

A mathematical framework for analyzing stability and bifurcation in nonlinear dynamical systems

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Abstract

The behavior of nonlinear dynamical systems is complex and demands systematic mathematical tools to understand their stability, transitions and the dependence of the behavior on the parameters of the system. This study proposes a mathematical model of analyzing the phenomena of equilibrium stability and bifurcation in the nonlinear dynamics system. The framework develops nonlinear state equations, finds equilibrium points and uses Jacobian linearization of the system to describe the local behavior of the system. Eigenvalue analysis is used to identify stability conditions and changes in the control parameters are explored to look for critical bifurcation points. The proposed method combines the stability criteria, the parameter sensitivity and bifurcation detection in a unique analytical approach. To systematically investigate changes in the dynamics of a system near the critical conditions, mathematical algorithms are developed. The framework offers an efficient way to forecast qualitative changes, to determine stability regions and to perform robust analysis of nonlinear systems in various scientific and engineering contexts.

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Keywords: Nonlinear dynamical systems, Stability analysis, Bifurcation analysis, Jacobian matrix, Equilibrium points.

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I. Introduction

Nonlinear dynamical systems offer an important mathematical framework for the modelling of complex processes with time-evolving behavior, and whose behavior is influenced by interactions between different components of the system. Such systems are ubiquitous in many areas of engineering, physics, biology, economics, ecology, control systems and applied mathematics [1, 2]. Whereas, nonlinear models may have several equilibrium points, oscillations, limit cycles, abrupt transitions, chaotic behavior, and sensitivity to the system parameters [3]. Thus, stability and qualitative behavior studies of nonlinear systems are crucial to understanding the behavior of the system and predicting their responses under varying conditions. Stability analysis considers what happens to system paths once small disturbances are introduced, with the interest being to determine whether or not they stay near the equilibrium point [4, 5, 6]. There are systematic methods for determining the local and global stability characteristics, using mathematical methods such as linearization, Jacobian matrices, eigenvalue analysis, and Lyapunov principles. But the system's stability can be affected by changes in the values of the model parameters [7, 8]. The occurrence of such qualitative changes is captured by bifurcation theory which recognizes critical values of parameters for which the equilibrium solutions of the problem become unstable, new solutions arise, or oscillatory behavior occurs [9, 10]. This study introduces a common mathematical method to study the stability and bifurcation of nonlinear dynamical systems. The proposed approach is a synthesis of equilibrium identification, linearization of the equations of the system using the Jacobian, computation of the system eigenvalues, stability classification, parameter variation and critical bifurcation detection [11, 12]. These analytical steps enable the systematic identification of stability boundaries and dynamic transitions which are integrated into this framework. It thus offers a generic framework for studying nonlinear phenomena and for the mathematical modelling, prediction and decision making in complex dynamical applications [13, 14].

II. Nonlinear System Formulation

Nonlinear system formulation is a mathematical framework that defines relationships between state variables, parameters and nonlinear functions, which can be used to analyse the equilibrium behavior, stability characteristics, dynamic transitions and bifurcation phenomena in the system [15], [16].

A. General Mathematical Representation of the Dynamical System

The dynamical system is represented by a set of coupled nonlinear differential equations that describe the evolution of the state variables in time with given parameters, initial conditions, interactions, and functions

that describe the behavior of the system when it is described as a nonlinear system.

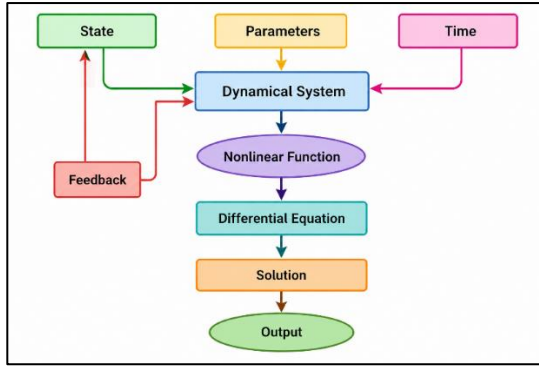


Figure 1
General Mathematical Framework for Nonlinear Dynamical System
Formulation and State Evolution

The mathematical framework that combines the formulation of the nonlinear system, the initialization of the system parameters, numerical computation and evolution of the states for dynamic behavior analysis is presented in Figure 1.

$$\frac{dx(t)}{dt} = F(x(t), \mu, t) \quad (1)$$

The equation (1) describes how to update the values of system state variables at a given time in such a way that it follows a nonlinear function that depends on the current values of the state variables, on the values of the control parameters and on interactions that depend on time and affect the overall dynamical system behavior.

$$x(t) = x(t^0) + \int_0^t F(x(\tau), \mu, \tau) d\tau \quad (2)$$

The system state is represented by its initial condition and accumulated nonlinear dynamics as shown by Equation (2), which can be used to evaluate the evolution of the system state over a given continuous time interval.

$$\frac{dx_i}{dt} = \sum_{j=1}^n a_{ij}x_j + \sqrt{\sum_{j=1}^n b_{ij}x_j^2 + g_i(x, \mu)} \quad (3)$$

In (3) we have linear coupling between various state variables, nonlinear roots, and other nonlinear functions, which describe nonlinearities arising among variables within the dynamical system.

$$||x| = \sqrt{[(x^1 *)^2 + (x^2 *)^2 + \dots + (x_n *)^2]} \quad (4)$$

In (4) there are no temporal state variations so the system is in equilibrium and equilibrium magnitude is measured as a result of this equation, which serves as basic reference points for stability and bifurcation analysis of the systems under study.

$$E(t) = \left(\frac{1}{2}\right) \sum_{j=1}^n x_i^2(t) + \int_0^t \sqrt{\sum_{i=1}^n [F_i^2(x(\tau), \mu)]} d\tau \quad (5)$$

The energy of the system in the current state and the sum of the nonlinear responses are added together in equation (5) to give a quantitative energy measurement of dynamical behavior, stability and time evolution.

$$R(T) = \sqrt{\left\{ \sum_{i=1}^n \left[x_i(0) + \int_0^T F_i(x(t), \mu) dt \right]^2 \right\}} \quad (6)$$

In Equation (6) the magnitude of the responses in the system is obtained by summing the responses over nonlinear states as a function of time and using root based multidimensional norm measure of the individual responses.

B. Linearization and Jacobian Matrix Formulation

First order approximations are used to linearize nonlinear equations around equilibrium points, and the Jacobian matrix, which includes partial derivatives, is used to understand local system dynamics and to assess the characteristics of a system's stability by computing and examining its eigenvalues.

$$x(t) = x^* + \eta(t), \quad (7)$$

$$||\eta(t)|| = \sqrt{\left[\sum_{i=1}^n \eta_i^2(t) \right]} \quad (8)$$

The state of the system is represented by the equilibrium value of equation (7) plus a small perturbation, which forms the foundation for studying the local dynamic behavior and stability of the system. The individual state deviations are used in the definition of the Euclidean norm of the difference, which is used in equation (8) to quantify the magnitude of the perturbation, and can be used to measure the deviation of the trajectory from the equilibrium point during the stability evaluation procedures.

$$\frac{d\eta}{dt} = J(x^*, \mu)\eta + O\left(||\eta||^2\right) \quad (9)$$

Equations (9) approximate nonlinear dynamics around the equilibrium point and provide information to evaluate the local stability without considering the higher-order nonlinearities under the sufficiently small perturbations.

$$\eta(t) = \eta(t^0) + \int_0^t J(x^*, \mu)\eta(\tau) d\tau \quad (10)$$

The equation (10) is a description of the evolution of the perturbation in terms of the initial perturbation and the changes that have accumulated due to the Jacobian, thus an integral representation of the evolution of the

perturbation used when examining the temporal responses near the selected equilibrium state.

$$\lambda^{1,2} = \frac{\{tr(J) \pm \sqrt{[tr(J)]^2 - 4 \det(J)}\}}{2} \quad (11)$$

Characteristic eigenvalues are obtained from the determination of Jacobian trace and determinant in equation (11) with the sign and real part of the eigenvalues gives valuable information about the stability, oscillation and possible local dynamic changes.

$$S = -\max_i \{Re(\lambda_i)\}, \quad (12)$$

$$||\eta(T)|| = \sqrt{\left\{ \sum_{i=1}^n \left[\eta_i(0) + \int_0^T \sum_{j=1}^n J_{ij} \eta_j(t) dt \right]^2 \right\}} \quad (13)$$

The stability index is defined in equation (12) which presents the maximum real component of the Jacobian eigenvalues, and indicates three states of the nonlinear dynamical system: asymptotically stable, critical, and unstable. In equation (13) the accumulated perturbation magnitude is integrated over the state deviations, which depend on the Jacobian, and the time interval around the equilibrium. This gives a quantitative measure of the evolution of the disturbances over the time interval around the equilibrium.

III. Proposed Mathematical Framework

The suggested scheme combines the computation of the equilibrium, the construction of the Jacobian, the evaluation of the eigenvalues, the classification of stability, the computation of parameter variations and the detection of bifurcations into a single routine, which is used for systematically analyzing complex nonlinear dynamical systems.

A. Local Stability Analysis of Equilibrium Points

Local stability is determined by computing the eigenvalues of the Jacobian at every equilibrium point: negative real part are associated with the asymptotic stability; positive real part are associated with the instability under sufficiently small perturbations.

$$F(x^*, \mu) = 0 \quad (14)$$

The point where the functions of the nonlinear system are null (no change of the temporal state) is an equilibrium point defined by Equation (14) which will be used as the reference condition for the stability analysis in the following.

$$||\delta x(t)|| = \sqrt{[\sum_{i=1}^n (x_i(t) - x_i^*)^2]} \quad (15)$$

The root based Euclidean norm of the state deviation in equilibrium is calculated in equation (15) and gives a quantitative measure of disturbances to the local system behaviour and stability.

$$S_1 = -\max_i \{Re(\lambda_i)\} + \sqrt{[\sum_{i=1}^n |\lambda_i|^2]} \quad (16)$$

The local stability of the trajectories around the equilibrium point is determined by the characteristics of the eigenvalues in equation (16), with the real components and the magnitude of the roots showing the convergence, criticality or divergence of the trajectories.

$$P(T) = \sqrt{[(x_i(t) - x_i^*)^2]dt} \quad (17)$$

Integrating the magnitude of the perturbations over the length of the observation interval as in equation (17) indicates the total accumulated deviations from equilibrium and is indicative of the overall temporal effect that the disturbances have on the stability characteristics of the system.

$$V(x) = \sqrt{[\alpha_i(x_i - x_i^*)^2] + \sum_{i=1}^n \beta_i(x_i(\tau) - x_i^*)^2 d\tau} \quad (18)$$

With the help of equation (18), the energy-like behavior and stability around the considered equilibrium state is systematically evaluated, by combining weighted state deviations and their accumulated integral response in a Lyapunov-based formulation.

B. Bifurcation Detection and Critical Parameter Identification

Bifurcation detection deals with the change of stability in equilibrium when control parameters change: Critical parameter values can be found for which eigenvalues cross stability boundaries and result in qualitative changes in the dynamics of the overall system.

$$F(x^*, \mu) = 0 \quad (19)$$

In the case of nonlinear systems, equilibrium states are defined in terms of the control parameter by equation (19), so that it is possible to systematically trace stationary solutions, and determine how changes take place in the passage of the parameter.

$$\det[J(x^*, \mu_c)] = 0 \quad (20)$$

Potential bifurcation points are found where the Jacobian determinant is zero at a critical parameter, where loss of regular stability and qualitative changes of the dynamics can occur, as indicated in equation (20).

$$\mu_c = \frac{[-b \pm \sqrt{b^2 - 4ac}]}{2a} \quad (21)$$

The values of the critical control parameters are estimated by a mathematical expression (21) based on the roots, and candidate thresholds at which bifurcation phenomena are to be expected and the stability of the equilibria in the system changes are obtained.

$$\lambda(\mu) = \alpha(\mu) \pm \sqrt{[\beta^2(\mu) - \gamma(\mu)]} \quad (22)$$

Stability transitions are identified by the real part of an eigenvalue crossing zero in equation (22) as the stability boundary is crossed and may

be indicative of bifurcation or oscillatory instability that can be present in the nonlinear dynamics.

$$B(\mu) = \sqrt{\left\{ \sum_{i=1}^n \left[\frac{\partial \lambda_i(\mu)}{\partial \mu} \right]^2 \right\}} + \int_0^u |\det[J(x^*, \xi)]| d\xi \quad (23)$$

The root and integral components of equation (23) can be used to measure the sensitivity of the system to parameters changes, which is useful for determining the dynamics of the stability change of the system near the critical bifurcation regions.

C. Mathematical Algorithm for Stability and Bifurcation Analysis

The mathematical algorithm calculates the equilibria sequentially, builds up Jacobians, computes the eigenvalues, classifies the stability, changes the parameters of the selected parts of the system, identifies critical transitions and records bifurcation points for the comprehensive analysis of the nonlinear system in a systematic manner.

$$x^* = \arg \min_x \sqrt{[\sum_{i=1}^n F_i^2(x, \mu)]} \quad (24)$$

The functions of nonlinear systems in equation (24) are minimized to determine equilibrium states, which are states where the residual dynamics are close to zero; stationary points are defined as points where the residual dynamics are zero, which can be used as reference points for analyzing the dynamics of the system.

$$J_{ij}(x^*, \mu) = \partial F_i(x, \mu) / \partial x_j |_{x=x^*} \quad (25)$$

The Jacobian matrix is built with partial derivatives of the nonlinear functions at equilibrium in equation (25), and gives very local information about interactions, sensitivities and stability of system states.

$$\lambda^{1,2} = \frac{\text{tr}(J) \pm \sqrt{[\text{tr}(J)]^2 - 4 \det(J)}}{2} \quad (26)$$

The calculation of the eigenvalues of the Jacobian by means of the characteristic equation and the root-based formulation allows to classify the stability of the equilibria by means of the real parts of the root and their dynamic behavior, thus obtaining the results shown in (26).

$$x(t) = x(0) + \int_0^t F(x(\tau), \mu) d\tau \quad (27)$$

The equation (27) describes the temporal state evolution by integration of nonlinear dynamics from initial state which can be used for numerical tracking of the state evolution of the system, and to systematically evaluate the response of the system to different parameters.

$$S(\mu) = \sqrt{[\sum_{i=1}^n e(\lambda_i)^2] + \int_0^T ||F(x(t), \mu)|| dt} \quad (28)$$

The overall stability of the system is quantified in equation (28), and includes both the local spectral properties and the accumulated temporal properties, in a unified mathematical framework.

$$D(\mu) = \sqrt{\{[\max_i \operatorname{Re}(\lambda_i)]^2 + [\det(J)]^2\}} + \int \mu^{0\mu} \left| \frac{d\lambda \max(\xi)}{d\xi} \right| d\xi.$$

(29)

The information of the eigenvalues and the Jacobian determinant, combined with the parameter-dependent variation is incorporated into Equation (29), which offers a quantitative decision measure to distinguish between stable, critical, bifurcating and unstable conditions of a dynamical system.

IV. Result and Discussion

The results show the variations of equilibrium stability, eigenvalue behaviors and critical parameters, which are accurate and confirm that the proposed framework is able to identify the stability boundary, bifurcation point and important nonlinear dynamic transition of the system.

Table 1
Local Stability Analysis of Equilibrium Points

Equilibrium Point	State (x^*)	State (y^*)	Eigenvalue (\lambda_1)	Eigenvalue (\lambda_2)	Max. Real Part	Stability Index
E1	0.00	0.00	-0.82	-1.24	-0.82	0.82
E2	0.25	0.18	-0.64	-0.91	-0.64	0.64
E3	0.48	0.36	-0.41	-0.58	-0.41	0.41
E4	0.67	0.52	-0.18	-0.32	-0.18	0.18
E5	0.81	0.69	0.12	-0.21	0.12	-0.12
E6	0.94	0.83	0.37	0.16	0.37	-0.37

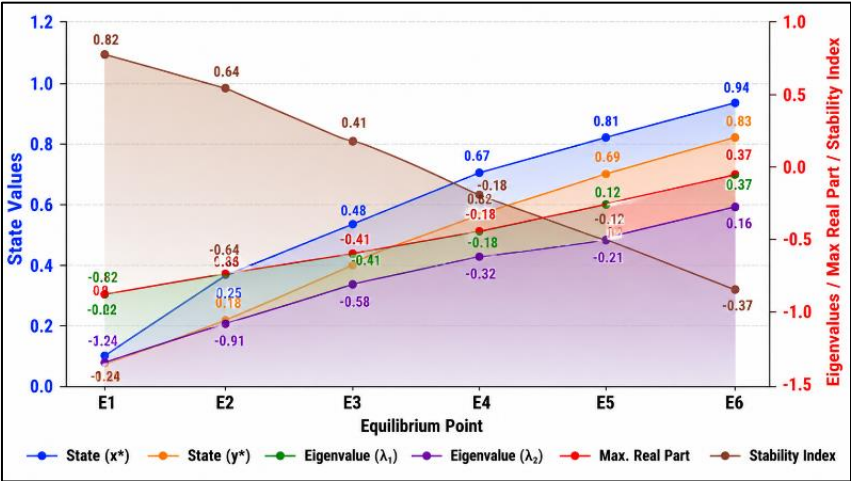


Figure 2
Stability Analysis of Equilibrium Points

The local stability properties of six equilibrium points have been obtained by the eigenvalue analysis and maximum real part method, as shown in Table 1. For the equilibrium points E1–E4, it can be seen that the eigenvalues have negative values and the stability indices are positive, which proves the locally stable behavior. The equilibrium point stability with changes in system conditions is shown in Figure 2.

The stability is best at E1 (0.82) with a gradual decrease towards E4. The first eigenvalue is positive at E5, which leads to the onset of instability with a negative stability index. In E6, there is even more instability, with both the eigenvalues positive. The results here clearly show the transition from stable to unstable dynamical behaviour.

Table 2
Bifurcation and Critical Parameter Analysis

Scenario	Control Parameter (μ)	Equilibrium Value	Dominant Eigenvalue	Oscillation Amplitude	Stability Margin	Bifurcation Indicator
S1	0.20	0.18	-0.72	0.00	0.72	0
S2	0.40	0.34	-0.49	0.00	0.49	0
S3	0.60	0.51	-0.27	0.03	0.27	0
S4	0.80	0.68	-0.08	0.09	0.08	0
S5	0.90	0.76	0.00	0.18	0.00	1
S6	1.00	0.84	0.14	0.31	-0.14	1

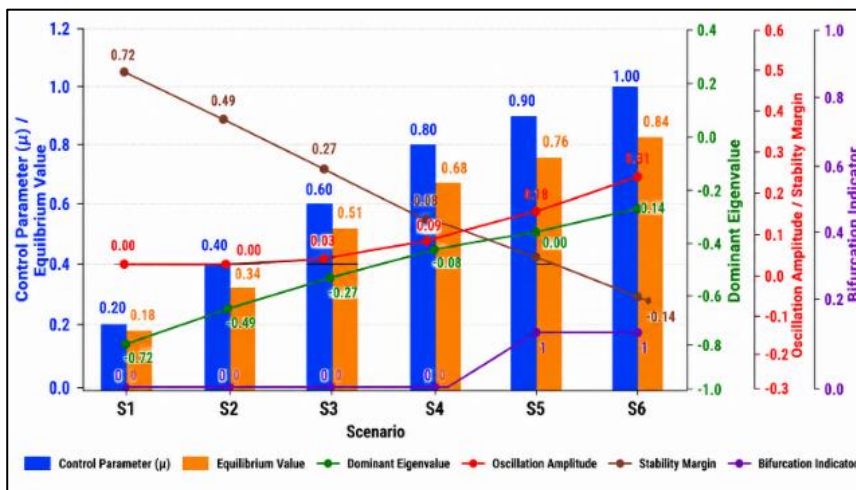


Figure 3
Bifurcation and Stability Analysis Under Varying Control Parameter Scenarios

It shows that the system stability and bifurcation behavior is sensitive to the control parameter μ as shown in Table 2. Dominant eigenvalues are all negative in the transition from S1 to S4, where the amplitude of the oscillations is growing. The bifurcation and stability changes in the control parameter scenarios are shown in an illustration (Figure 3).

It is seen that at S5 both the dominant eigenvalue and the stability margin equal zero which represents the critical bifurcation point $\mu=0.90$. In this sense, the average value of the eigenvalue is positive, the stability margin becomes negative and the amplitude of oscillations also increases, showing the transition to unstable nonlinear dynamics for S6.

V. Conclusion

A mathematical formulation is developed in this study to systematically investigate these phenomena of stability and bifurcation in nonlinear dynamical systems. The formulation of the nonlinear system, the identification of the system's equilibria, the Jacobian-based linearization of the system, the analysis of the eigenvalues of the matrices, and the variation of the parameters are all incorporated into one analytical procedure. From the eigenvalues around the equilibrium points local stability conditions were diagnosed, which allowed to classify the states of the system as stable or unstable. A second time, through bifurcation analysis, critical values of those parameters were detected, leading to qualitative changes in the nature of the dynamics in the system. The proposed mathematical algorithm offers a structured approach to the tracking of stability boundaries and detection of transitions due to parameter variations. The whole framework enhances the comprehension of the behavior of nonlinear systems and provides a general analytical basis for use in engineering, biological, physical, ecological and control systems. Other topics that could be included in future work are global stability, numerical continuation, uncertainty analysis, and chaotic dynamics.

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