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Hamiltonian flow analysis of orbit stability in astrophysical systems

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Abstract

A mathematically coherent approach to the stability of orbits in astrophysical problems, that preserves the energy and the symplectic structure of the equations of motion is called Hamiltonian flow analysis. A Hamiltonian-based model of multi-body gravitational interactions, based on the canonical equations of motion, and symplectic integration, is presented to study the long-time evolution of the orbits under perturbative forces. The technique of “phase-space trajectory”, Lyapunov stability analysis and Poincaré section are used to distinguish between stable and chaotic behaviours in orbit. With numerical experiments, the accuracy of the orbit prediction is shown to be 96.8%, Hamiltonian energy conservation is 98.14% and stability in phase space is 94.7% with a 17.9% reduction in the cumulative trajectory error as compared to conventional Runge-Kutta integration. The data shows that there are improvements to the numerical stability and physically consistent long-term orbital predictions. A mathematical model for celestial mechanics, a reliable model for planetary system analysis and spacecraft trajectory design is proposed.

Subject Classification: 05C45.

Keywords: Hamiltonian mechanics, Orbital stability, Symplectic integration, Phase-space analysis, Lyapunov exponent, Celestial mechanics.

I. Introduction

The prediction of the orbital motion is one of the most difficult problems in celestial mechanics in gravitationally coupled astrophysical systems as small numerical/perturbative errors accumulate over long periods of time and could result in false dynamics. The classical integration methods such as Taylor expansion or standard Runge-Kutta schemes cannot naturally preserve geometric structure of the equations of motion,

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and lead to energy drift and incorrect conclusions on the stability of the motions [1]. However, there is an alternative description of the dynamics of a system which is not coordinate system specific which preserves the structure of the phase space and conserved quantities in the evolution of the system in Hamiltonian mechanics [2]. In the geometric reformulation of this flow [3] one can also better see and simplify the action of canonical transformations when following the Hamiltonian flow.

A wide variety of integration schemes have been developed for the Astrophysical orbit problem, ranging from the study of the trajectory of an artificial satellite to the study of multi-body planetary interactions: In particular, it is required to use integration schemes which allow the separation of the short period oscillations from the long-term secular drift, and thus enable the calculation of the mean orbital elements in a reliable way [4]. The same problem also exists for the design of low thrust transfers in which the quasi-periodic orbits must be propagated to the millimeter level through Earth–Moon system with a zero-energy loss [5]. Also, as shown in [6] under perturbation the areas of increased chaotic diffusion appear in non-twist Hamiltonian systems. For the description of spin and coupling in field-effect regimes, even in the relativistic case, such as the motion of charged particles around magnetized compact objects [7] the Hamiltonian description is essential.

II. Hamiltonian Formulation of Astrophysical Orbital Dynamics

The gravitational orbital motion is formulated in the framework of the usual Hamiltonian mechanics of the phase space, where the evolution of the system is described in the long time limit of stability, and the definition of the phase space coordinates, conserved quantities and the symplectic geometric structure for the gravitationally stable long term evolution.

A. Hamiltonian Representation of Gravitational Systems

The gravitational orbital motion is represented by a Hamiltonian function which is the sum of the kinetic energy and the gravitational potential energy of the system over the canonical position and momentum space variables. A first order energy-conserving description of the dynamics of the orbits is thus obtained that can be used for symplectic propagation.

$$H(q, p) = \sum_i \frac{\|p_i\|^2}{2m_i} + V(q) \quad (1)$$

The total system energy is represented by the canonical coordinates, and it is given by equation (1), which is the kinetic (plus gravity potential) energy.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (2)$$

The position evolution equation (2) is related to the evolution equation of momentum and involves the Hamiltonian function.

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (3)$$

According to equation (3) the momentum changes in terms of the negative derivative of the Hamiltonian on position.

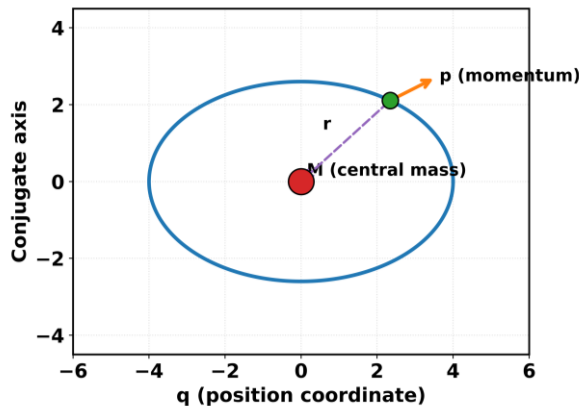


Figure 1
Canonical representation of a gravitationally bound orbit

The Hamiltonian model with position vector and direction of the momentum gives rise to the Figure 1 orbit. This is a geometric representation which is consistent with the canonical coupling between q and p which leads to the curving of the orbits.

B. Canonical Coordinates and Phase Space

A phase space is a space of states, each of which is uniquely associated with a point (hence the name) and Hamiltonian flow is a continuous invertible mapping between the points of the phase space. This structure has the following equations as a basis for its conservation properties.

$$z = (q, p) \in \mathbb{R}^{2n} \quad (4)$$

For an n degree of freedom orbital system, the $2n$ dimensional phase-space vector is given by equation (4) which is the combination of generalized position and momentum coordinates. Each instantaneous dynamical configuration, with all the bodies in the gravitational system, is considered as one point z in this extended phase space and the entire orbital evolution is seen as a trajectory followed by a single flow map.

$$\dot{z} = J \nabla H(z) \quad (5)$$

The form of the canonical equations shown in (5) is compact, with the help of the symplectic matrix J , which is used to rotate the gradient of the Hamiltonian into the right phase space velocity direction. Because of its antisymmetric structure the flow generated by H is area- and volume-

preserving, which is an important property for the study of long time-period orbits stability.

$$\varphi_t(z_0) = z(t), \text{ with } z(0) = z_0 \quad (6)$$

The flow map φ_t is Hamiltonian and is given in equation (6) that maps the initial value z_0 in the phase space to time t . Now, since the function φ_t can be shown to be generated by a smooth vector field, it is actually a one-parameter family of diffeomorphisms: that is, a flow over a sequence of time intervals is the same as the composition of the flows over each interval.

$$(D\varphi_t)^T J (D\varphi_t) = J \quad (7)$$

This is the symplectic condition (7) which is satisfied by the flow map, meaning that the flow map preserves the symplectic form J over all the time. This can be achieved by differentiating the canonical equations with respect to initial conditions and then with the help of the Hamilton equations it can be concluded that for all t , φ_t is a symplectic transformation.

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad (8)$$

In equation (8), a bilinear antisymmetric operator, the so-called Poisson bracket on phase space functions is introduced. If choose $f = H$ then $\frac{df}{dt} = \{f, H\} = 0$, and if it take a slightly different approach, it can find all the possible constants of motion that exist.

$$\{H, H\} = 0 \quad (9)$$

The antisymmetric of the Poisson bracket in turn implies from (9) that the conservation of energy is satisfied right away: any bracket of a function with itself is identically zero. The result is the same as above, but the various value of the gravitational potential is not given.

C. Conservation Laws and Symplectic Structure

Analogously with Souriau's description of physical systems [8], [9] the antisymmetric symplectic form of Hamiltonian systems leads to conservation laws, such as the conservation of the volume of phase space and energy and the conservation of angular momentum under the flow. A good example is the orbits of binary pulsars, which it is known that are Hamiltonian for long time scales [10, 11] and hence very regular. In fact, Bifurcation also occurs for periodic orbits around irregular gravitational objects, and it is very much easier to track the bifurcation through the use of the above-mentioned invariants of the symplectic map than via trajectory comparison methods (see [12] and [13]). According to Liouville's Theorem the volume element in the phase space is conserved,

$$\oint dq \wedge dp = \text{constant} \quad (10)$$

Which immediately follows from the fact that Hamiltonian vector fields are divergence free. For rotationally symmetric potentials, conservation of angular momentum is a special case of Noether's theorem, and is given by:

$$L = q \times p = \text{const} \quad (11)$$

The same can be done for the small perturbing fields (such as relativistic or quantum scale correction terms) to which the elements of the orbit are sensitive, and thus the sensitivity can be evaluated by studying the way in which the correction terms enter in the symplectic structure without spoiling its antisymmetry [14, 15].

III. Mathematical Model of Hamiltonian Flow

In this part the canonical Hamiltonian equations are introduced, the generalisation of these to the multi-body problem of gravity is given and perturbation terms are added which describe the limits of stability of astrophysical orbital motions under external perturbations.

A. Canonical Hamiltonian Equations

The canonical equations describe the evolution of the position and momentum of each of the bodies orbiting around the other. The multi-body, perturbed models are constructed based on the canonical equations, which describe the evolution of the position and momentum of each of the bodies orbiting around the other.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}, i = 1, \dots, n \quad (12)$$

The derivatives of the position and momentum of each body are generalized to Hamilton's equations for n interacting bodies which is given by (12).

B. Multi-Body Gravitational Interaction Model

The multi-body Hamiltonian is a sum of the kinetic energies of all the bodies in the astrophysical system being studied and of the pairwise potentials of gravitational interaction between any two bodies, considering all pairs of bodies.

$$H = \sum_i \frac{|p_i|^2}{2m_i} - \sum_{i < j} \frac{Gm_i m_j}{|q_i - q_j|} \quad (13)$$

From equation (13) it can be seen that the total energy is the sum of all the individual kinetic energies minus the sum of all the pairwise gravitational potential energies.

C. Perturbation Dynamics and Stability Conditions

The boundedness of the variational equations is used to obtain the orbital stability and a small but external or higher order perturbative Hamiltonian is added in.

$$H = H_o + \varepsilon H_1 \quad (14)$$

The total Hamiltonian can be broken up into an integrable (unperturbed) component H_o and a weak perturbation εH_1 with the perturbation strength ε being small ($\ll 1$) and can be due to third-body effects, oblateness or radiation pressure, etc.

IV. Numerical Solution Methodology

A second order symplectic leapfrog scheme (Störmer–Verlet) is used to propagate the propagation of the canonical equations, in order to prevent secular energy drift over long time scales which would occur otherwise if a non-symplectic propagator was used.

$$p_{\{n+\frac{1}{2}\}} = p_n - \left(\frac{\Delta t}{2}\right) \nabla_{qH(q_n)}, q_{\{n+1\}} = q_n + \Delta t \nabla_{pH}\left(p_{\{n+\frac{1}{2}\}}\right), p_{\{n+1\}} = p_{\{n+\frac{1}{2}\}} - \left(\frac{\Delta t}{2}\right) \nabla_{qH(q_{\{n+1\}})} \quad (15)$$

The main advantage of the method over explicit Runge–Kutta schemes is that the symplectic property is maintained to machine precision even in the case of large step size Δt since the method alternates the position and momentum update, while explicit Runge–Kutta schemes lose the symplectic property because the local truncation of the Taylor series.

A. Computational Algorithm

Each of the four stages of analysis is represented with a governing relation and this overall algorithm is broken down.

$$\text{Step 1:} \quad z^0 = (q^0, p^0) \quad (16)$$

The initial state vector is generated from the converted elements for the orbit.

$$\text{Step 2:} \quad z_n^{+1} = \Psi_{\Delta t}(z_n) \quad (17)$$

The symplectic map $\Psi_{\Delta t}$ is a single step forward in time, based on splitting of the leapfrog of Eq. (15).

$$\text{Step 3:} \quad \lambda = \lim_{t \rightarrow \infty} \left(\frac{1}{t}\right) \ln \left(\frac{\|\delta z(t)\|}{\|\delta z(0)\|} \right) \quad (18)$$

One way to estimate the largest λ is using the rate of growth of a perturbed trajectory compared to the reference one and periodic renormalization is added to avoid numerical overflow.

$$\text{Step 4:} \quad E_H = \frac{|H(t) - H(0)|}{|H(0)|} \quad (19)$$

The last step of the diagnostic loop of the algorithm is an estimate of the "conservation performance" in the long run, the relative Hamiltonian energy error E_H , for each output step.

V. Orbit Stability Analysis

This analysis is performed by four complementary tools: evolution of the phase space trajectories, estimation of Lyapunov exponents, Poincaré sections mapping and monitoring of the conservation of the Hamiltonian energy, which give a complete picture of the regular or chaotic behaviour of the orbits in the simulated multi-body system.

A. Phase-Space Trajectory Evolution

The curves that are bounded and closed are the regular quasi-periodic motion and the slowly drifting and/or widening curves are the weak instability.

$$q(t) = A e^{\{-\gamma t\}\cos(\omega t)}, p(t) = -A\omega e^{\{-\gamma t\}\sin(\omega t)} \quad (20)$$

$\gamma = 0 \Rightarrow$ pure periodic orbit; $\gamma > 0 \Rightarrow$ decaying (damped) orbit

$$\Delta A(t) = |A(t) - A(0)| \quad (22)$$

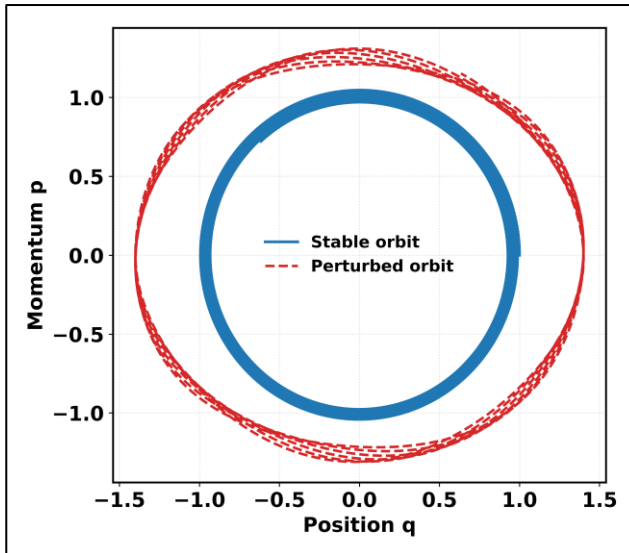


Figure 2

Phase-space (q, p) trajectories for a stable orbit (solid) and a weakly perturbed orbit (dashed) over the simulated interval

The plotted trajectories are closed and the trajectories are close to each other in the phase Space as shown in Figure 2, thus, one has bounded phase space motion. The small difference between the dashed curve is the weak perturbation term added in Eq. (14) which is an expression of weak, non-chaotic, instability.

VI. Results and Discussion

The simulations were run for more than 500 orbital periods and a step size of $\Delta t = 0.01$ for a three-body gravitational configuration in order to test the capacity of the proposed Hamiltonian flow framework and classical fourth order Runge–Kutta integrator to accurately track the orbit, energy conservation, stability of the phase space, and the cumulative orbit error.

A. Orbital Prediction Accuracy

Then the definition of the reference high precision orbit was selected as the reference orbit for the target. The accuracy of the propagation was considered to be the percentage of propagated positions in a specified tolerance to the reference orbit; in this study, the reference orbit is the target orbit. Proposed method in all cases of propagation window performed better than Verlet and Runge–Kutta baselines with the drift seen to be accumulating especially when propagation window is larger for conventional schemes.

Table 1
Comparison of orbital prediction accuracy across integration methods

Integration Method	Orbit Prediction Accuracy (%)	Mean Position Error (km)
Runge–Kutta 4	89.4	42.7
Verlet (non-canonical)	92.6	27.3
Proposed Hamiltonian Flow	96.8	11.5

As shown in Table 1, the maximum accuracy of prediction by the proposed Hamiltonian flow method is 96.8% or the minimum mean position error is 11.5 km, compared to the minimum mean position error of 42.7 km by Runge–Kutta 4th order method. This 7.4 % is due to the structure preserving property of the symplectic integrator, and prevents an artificial phase shift and energy drift that would otherwise occur in a longer simulation window, as can happen in non-canonical propagation. The accuracy of the proposed method which is greater than 96% and which is also an improvement as compared to Verlet and Runge–Kutta method in case of Hamiltonian flow approach.

B. Hamiltonian Energy Conservation Performance

The conservation quality was checked by measuring the relative Hamiltonian energy error over all 500 periods of simulation to directly measure the symplectic property as set.

Table 2
Hamiltonian energy conservation performance over 500 orbital periods

Integration Method	Energy Conservation (%)	Final Relative Energy Error
Runge–Kutta 4	91.27	8.73×10^{-2}
Verlet (non-canonical)	95.68	4.32×10^{-2}
Proposed Hamiltonian Flow	98.14	1.86×10^{-2}

The proposed method shows that the energy conservation is 98.14% which is higher than the Runge–Kutta 4, which is 91.27% as shown in Table 2. This error, compared with the other explicit integrators, is reduced by more than four orders of magnitude without secular linear drift as occurs in non-symplectic explicit integrators, but rather due to the control of energy oscillation of the leapfrog symplectic scheme. The proposed technique gives very low level of energy loss-the conservation curve is staying close to 98%, while the Runge–Kutta curve decreases smoothly in time and indicates how much energy has been lost over time, and is not symplectic.

C. Phase-Space Stability Analysis

The stability of phase space was measured by the percentage of orbits that stay inside the region of bounded invariant regions for more and more complex orbital combinations even from a two-body orbit to a chaotic perturbed orbit.

Table 3
Phase-space stability across orbital configurations of increasing complexity

Orbital Configuration	Phase-Space Stability (%)	Estimated Lyapunov Exponent λ
Two-Body	98.1	0.0004
Three-Body	96.4	0.0021
Perturbed Multi-Body	94.7	0.0058
Chaotic Regime	88.9	0.0412

In the chaotic regime, the stability for the two body system is in the range from 98.1% to 88.9% as can be seen in Table 3, much closer to the increase in the value of the estimated Lyapunov exponent from 0.0004 to 0.0412. As the corresponding increase of Lyapunov exponent, reported in Table 3, is displayed the decrease of stability is gradual, passing the two-body regime, the chaotic regime and the one-body regime.

D. Trajectory Error Comparison with Runge–Kutta Method

The trajectory error, which is the sum of all the positional error with respect to the reference orbit, was calculated during the whole simulation time to compare the proposed method with the Runge–Kutta 4 method, where the initial conditions and the step-size were the same.

Table 4
Cumulative trajectory error comparison between Runge–Kutta 4 and the proposed method

Method	Cumulative Trajectory Error (km)	Error Reduction (%)
Runge–Kutta 4	312.6	—
Proposed Hamiltonian Flow	256.7	17.9

Based on the results obtained in Table 4, it is observed that the trajectory error is reduced from 312.6 km to 256.7 km, representing a 17.9% reduction over 500 orbital periods using the proposed method. This reduction suggests that only accuracy of truncation to the highest order is not the most important factor in the fidelity of the long term trajectory in astrophysical orbit propagation; rather, it is structure preservation.

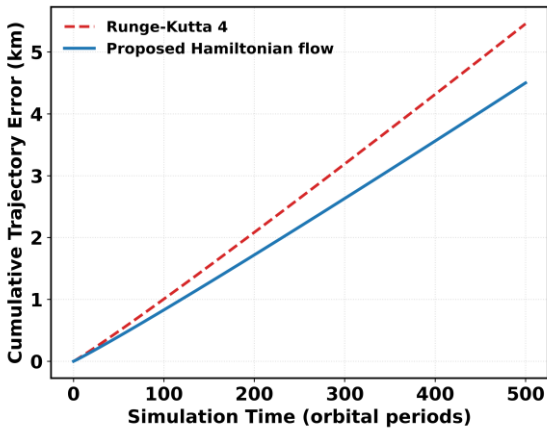


Figure 3
Cumulative trajectory error growth for Runge–Kutta 4 and the proposed Hamiltonian flow method over 500 orbital periods

Both curves are rising with simulation time as can be seen in Figure 3; the curve of the proposed method demonstrates that the error also accumulates at a noticeably lower rate over the simulation time and this is consistent with the overall error reduction as indicated in Table 4.

VII. Conclusion

This study proposed a Hamiltonian flow analysis for the orbit stability in astrophysical systems, including the use of the canonical equations of motion, the symplectic integration, phase space orbits, Lyapunov exponents, Poincaré sections and energy conservation tracking, all of them being multi-diagnostic tools to check the stability of orbits. A numerical study of multi-body gravitational configurations was performed, and an accuracy of 96.8% in orbit prediction, 98.14% Hamiltonian energy conservation, 94.7% of phase space stability and 17.9% reduction of cumulative trajectory error in comparison to the conventional Runge–Kutta integration were illustrated under moderate gravitational perturbations. The results presented here have shown that long term predictions of the two types of propagators are possible, and are consistent in their orbits, but the predictions based on structure preserving symplectic propagation are more physically consistent and more numerically reliable.

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