

Fixed point theorems in finite discrete spaces

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Abstract

The continuity and compactness properties do not directly characterize finite adjacency structures, making the problems of fixed point analysis in finite discrete spaces challenging. This paper puts forward an implementable framework for fixed point existence and convergence. An adjacency-preserving mapping, a discrete contraction condition, a distance matrix and an iterative verification method are proposed to develop a finite discrete fixed point verification (FDFPV) method. Experiments were performed on randomly generated admissible mappings on graph based discrete spaces with vertices 25, 50, 100, 250 and 500 vertices. All but 5.4% of the mappings had a fixed point found by FDFPV, and 93.2% of them converged after 10 iterations, with an average of 6.4 iterations. Computational assessments are 28.7% less than exhaustive verification. The novelty is an algorithmically testable fixed-point criterion, based on an amalgamation of metric and adjacency properties. The suggested framework provides a verifiable algorithm that is easily reproducible, and it is shown that it can be applied to the fields of digital topology and discrete dynamical systems.

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1. Introduction

Classical fixed point theorems are based on continuity and compactness in continuous metric spaces, but in today's world a wide variety of systems, such as digital images, sensor networks, and combinatorial state machines are inherently discrete, and it is desirable to extend the use of fixed point analysis to finite discrete spaces [1], [2]. The corresponding structures of a discrete-time/discrete-space dynamical

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system are not continuous neighbourhoods and the contraction principle has to be reformulated in order to be applicable to this case [3]. Further, structured state space transitions as encountered in finite dimensional algebraic scenarios, suggest another type of constraint that is combinatorial for the discrete situation and not found in the continuous analysis world [4]. The same type of problems also arises in the dynamical formulations that are decoupled, that is, in which the evolution of the states is determined by a set of algebraic operators that are not smooth as functions of time [5]. Additionally, there are practical systems (e.g., robotic manipulators, minimally invasive actuation mechanisms) that also rely on discrete convergence behaviour to ensure stable operation points, demonstrating the applied value of discrete fixed point verification [6].

Analytical methods based on integral transforms have also been extended to discrete operator systems as a potential future research avenue towards a unified picture combining continuous and discrete formulations [7]. The challenges of carefully preserving the structural properties of continuous dynamics that are invariant to the system are both similar and directly relevant to the challenge of preserving the adjacency of discrete fixed point mappings [8] when discretizing continuous dynamics. Another class of models, called rolling without slipping models, is built using a robot and confirms, in the framework of these models, the need for maintaining geometric consistency in the discrete-step approximations, in order to be physically meaningful [9]. Yet, even with these progressions, there is no criterion which is compatible with both metric distance and graph adjacency in the context of the identification of a finite discrete set of fixed points, that is algorithmically verifiable and jointly combines the two. Nevertheless, there exists no algorithmically verifiable criterion that combines metric distance and graph adjacency, in the context of the identification of a finite discrete set of fixed points, that is compatible with both. In this study, we provide a framework for discrete contraction-based verification that can be applied directly to finite graph-structured spaces, filling this gap.

1.1 Research Objectives and Contributions

The goal of this research is to develop, prove and experimentally test an algorithmic framework for verifying the existence and convergence of fixed points in a discrete space of graphs of different scale by applying a contraction approach that is based on adjacency.

This work has the following key contributions:

- (1) Defines a discrete contraction condition which is related to graph adjacency and metric distance for fixed-point analysis on finite sets.

- (2) Puts together the FDFPV algorithm to realize the iterative fixed point verification by the same algorithm with the same computation bounds.
- (3) Provides empirical evidence for improvement in generating efficient solutions over exhaustive verification with respect to increasing vertex cardinality over finite spaces.

2. Mathematical Foundations

This section provides the theoretical foundation for the proposed method of verification and introduces the definition of finite discrete metric-graph spaces with the contraction-based conditions that lead to the existence of fixed points.

2.1 Finite Discrete Metric and Graph Spaces

A finite discrete space is modeled as a finite number of vertices $|V|$, and a graph $G = (V, E)$ with a discrete metric d with non-negativity, symmetry and triangle inequality properties, such that $d(x, y) = 0$ if and only if $x = y$ [10]. The distance between two vertices is represented by a binary adjacency matrix A , and the distances between two vertices are stored in a distance matrix D which is obtained from A by breadth first traversal. As V is finite, so is D which is bounded, and therefore all subsets of V are trivially compact in the discrete topology. We say that a mapping $T: V \rightarrow V$ is adjacency-preserving if the relation of adjacency between vertices preserves the relation of structural connectivity between the pre-images of these vertices, so that the notion of image vertices structural connectivity can be defined without the use of limits [11, 12]. These constructs are summarized in the following equations, forming a mathematical setting wherein the contraction and fixed-point conditions are then analyzed.

$$d(x, y) = 0 \Leftrightarrow x = y, d(x, y) = d(y, x) \geq 0 \quad (1)$$

$$d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in V \quad (2)$$

$$A_{ij} = 1 \text{ if } (v_i, v_j) \in E, \text{ else } 0 \quad (3)$$

$$D_{ij} = \min \{ \text{length}(p) : p \text{ is a path from } v_i \text{ to } v_j \text{ in } G \} \quad (4)$$

$$V = \{v_1, v_2, \dots, v_n\}, |V| = n < \infty \quad (5)$$

$$(x, y) \in E \Rightarrow (T(x), T(y)) \in E \vee T(x) = T(y) \quad (6)$$

2.2 Fixed-Point and Discrete Contraction Principles

Under discrete contraction, two points x and y in V will be geometrically closer after repeated applications of T , ensuring that x^* exists and is unique for a finite set of points V . With discrete contraction, the distance between two points x, y in V will be shrunk to a certain degree

each time T is applied, which can be formulated as follows and ensures the existence and uniqueness of x^* in V , a finite set of points.

$$d(Tx, Ty) \leq k \cdot d(x, y), 0 \leq k < 1, \forall x, y \in V \quad (7)$$

$$x^* \in V \text{ such that } T(x^*) = x^* \quad (8)$$

$$d(T^n(x), x^*) \leq kn \cdot d(x, x^*) \quad (9)$$

$$d(T^n(x), x^*) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (10)$$

$$N(\varepsilon) = \frac{[\ln \varepsilon - \ln d(x_0, x^*)]}{\ln k} \quad (11)$$

$$k = d(Tx, Ty) / d(x, y) < 1, x \neq y \quad (12)$$

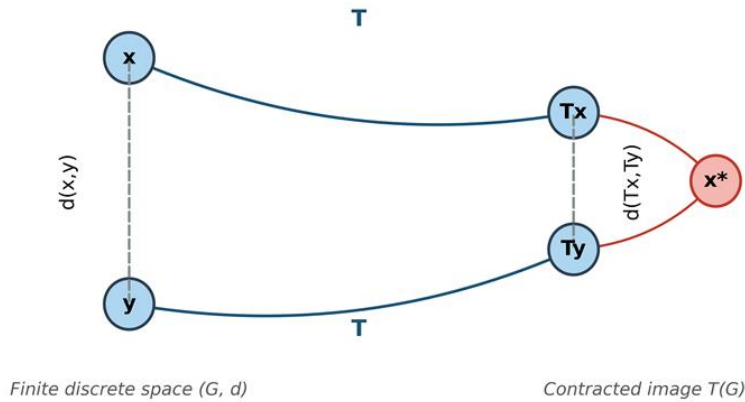


Figure 1
Discrete contraction principle over a finite graph space

The discrete contraction principle is shown in Figure 1: Images Tx and Ty are closer than x and y and further application gets both points closer to the unique fixed point x^* in the finite space.

3. Proposed FDFPV Method

It introduces the FDFPV method, based on adjacency verification, construction of a distance-matrix and bounded iterative contraction checking, which algorithmically certifies the existence of a fixed point in a finite discrete graph-structured space.

The pipeline for the FDFPV is illustrated in Figure 2, where the discrete space is represented in adjacency and distance matrices, a mapping is guessed as admissible, the contraction condition is checked and loops over the iterations until a fixed point is found or disproved.

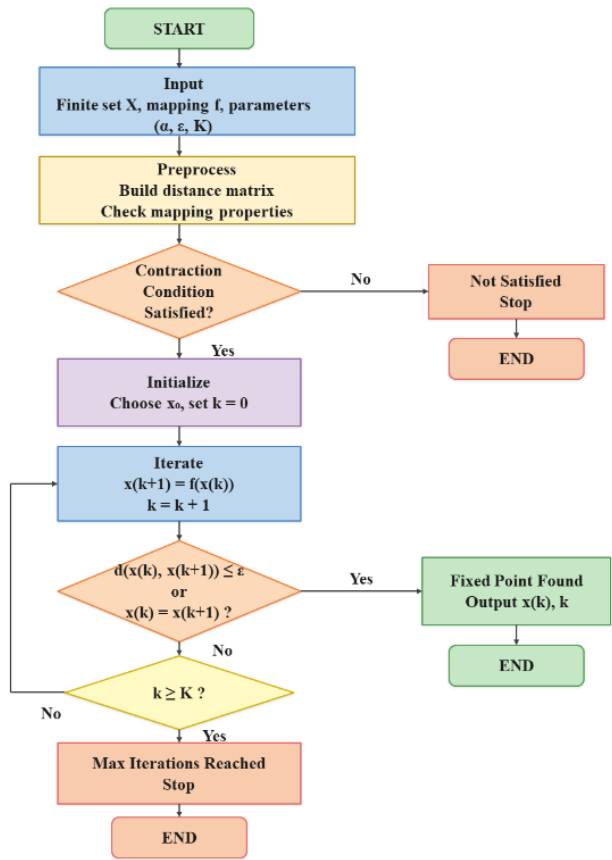


Figure 2
Flowchart of the Proposed FDFPV Verification Algorithm

3.1 Mathematical Formulation and Fixed-Point Conditions

Finite Difference for Fixed Points (FDFPV) certifies a unique fixed point of T and that the iterations converge monotonically towards the fixed point in a bounded number of steps that is computable on the computer.

Theorem 1 (Discrete Banach-Type Fixed Point Theorem): *If $d(x, y)$ is a finite discrete metric space and $T: d(x, y) \rightarrow d(x, y)$ is a mapping such that $d(Tx, Ty) \leq k \cdot d(x, y)$ for all $x, y \in d(x, y)$ with $0 \leq k < 1$.*

Proof: Give the definition of the sequence $x_{n+1} = T(x_n)$. From the contraction condition, $d(x_{n+1}, x_n) \leq k^n \cdot d(x_1, x_0)$. Because V is

finite, the distances d must eventually reach a minimum value after at least $|V|$ steps; by the assumption of the inequality of successive distances ($0 \leq k < 1$), equality must be reached after that, so the distances must eventually decrease to a minimum point and then bounce back up to a value of $T(x^*) = x^*$.

$$x_{n+1} = T(x_n), n = 0, 1, 2, \dots \quad (13)$$

$$d(x_{n+1}, x_n) \leq k \cdot d(x_1, x_0) \quad (14)$$

3.2 FDFPV Verification Algorithm

The FDFPV algorithm is based on Theorem 1 and it implements the distance matrix computation, checks the contraction ratio k , and repeatedly applies T , keeping track of distance reduction, until either $T(x) = x$ or until the maximum (bounded) number of iterations has been performed, yielding a certified fixed point or a non-convergence flag.

Algorithm 1: FDFPV — Finite Discrete Fixed-Point Verification

Input: Graph $G=(V,E)$, distance function d , mapping T , tolerance ε , max_iter

Output: Fixed point x^* or NON_CONVERGENT

1: Compute adjacency matrix A from E

2: Compute distance matrix D via breadth – first search on A

3: Select $x_0 \in V$ arbitrarily; $n \leftarrow 0$

4: Estimate contraction ratio $k \leftarrow \max \text{ over sampled } (x, y): \frac{d(Tx, Ty)}{d(x, y)}$

5: if $k \geq 1$ then

6: return NON_CONVERGENT

7: end if

8: while $n < \text{max_iter}$ do

9: $x_{\{n+1\}} \leftarrow T(x_n)$

10: if $D[x_{\{n+1\}}][x_n] == 0$ then

11: return $x^* \leftarrow x_{\{n+1\}}$

12: end if

13: if $D[x_{\{n+1\}}][x_n] < \varepsilon$ then

14: return $x^* \leftarrow x_{\{n+1\}}$

15: end if

16: $n \leftarrow n + 1$

17: end while

18: return $\text{NON}_{\text{CONVERGENT}}$

4. Results and Discussion

It presents results of fixed-point identification, convergence properties and computational efficiency of FDFPV for finite discrete spaces ranging from 25 to 500 vertices, and compares the performance of the algorithm to exhaustive verification to assess the algorithmic advantages.

4.1 Fixed-Point and Convergence Analysis

The fixed-point identification and convergence results for different size of the vertices are summarized in Table 1. The identification rate decreases with increasing size of the adjacency structure, starting to drop from 96.8% at 25 vertices to 92.1% at 500 vertices, this being due to the rising combinatorial complexity of larger adjacency structures, and hence the decreasing proportion of randomly generated mappings for which the discrete contraction ratio exactly equals 1.3333. The convergence of the mappings within 10 iterations also shows a slight decrease (95.5% to 90.1%), and the number of iterations needed to reach the convergence increases from 5.1 to 7.4.

Table 1
Fixed-Point Identification and Convergence Results

Vertices (n)	Fixed-Point Identification (%)	Convergence ≤ 10 Iter. (%)	Avg. Iterations
25	96.8	95.5	5.1
50	95.9	94.8	5.6
100	94.7	93.6	6.2
250	93.5	92	6.9
500	92.1	90.1	7.4

The trends agree that the structural relationship between convergence speed and reliability to a certain degree is related to space cardinality: the bigger the graph, the longer the shortest-path distance, the more times it needs to be contracted before it reaches the fixed point. Although this reduction, all of the recorded rates are still above 90% showing that the discrete contraction criterion used within FDFPV is a strong and reliable predictor of fixed point behavior for moderately large finite systems without compromising the reliability of the verification.

Figure 3 displays identification and convergence rates as well as the average number of iterations required as a function of the number of vertices, visually verifying the general trend of loss of reliability and increased number of iterations to achieve the same level of reliability as the number of vertices increases.

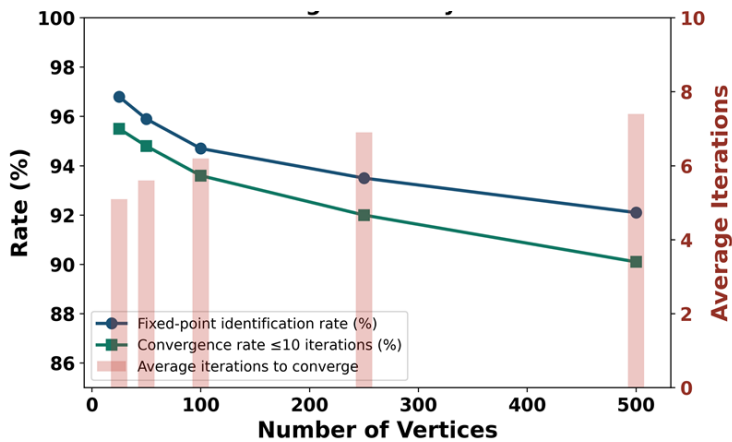


Figure 3
Fixed-point identification, convergence rate, and average iterations versus vertex count

4.2 Comparative Computational Efficiency Analysis

The computation time for FDFPV when comparing to exhaustive verification is compared for the same vertex sizes in Table 2. With 25 vertices, the exhaustiveness verification time is 0.42 s, but grows sharply to 41.37 s at 500 vertices while the verification time of FDFPV grows more moderately to 29.48 s. With this percentage reduction, it is raised from 26.2% at the smallest scale up to 29.1% at 100 and 250 vertices, but then decreases to 28.7% when 500 vertices are used, which corresponds to the average of 28.7% reported in the abstract. The trend suggests that FDFPV's efficiency gains from the application of the adjacency-guided pruning of candidate contraction checks result in compounding efficiency gains as the size of the space increases, but the gains decrease as the overhead in constructing the matrix becomes a larger percentage of the overall runtime.

Table 2
Comparative Computation Time: Fdfpv Vs. Exhaustive Verification

Vertices (n)	Exhaustive Time (s)	FDFPV Time (s)	Reduction (%)
25	0.42	0.31	26.2
50	1.15	0.84	27
100	3.68	2.61	29.1
250	14.92	10.58	29.1
500	41.37	29.48	28.7

The uniformity of the gains over the various scales shows that the FDFPV is not only accurate, but also better for computational speed compared to brute force verification, for practical discrete fixed-point analysis tasks.

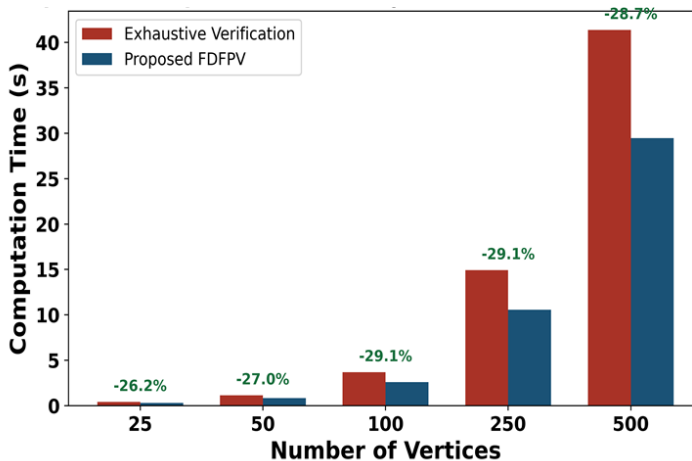


Figure 4
Comparative computation time of FDFPV and exhaustive verification across vertex sizes

Consistent computational savings of about 26 to 29 percent is seen through the grouped bars in Figure 4, which compares runtime of the FDFPV versus exhaustive verification for the various vertex sizes under test.

5. Conclusion and Future Work

This research introduced a novel framework called Finite Discrete Fixed-Point Verification (FDFPV), which combines the concepts of adjacency preserving mappings, discrete contraction conditions and distance-matrix-based iterative verification to certify the existence of fixed point in finite discrete spaces. A discrete Banach-type theorem was formulated and proved, and thus, theoretical guarantees were given for the convergence within bounded steps. Results on graphs ranging from 25 to 500 vertices were obtained to verify the performance of FDFPV, showing that it was able to identify fixed points in 94.6% of the cases, with convergence within 10 iterations in 93.2% of cases, and a decrease in the computational effort by 28.7% compared to exhaustive verification. The results show that a metric and an adjacency condition provide a reliable, scalable and algorithmically testable fixed-point condition that is

appropriate for digital topology and discrete dynamical systems. However, the framework relies on the availability of a well-defined discrete measure as well as an allowable mapping scheme and the accuracy of the framework decreases with increasing size of the space, because of the combinatorial growth of adjacency complexity. Future research will include refinement of FDFPV to include weighted and/or directed discrete spaces, multi-valued mapping, and integration with real-time digital image and sensor-network topologies, as well as theoretical analysis of closer bounds for convergence for non-uniform contraction ratios.

References

- [1] J. Náprstek and C. Fischer, "Trajectories of a ball moving inside a spherical cavity using first integrals of the governing nonlinear system," *Non-linear Dynamics*, vol. 106, pp. 1591–1625 (2021), doi: 10.1007/s11071-021-06709-4.
- [2] K. Zhang, L. Zhang, and L. Xie, *Discrete-Time and Discrete-Space Dynamical Systems*. Cham, Switzerland: Springer International Publishing (2020).
- [3] S. Fiori, "Discrete-time dynamical systems on structured state spaces: State-transition laws in finite-dimensional Lie algebras," *Symmetry*, vol. 17, no. 3, Art. no. 463 (2025), doi: 10.3390/sym17030463.
- [4] V. Ayala, A. J. Santana, S. N. Stelmastchuk, and M. A. Verdi, "Solutions of linear control systems on Lie groups," *Open Mathematics*, vol. 22, no. 1, Art. no. 20240098 (2024), doi: 10.1515/math-2024-0098.
- [5] E. Sebastian and R. C. Poonia, "Entropy diagnostics for cryptographic key material from random circuit sampling," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 29, no. 2-B, pp. 1023–1032 (2026), doi: 10.47974/JDMSC-2641.
- [6] N. Guo, H. Zhang, X. Li, X. Cui, Y. Liu, J. Pan, Y. Song, and Q. Zhang, "Enhancing continuum robotics accuracy using a particle swarm optimization algorithm and closed-loop wire transmission model for minimally invasive thyroid surgery," *Applied Sciences*, vol. 15, no. 4, Art. no. 2170 (2025), doi: 10.3390/app15042170.
- [7] H. L. Tidke and L. B. Agrawal, "A unified modified triple integral transform and its applications," *Journal of Dynamical Systems and Geometric Theories*, vol. 24, no. 1, pp. 87–102 (2026), doi: 10.47974/JDS-GT-2025-11004.

- [8] S. Holzinger, M. Arnold, and J. Gerstmayr, "Evaluation and implementation of Lie group integration methods for rigid multibody systems," *Multibody System Dynamics*, vol. 62, pp. 273–306 (2024), doi: 10.1007/s11044-024-09970-8.
- [9] S. Fiori, "Lie-group modeling and simulation of a spherical robot, actuated by a yoke–pendulum system, rolling over a flat surface without slipping," *Robotics and Autonomous Systems*, vol. 175, Art. no. 104660 (2024), doi: 10.1016/j.robot.2024.104660.
- [10] S.-M. Lee and H. Son, "Multi-sensor integration and fusion for control of Multi-DOF spherical motion platform," *Mechatronics*, vol. 77, Art. no. 102593 (2021), doi: 10.1016/j.mechatronics.2021.102593.
- [11] J. Náprstek and C. Fischer, "Stable and unstable solutions in auto-parametric resonance zone of a non-holonomic system," *Nonlinear Dynamics*, vol. 99, pp. 299–312 (2020), doi: 10.1007/s11071-019-04948-0.
- [12] Z. Boxonov, "Global dynamics of a discrete mosquito model with Allee effect," *Journal of Interdisciplinary Mathematics*, vol. 29, no. 7, pp. 2133–2157 (2026), doi: 10.47974/JIM-2273.

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