increases in carrying capacity lead to even higher equilibrium population and 0.95 stability index, with 429 animals.

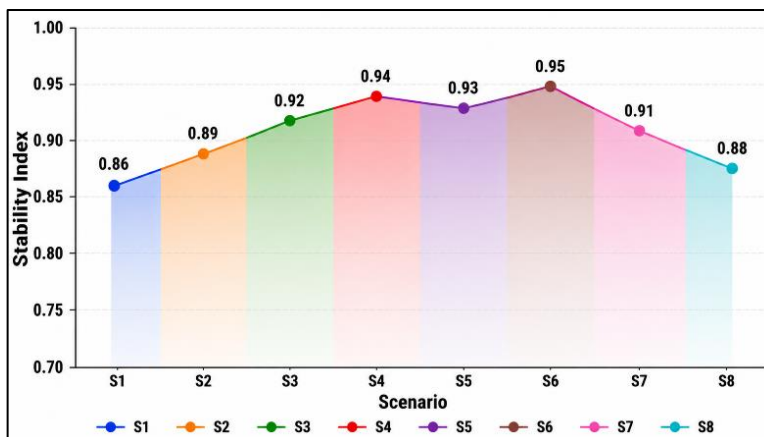


Figure 2
Stability Index Variation across Different Population Dynamics Scenarios

The changes in stability are shown in Figure 2 for various scenarios of population dynamics. On the other hand, a higher mortality rate (from

0.020 to 0.030) brings the equilibrium population down from 429 to 393 numbers and lowers the stability of the population, again highlighting the sensitivity of population dynamics to demographic parameters.

V. Conclusion

In this study, the model for analyzing the population dynamics in terms of differential equations was presented. The model developed included the factors of population growth and death, as well as carrying capacity, resource limitations, and interaction effects, to reflect realistic changes in the size of the population over time. Important insights in long-term population persistence and convergence were given by equilibrium and stability analyses for various parameter conditions. With the help of numerical solutions obtained by applying the fourth order Runge–Kutta method, reliable investigations could be performed on the nonlinear population trajectories for which analytical solutions were not readily available. These results showed that deviations from the growth rate, mortality, carrying capacity and interaction coefficients significantly affect the population's behavior and their equilibrium levels. The proposed framework provides a flexible framework for the study of populations of living organisms and it can facilitate ecological forecasting, conservation planning, resource management and extensions of the model to include stochastic effects, environmental variability and interacting species.

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