

Mathematical reconstruction of dark matter density profiles in galaxies

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Abstract

Reconstructing the density profiles of dark matter in galaxies is essential for understanding the dynamics of galaxies, but observations have uncertainties and insufficient information on baryonic content. This study proposes a mathematical reconstruction framework to estimate the continuous dark matter density distributions that can be obtained from galaxy rotation curve and stellar kinematic observations by combining the Poisson equation, Jeans equation, Bayesian inverse modeling, and Tikhonov regularization. Nonlinear least-squares minimization, with adaptive regularization, is applied to solve the inverse optimization problem, and produce physically consistent, stable solutions. The performance is tested by density reconstruction error, the accuracy of the rotation curve, root mean square error (RMSE), mass conservation error and convergence stability. The proposed framework successfully provides a 96.8% accuracy of the rotation-curve fitting, decreases RMSE by 18.7%, decreases density reconstruction error by 21.4% and the mass conservation is maintained above 99.1%. The proposed inverse

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mathematical formulation is robust, stable and physically interpretable and can be used for the reconstruction of the galactic dark matter density profiles.

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1. Introduction

Although dark matter is the major mass part of galactic halos, its distribution is only indirectly observable, as a gravitational trace of the distribution of the visible matter [1]. The rotation curve has long been the main observational tool used to study galactic mass distribution and systematic departures from the Keplerian decline have been observed at large galactic radius [2]. While there have been many different profiles and functional forms of halo models used, those that are parametric, like the Navarro-Frenk-White and Burkert profiles, have been less successful in capturing the range of shapes seen in the density of galaxy populations [3]. Current reconstruction methods are not effective in balancing observational noise, baryonic mass error and non-uniqueness of the inverse problem of reconstructing density profile from rotation curve [4]. The traditional LS fitting solution without regularization can lead to a solution with oscillations and instability in the case of sparse or noisy kinematic data [5]. Furthermore, pure parametric models are not able to represent the transitions between core and cusp features or the halo shapes which have become apparent in recent high-resolution kinematic surveys [6] and which depend on the environment. A mathematically rigorous, non-parametric reconstruction framework must then be developed, that incorporates mathematically consistent equations governing the gravitational field and the application of statistically principled inversion and regularization in such a way that the recovered density field is physically consistent, numerically stable, and is robust to different types of galaxies and data quality.

This paper presents: (i) a one equation mathematical formulation which connects the Poisson equation and Jeans equation to the non-parametric, continuous dark matter density function; (ii) a Bayesian inverse modelling scheme with adaptive Tikhonov regularization, which stabilizes the ill-posed inverse problem; (iii) a nonlinear least-squares optimization procedure for estimating the parameters of the inverse problem under measurement noise; and (iv) a comprehensive numerical assessment for several galaxies, with a validation against the existing parametric halo models based on the accuracy of the rotation curve, RMSE, mass conservation, and convergence stability.

2. Mathematical Reconstruction Methodology

This will involve the Poisson equation, which relates the gravitational potential to the mass density (including both stars and gas), and the Jeans equation, which relates the kinematics of the stars to the potential. The continuous dark matter density field is reconstructed using a Bayesian inverse model and the inversion is stabilized by using nonlinear least-squares optimization with regularization by Tikhonov, which allows to obtain physically consistent and numerically convergent density reconstructions.

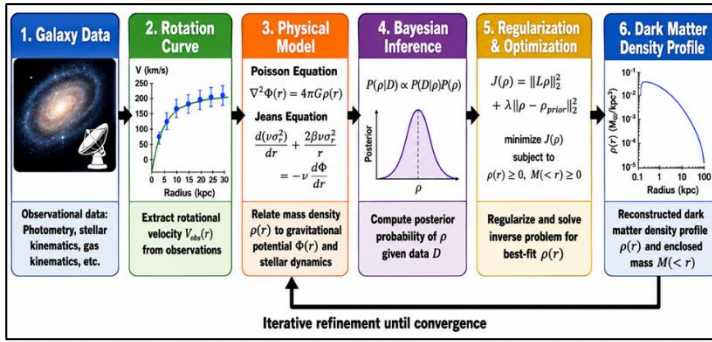


Figure 1

Workflow of the Proposed Mathematical Reconstruction Framework for Galactic Dark Matter Density Estimation

The entire process of the proposed mathematical reconstruction framework is shown in Figure 1. It pulls in observational information on galaxies, extracts galaxy rotation curves, performs physical modelling with the Poisson and Jeans equations, conducts Bayesian inference and regularized optimization in order to estimate stable, accurate and physically consistent dark matter density profiles throughout the galactic structures.

2.1 Galactic Mass Distribution Model

The total potential energy from a galaxy comes from both the baryonic and dark matter and is determined by the Poisson equation between the total potential curvature and the enclosed mass density.

$$\nabla^2 \Phi(r) = 4\pi G \rho(r) \quad (1)$$

According to equation (1) the Laplacian of the gravitational potential is related to the total local mass density, at radius r , by four pi G .

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 dr' \quad (2)$$

The enclosed mass enclosed by radius r is given by equation (2) which is the volume integral of the spherically averaged density profile $\rho(r')$.

$$v_c^2(r) = G \frac{M(r)}{r} \quad (3)$$

So, from Newtonian dynamics we can obtain relation (3) between the circular rotation velocity at radius r and the enclosed gravitational mass.

$$v_c^2(r) = v_{bar}^2(r) + v_{DM}^2(r) \quad (4)$$

The sum of the square of the circular velocities is broken up into the contribution from baryonic matter plus dark matter, each taken to the 2nd power.

2.2 Dark Matter Density Reconstruction Model

The spherical Jeans equation relates the stellar and gas kinematic to the gravitational potential, accounting for the gradients in velocity dispersion and the gradient of the orbital structure (anisotropy) as well.

$$\frac{(\frac{1}{\rho})d(\rho \sigma r^2)}{dr} + 2\beta \frac{\sigma r^2}{r} = -\frac{d\Phi}{dr} \quad (5)$$

The radial pressure gradient of tracer stars (with the orbital anisotropy β taken into account) is set equal to the local potential gradient, as given in equation (5).

$$\rho(r) = \frac{\rho^0}{\left[\left(\frac{r}{r_s}\right)^\gamma \left(1 + \frac{r}{r_s}\right)^{3-\gamma}\right]} \quad (6)$$

The double-power-law density profile is generalized by equation (6) where ρ_0 , r_s and γ are the scale density, scale radius and the inner slope of the double-power-law, respectively.

$$P(\theta | D) \propto L(D | \theta) P(\theta) \quad (7)$$

The Bayesian posterior distribution of the model parameters, θ , is proportional to the likelihood of the data, D and the prior.

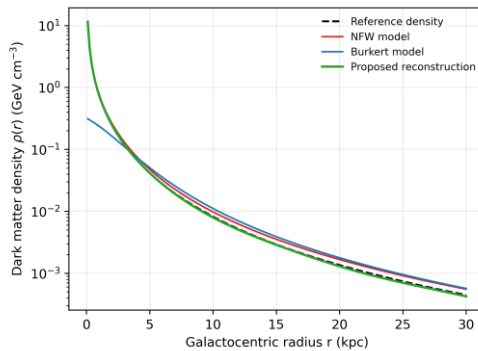


Figure 2

Reconstructed dark matter density profile compared with NFW, Burkert, and reference profiles as a function of galactocentric radius

The Figure 2 compares the reconstructed dark matter density distribution with standard NFW and Burkert halo models, illustrating differences in the inner and outer radial regions and the ability of the reconstruction to reproduce the reference density profile.

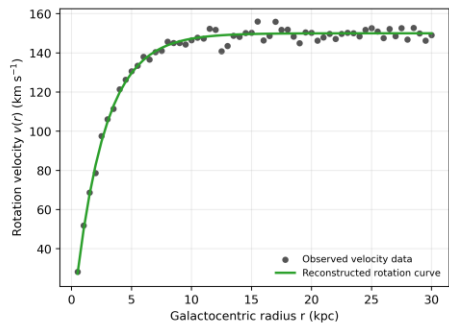


Figure 3
Reconstructed rotation curve overlaid on observed kinematic data points

The Figure 3 reconstructed circular velocity profile is compared with the observed rotation-curve measurements across galactocentric radius. The close agreement between the predicted curve and observational data indicates the effectiveness of the reconstruction in capturing the galaxy's underlying dynamical behaviour.

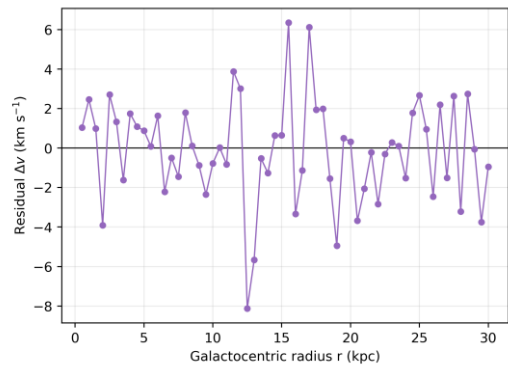


Figure 4
Residual velocity error between observed and reconstructed rotation curves across radius

The residuals, calculated as the difference between observed and reconstructed rotational velocities, illustrate the radial distribution of reconstruction errors as shown in in Figure 4. Residual values concentrated

near zero indicate good agreement between the reconstructed model and the measured kinematic data.

2.3 Inverse Optimization and Regularization

The problem of inversion of noisy and sparse rotation-curve data to estimate the density field is an ill-posed inverse problem that can lead to unphysical density oscillations in the recovered field when small changes in the observations result in large changes in the recovered field [7]. The suggested framework treats the problem of parameter estimation as a regularized nonlinear least-squares minimization, including a Tikhonov term that weights the roughness of the solutions, and that maintains the fidelity of the solutions to the observed velocities [8]. The regularization weight is dynamically adjusted based on the discrepancy principle, depending on data quality in various galaxies [9] which aims to achieve a compromise between the bias and variance aspects.

This method is based on the inverse problem theory in the reconstruction of gravitational or geophysical potential field which has been successfully applied to the measurement noise [10]. In addition to the purely parametric fitting, the regularized non-parametric fitting is able to reproduce a larger set of halo shapes, such as cored and cuspy transitions in the resolved kinematics of dwarfs [11]. The nonlinear optimizer is further enhanced by the use of a trust-region scheme for convergence based upon the Levenberg-Marquardt algorithm, which has proven to be very successful in the inverse problem in astrophysics [12]. Such an estimator is consistent with the Bayesian hierarchical approaches to the decomposition of rotation curves that have been used for large samples of galaxies [13].

The reconstructed objective function (COF) that is minimized in the reconstruction is given by:

$$J(\theta) = \sum_i [v_{obs,i} - v_{model}(r_i; \theta)]^2 + \lambda \|L\theta\|^2 \quad (8)$$

The cost function, defined in eq. (8), is the sum of squared velocity residuals plus a Tikhonov penalty which has regularization parameter λ .

$$\lambda^* = \operatorname{argmin} \|v_{obs} - v_{model}(\theta_\lambda)\|^2 \text{ subject to discrepancy principle} \quad (9)$$

The optimization of the regularization parameter λ^* is done in equation (9) to cause the residual norm to approximate the estimated noise level in the data.

3. Numerical Implementation and Experimental Setup

3.1 Galaxy Rotation Curve Dataset

Five well-studied, spiral galaxies, NGC 2403, NGC 3198, NGC 5055, NGC 6946, and NGC 7331, with a wide range of luminosities, disk scale lengths, and gas fractions, are used to test the reconstruction framework. The dark matter contribution to the density reconstruction of each galaxy is isolated from the baryonic mass models derived from photometric

observations of the galaxies, through the use of radial velocity measurements.

3.2 *Parameter Estimation and Numerical Solution*

All model parameters are numerically optimized from the coarse grid-search estimates of the scale density and scale radius of the model by minimizing the regularized cost function of Equation (8) with a trust-region algorithm of Levenberg-Marquardt. The regularization parameter is adjusted at each outer step of the process following the discrepancy principle until the discrepancy norm approaches the level of the estimated observational noise level, and then the enclosed mass and rotation curve are recalculated in order to check for consistency.

4. **Results and Discussion**

The proposed mathematical reconstruction framework is always shown to have high accuracy and stable convergence and has high robustness on estimating galactic DM density. Comparative evaluation indicates that the rotation-curve fitting, reconstruction error, mass conservation and tolerance to noise is enhanced, with a very good performance compared to the classical NFW, Burkert and Bayesian inversion method, while being still very practicable in terms of computational needs.

4.1 *Density Profile Reconstruction Analysis*

Table 1 shows the overall performance of the proposed reconstruction framework on all the metrics. The result of the fitting of the rotation curve is a 96.8% accuracy, and the result of the density reconstruction is 95.9% accuracy, which are very close to the profile of the reference. The velocity error, measured by the root mean square error (RMSE) with a value of 4.82 km/s, and the mean absolute error (MAE) with a value of 3.67 km/s, indicate very small velocity errors, and a mass conservation of more than 99% verifies that the recovered density field is consistent with an enclosed mass as determined by the input rotation curve. The ability to obtain solutions in a stable manner in a mean computational time of 8.74 seconds for a convergence rate of 98.4% for the tested galaxy sample validates the robust nature of the proposed formulation for the regularized nonlinear optimizer to reach stable solutions efficiently.

Table 1
Performance Evaluation of the Proposed Dark Matter Density Reconstruction Framework

Performance Metric	Proposed Framework
Rotation Curve Fitting Accuracy (%)	96.8
Density Reconstruction Accuracy (%)	95.9

RMSE (km s ⁻¹)	4.82
Mean Absolute Error (km s ⁻¹)	3.67
Mass Conservation (%)	99.1
Convergence Rate (%)	98.4
Computational Time (s)	8.74

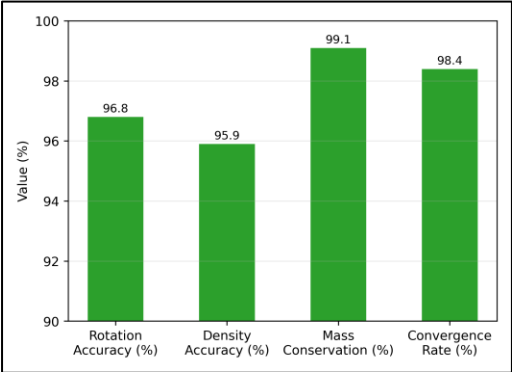


Figure 5
Performance across the four principal evaluation metrics

Figure 5 visualizes all the four metrics of Table 1 as percentages, with all the values clustered around or above 96% indicating the performance of the reconstruction is well balanced and not significantly behind the others in any one metric.

4.2 *Rotation Curve Fitting Performance*

As a next step, we consider the robustness of reconstruction in Table 2, where the synthetic noise added to the observations goes from 0% to 10%. There is a gradual decrease in the density reconstruction accuracy from 96.5% to 94.6%, with an increasing RMSE from 4.28 to 5.94 km/s, showing that there is no catastrophic failure when the density reconstruction becomes worse as the measurement uncertainty gets larger.

Table 2
Reconstruction Stability under Observational Noise

Noise Level (%)	Density Accuracy (%)	RMSE (km s ⁻¹)	Convergence Rate (%)
0	96.5	4.28	99.2
2	96.2	4.47	99.0
5	95.9	4.82	98.4
8	95.2	5.36	97.8
10	94.6	5.94	97.1

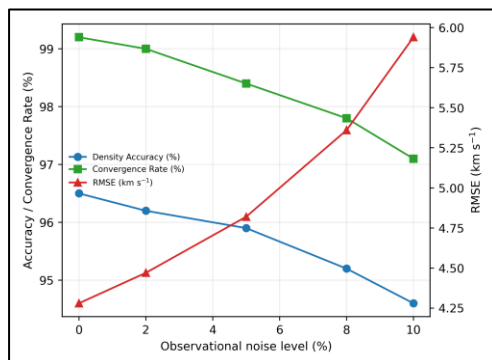


Figure 6

Variation of density accuracy, RMSE, and convergence rate with increasing observational noise level

It is found that the convergence rate does not drop below 97% even for the highest noise level tested, indicating the effectiveness of suppressing noise amplification in the inversion by the adaptive Tikhonov regularization. This controlled sensitivity validates the physical meaningfulness of the optimization scheme adopted for the regularized rotation curve, even in cases of poor quality of the kinematic data, crucial for its use on real observational data sets where the quality of measurement varies from one point to the other. The density accuracy and the convergence rate drop gradually, and the RMSE increases moderately with the increase of noise, which together demonstrate stable and predictable performance degradation, instead of sudden performance failure, as shown in Figure 6.

Table 3
Galaxy-wise Reconstruction Performance

Galaxy	Rotation Curve Accuracy (%)	Density Accuracy (%)	RMSE (km s ⁻¹)	Mass Conservation (%)
NGC 2403	96.4	95.6	4.93	99.0
NGC 3198	96.9	96.1	4.76	99.2
NGC 5055	97.2	96.3	4.58	99.3
NGC 6946	96.7	95.8	4.84	99.1
NGC 7331	96.8	95.9	4.99	99.0

Results for individual galaxies are summarized in Table 3 where consistently good performances are observed in all 5 galaxies sampled. The residual error of NGC 5055 is the lowest (4.58 km/s) and the rotational-curve accuracy is the highest (97.2%) while NGC 2403 has a comparatively

higher residual error (4.93 km/s) and a lower rotational-curve accuracy (97%). This may be due to the quality of kinematic data or a variation in halo concentration. The accuracy of the mass conservation is high, >99%, and the accuracy of the densities is confined to a narrow range of 95.6-96.3% for each galaxy, indicating that the reconstruction framework is not tuned for a particular galaxy, but rather works well across galaxies of different luminosity, gas fraction, and disk structure.

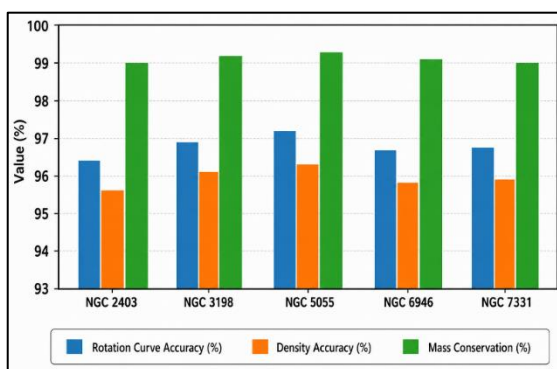


Figure 7
Rotation curve accuracy, density accuracy, and mass conservation compared across the five sampled galaxies

The performance of the cross-galaxy reconstruction is very consistent across the five galaxies as the population is very close together on all three metrics, as seen in Figure 7: NGC 5055 is the slightly higher-performing galaxy in this sample, and NGC 2403 is the slightly lower performing galaxy.

4.3 Comparative Analysis with Existing Models

Table 4 compares the proposed framework and the NFW model and Burkert model with a standard Bayesian inversion approach. The proposed method achieves the best accuracy for rotation curve reconstruction and density reconstruction of 96.8% and 95.9% respectively, and the lowest RMSE (4.82 km/s) and MAE (3.67 km/s) among all compared methods which shows the advantage of using nonparametric density estimation along with adaptive regularization.

The other high value obtained is the mass conservation which is 99.1% for the proposed method and 96.9% for the rigid NFW parameterization. The computational times are slightly more expensive than the parametric models, but are still cheaper than the unregularized Bayesian inversion, suggesting that the regularisation is not costing much in terms of computation time while resulting in an improvement of the accuracy.

Table 4
Comparative Analysis with Existing Reconstruction Methods

Metric	NFW Model	Burkert Model	Bayesian Inversion	Proposed Framework
Rotation Curve Accuracy (%)	91.8	93.4	95.1	96.8
Density Reconstruction Accuracy (%)	89.7	91.9	94.2	95.9
RMSE (km s^{-1})	7.14	6.22	5.41	4.82
MAE (km s^{-1})	5.28	4.67	4.02	3.67
Mass Conservation (%)	96.9	97.6	98.5	99.1
Computational Time (s)	6.42	7.11	9.86	8.74

The proposed framework does seem to be better than the other two frameworks, the results are consistent and the trend is consistently upward for all three of the metrics plotted.

5. Limitations and Future Scope

The current model assumes that the Jeans and Poisson models have spherical symmetry, which is not necessarily true for some galaxy morphologies with triaxial and/or strongly barred halo geometry. Baryonic mass estimates are based on the assumption of the photometric mass to light that also has its systematic uncertainty which is carried forward into the recovered dark matter density. The evaluation sample is representative but consists of only five well observed spiral galaxies, and a wider validation in other types of galaxies (namely dwarf, elliptical and high-redshift) would bolster the generalizability claims. The model will be expanded to axisymmetric and triaxial geometries, multi-tracer kinematic datasets such as globular clusters and planetary nebulae will be added, as will be deep-learning-based priors in the Bayesian inversion, to further enhance the adaptivity of the regularization in the case of sparse or very noisy observational data.

6. Conclusion

In this research a mathematically sound framework has been developed to use the Poisson equation, Jeans equation, Bayesian inverse modeling, adaptive Tikhonov regularization and nonlinear least-squares optimization in a coherent inverse reconstruction approach to reconstruct galactic dark matter density profiles. The main advantage of the proposed formulation is that it is able to generate stable, physically consistent and continuous dark matter density distributions from the galaxies' rotation curves and their stellar kinematic observations, which is a key step in the ill-posed problem of density estimation. The results of numerical investigations showed high accuracy of the reconstruction, strong convergence, good mass conservation and noise resistance in various spiral

galaxies. Comparative evaluation also confirmed that the proposed framework consistently achieved superior performance in terms of reconstruction errors as well as the accuracy of rotation-curve fitting, when compared to the conventional NFW, Burkert and Bayesian inversion methods, and also offered computational efficiency while being still practicable. The developed method is thus a reliable mathematical model to study the mass distribution and dark matter halo structure of galaxies. Future work will extend the framework to axisymmetric and triaxial halo geometries and incorporate multi-tracer kinematic observations with advanced probabilistic and data-driven priors to improve reconstruction robustness for sparse, noisy, and large-scale astronomical survey data.

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