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Weakly (K, L)-continuous functions in ideal bitopological spaces

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Abstract

Using the notion of ideal bitopological space we introduce the (*k, l*)-weakly (*K, L*)-continuous functions and obtain several properties and looking for its relationships with another type of continuity. Also we define the notion of *b*-(*K, L*)-compact relative and we prove the preservation of this notion under the (*k, l*)-weakly (*K, L*)-continuous functions.

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1. Introduction

The notion of a bitopological space was first introduced by Kelly in 1963 [1]. Following Kelly's work, many topologists have focused on generalizing various concepts from single topological spaces within the framework of bitopological spaces. We can see, for example, the recent works of Wasi and Rajihy [2], as well as Nassar and Ismael [3]. Generalized open sets are highly significant in General Topology and have now become a key area of research for topologists worldwide. Currently, leveraging the concept of an ideal topological, various types of continuous functions have been investigated [4],[5], [6]. The study of ideal topological space was proposed and studied by Kuratowski [7] whereas Vaidyanathaswamy [8] introduced the local function $S^*(E,\alpha)$, briefly S^*, associated to each subset $S \subseteq E$ in a structure (E,α,K), with α a topology and K an ideal on E. The concept of K-open sets relative to an ideal K in a topological space was introduced by Jankovic and Hamlet in 1990 [9]. In this paper a bitopological space $(E,\alpha_1,\alpha_2,K,L)$ mean an ordered quintuple, with α_1,α_2 topologies and K,L ideals considered on E. In this frame work, we define (k,l)-weakly (K,L)-continuous functions $(E,\alpha_1,\alpha_2,K,L) \overset{\Phi}{\to} (F,\beta_1,\beta_2)$ and study characterization for this type of continuity. Additionally we give relationships with other types of functions.

2. Preliminaries

In the sequel the ordered pair (E,α) (or briefly E) denote a topological space and $\overline{S}$ (resp. S°) is the closure (resp. interior) of $S \subseteq E$ in α. Also, being K an ideal on E, the ordered triplet (E,α,K) denote an ideal topological space. S is said to be regular open [10] (resp. preopen[11]) if $S = (\overline{S})^\circ$ (resp. $S \subseteq (\overline{S})^\circ$). Taking their complements, we have the following terminology, regular closed (resp. preclosed) set is the complement of a regular open (resp. preopen) set. A subset $S \subseteq E$ is K-open [9], if $S \subseteq (S^*)^\circ$ (resp. S is K-closed, if $E \backslash S$ is an K-open set). The K-closure (resp. K-interior) K-$\overline{S}$ (resp. K-A°) of a subset $S \subseteq E$ can be defined in similar way (*mutatis mutandis*) as closure $\overline{S}$ (resp. interior S°) of S. These classes of subsets generate the following families of subsets in a structure (E,α,K):

- $KO_{(E,\alpha)} = \{S \subseteq E: S \text{ is } K - \text{open}\}$,
- $KC_{(E,\alpha)} = \{S \subseteq E: S \text{ is } K - \text{closed}\}$,
- $RO_{(E,\alpha)} = \{S \subseteq E: S \text{ is regular open set}\}$,
- $RC_{(E,\alpha)} = \{S \subseteq E: S \text{ is regular closed set}\}$,
- $SO_{(E,\alpha)} = \{S \subseteq E: S \text{ is semi open set}\}$,
- $SC_{(E,\alpha)} = \{S \subseteq E: S \text{ is semi closed set}\}$,
- $PO_{(E,\alpha)} = \{S \subseteq E: S \text{ is pre open set}\}$,
- $SPO_{(E,\alpha)} = \{S \subseteq E: S \text{ is semi} - \text{pre open set}\}$,
- $SPC_{(E,\alpha)} = \{S \subseteq E: S \text{ is semi} - \text{pre closed set}\}$.

Also, for avoid confusions given a structure (E, α_1, α_2), we denoted $\overline{S}^k = cl_{\alpha_k}(S)$ (resp. $S^{\circ k} = int_{\alpha_k}(S)$) as the closure (resp. interior) of $S \subseteq E$ in a space (E, α_k) for $k = 1,2$.

Definition 2.1: Given (E, α_1, α_2) a subset $S \subseteq E$ is called:

(1) (k, l)-preopen [12] if $S \subset (\overline{S}^l)^{\circ k}$;

(2) (k, l)-regular open [10] if $S = (\overline{S}^l)^{\circ k}$,

with $k, l = 1,2$ and $k \neq l$.

The complements of the sets described in the above definition are called (k, l)-preclosed and (k, l)-regular closed, respectively. Furthermore, we employ the following notations:

- (k, l)-$PO_{(E,\alpha_1,\alpha_2)}$ the (k, l)-preopen sets,
- (k, l)-$RO_{(E,\alpha_1,\alpha_2)}$ the (k, l)-regular open sets,
- (k, l)-$PC_{(E,\alpha_1,\alpha_2)}$ the (k, l)-preclosed sets,
- (k, l)-$RC_{(E,\alpha_1,\alpha_2)}$ the (k, l)-regular closed sets.

Definition 2.2: Given $(E, \alpha_1, \alpha_2, K, L)$, then:

(1) the elements belong to $KO_{(E,\alpha_1)} \cup LO_{(E,\alpha_2)}$ are called b-(K, L)-open sets.

(2) $S \subseteq E$ is called b-(K, L)-closed set if $E \setminus S$ is b-(K, L)-open

From this definition, we have:

$$b - (K, L)O_{(E,\alpha_1,\alpha_2)} = \{A \subseteq E: A \in KO_{(E,\alpha_1)} \cup LO_{(E,\alpha_2)}\};$$
$$b - (K, L)C_{(E,\alpha_1,\alpha_2)} = \{A \subseteq E: E \setminus A \in KO_{(E,\alpha_1)} \cup LO_{(E,\alpha_2)}\}.$$

Definition 2.3: In a bitological structure $(Z, \gamma_1, \gamma_2, K, L)$;

(1) (γ_1, γ_2)-$(K, L)\ cl(S) = (K \setminus \overline{S}) \cap (L \setminus \overline{S})$,

(2) (γ_1, γ_2)-$(K, L)\ int(S) = (K \setminus S^{\circ}) \cup (L \setminus S^{\circ})$,

for all $S \subseteq Z$.

Lemma 2.4: *Given* $(E, \alpha_1, \alpha_2, K, L)$ *and* $S \subseteq E$*:*
(1) (γ_1, γ_2)-(K, L) $int(S)$ is b-(K, L)-open;
(2) (γ_1, γ_2)-(K, L) $cl(S)$ is b-(K, L)-closed;
(3) S is b-(K, L)-open iff $S = (\gamma_1, \gamma_2)$-(K, L) $int(S)$;
(4) S is b-(K, L)-closed iff $S = (\gamma_1, \gamma_2)$-(K, L) $cl(S)$;
(5) (γ_1, γ_2)-(K, L) $int(E \backslash S) = E \backslash (\gamma_1, \gamma_2)$-$(K, L)$ $cl(S)$;
(6) (γ_1, γ_2)-(K, L) $cl(E \backslash S) = E \backslash (\gamma_1, \gamma_2)$-$(K, L)$ $int(S)$.

Lemma 2.5: *Given* $(Z, \gamma_1, \gamma_2, K, L)$ *and* $S \subseteq Z$. x *belong to* (γ_1, γ_2)-(K, L) $cl(S)$ *iff* $U \cap S \neq \emptyset$ *for all* b-(K, L)*-open set* U *containing* x.

Definition 2.6: [13] Given $(E, \alpha_1, \alpha_2, K, L)$ and $S \subseteq E$. For $k \neq l$, (k, l)- $cl_\theta(A)$ denote the (k, l)-θ-closure of S and $x \in (k, l)$- $cl_\theta(A)$ iff $S \cap \overline{U}^l \neq \emptyset$ for all $U \in \alpha_k$ such that $x \in U$.

Definition 2.7: [1] Given $(E, \alpha_1, \alpha_2, K, L)$ and $S \subseteq E$. If (k, l)- $cl_\theta(S) = S$, S is a (k, l)-θ-closed set. Complements of (k, l)-θ-closed sets are (k, l)-θ-open sets. (k, l)- $int_\theta(S)$ denote the (k, l)-θ-interior of S, and $x \in (k, l)$- $int_\theta(S)$ iff x belong to some $U \in \alpha_k$ such that $\overline{U}^l \subset S$.

Lemma 2.8: [13] *Given* (E, α_1, α_2) *and* $S \subseteq X$. *Then*
(1) (k, l)- $int_\theta(E \backslash S) = X \backslash (k, l)$- $cl_\theta(S)$;
(2) (k, l)- $cl_\theta(E \backslash S) = X \backslash (k, l)$- $int_\theta(S)$.

Lemma 2.9: [13] *Given* $(E, \alpha_1, \alpha_2, K, L)$ *and* $U \subseteq E$. *Then*

$$U \in \alpha_l \Rightarrow (k, l) - cl_\theta(U) = \overline{U}^k.$$

Definition 2.10: [14] Given $(E, \alpha_1, \alpha_2, K, L)$ and (F, β_1, β_2) a bitopological space. A function $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\rightarrow} (F, \beta_1, \beta_2)$ is called b-(K, L)-continuous if the condition $\phi(x) \in V \in \beta_1 \cup \beta_2$, with $x \in E$, implies that $\phi(U) \subseteq V$ and $x \in U$ for some b-(K, L)-open set U.

3. (k, l)-weakly (K, L)-continuity

Definition 3.1: A function $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\rightarrow} (F, \beta_1, \beta_2)$ is called (k, l)-weakly-(K, L)- continuous if the condition $\phi(x) \in V \in \beta_k$, with $x \in E$, implies that $\phi(U) \subseteq \overline{V}^l$ and $x \in U$ for some b-(K, L)-open set U.

Proposition 3.2: b-(K, L)-continuity implies (k, l)-weakly (K, L)-continuity.

Proof: This result directly follows from Definitions 2.10 and 3.1.

The subsequent example demonstrates that, in general, the converse of Proposition 3.2 fails.

Example 3.3: For $E = \mathbb{R}$, $\alpha_1 = \alpha_2 = \{\emptyset, \mathbb{R}, \mathbb{R}\backslash\mathbb{Q}\}$, $K = L = \{\emptyset\}$, $F = \{1,2,3\}$ and $\beta_1 = \beta_2 = \{\emptyset, F, \{1\}, \{1,2\}\}$. Define $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\to} (F, \beta_1, \beta_2)$ as

$$\phi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Observe that K-open sets in (E, α_1) and (E, α_2) are the preopen sets. Furthermore, the function ϕ is (i, j)-weakly (K, L)-continuous, however ϕ is not b-(K, L)-continuous.

Theorem 3.4: *Given* $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\to} (F, \beta_1, \beta_2)$*, then:*

(1) ϕ (k, l)-weakly (K, L)-continuous;

(2) $\forall B$, (α_1, α_2)-(K, L) $cl(\phi^{-1}((\overline{B}^k)^{\circ_l})) \subset \phi^{-1}(\overline{B}^k)$;

(3) $\forall V$ $(k, l) - regularclosed$, (α_1, α_2)-(K, L) $cl(\phi^{-1}((V)^{\circ_k})) \subset \phi^{-1}(V)$;

(4) $\forall G \in \beta_l$, (α_1, α_2)-(K, L) $cl(\phi^{-1}(G)) \subset \phi^{-1}(\overline{G}^k))$;

(5) $\forall G \in \beta_k$, $\phi^{-1}(G) \subset (\alpha_1, \alpha_2)$-$(K, L)$ $int(\phi^{-1}((\overline{G}^l)))$.

are equivalent.

Proof: (1) $\Rightarrow$ (2): Given $x \in E\backslash\phi^{-1}(\overline{B}^k)$, then $\phi(x) \in F\backslash\overline{B}^k$. Thus, for some V, $\phi(x) \in V \in \beta_k$ and $V \cap B = \emptyset$. Therefore, $V \cap (\overline{B}^l)^{\circ_k} = \emptyset$ and so $\overline{V}^l \cap (\overline{B}^k)^{\circ_l} = \emptyset$. Then $x \in U$ and $\phi(U) \subset \overline{V}^l$, for some b-(K, L)-open set U. Hence $U \cap \phi^{-1}((\overline{B}^k)^{\circ_l})) = \emptyset$ and $x \in X\backslash(\alpha_1, \alpha_2)$ - $(K, L)Cl(\phi^{-1}(jInt(iCl(B))))$. Thus, (α_1, α_2)-(K, L) $cl(\phi^{-1}((\overline{B}^k)^{\circ_l}) \subset \phi^{-1}(\overline{B}^k)$.

(2) $\Rightarrow$ (3): Note that $B = \overline{(B^{\circ_l})}^k$, for every (k, l)-regular closed set $B \subseteq F$. Then

$$\begin{aligned} (\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}((B)^{\circ_l})) &= (\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}((\overline{((B)^{\circ_l})}^k)^{\circ_l}) \\ &\subseteq \phi^{-1}(\overline{(B)^{\circ_l}}^k) = \phi^{-1}(B). \end{aligned}$$

(3) $\Rightarrow$ (4): Note that $\overline{V}^k$ is (k, l)-regular closed, for any $V \in \beta_l$. Then

$$\begin{aligned} (\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}(V)) &\subseteq (\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}((\overline{V}^k)^{\circ_l}) \\ &\subseteq \phi^{-1}(\overline{V}^k). \end{aligned}$$

(4) $\Rightarrow$ (5): Assuming that $G \in \beta_k$, we have $F\backslash\overline{G}^l \in \beta_l$ and (α_1, α_2)-(K, L) $cl(\phi^{-1}(F\backslash\overline{G}^l)) \subset \phi^{-1}(\overline{(F\backslash\overline{G}^l)}^k)$. Then

$$\begin{aligned} E\backslash(\alpha_1, \alpha_2) - (K, L) -\ int(\phi^{-1}(\overline{G}^l)) &\subseteq E\backslash\phi^{-1}((\overline{G}^l)^{\circ_k}) \\ &\subseteq E\backslash\phi^{-1}(G). \end{aligned}$$

Therefore, $\phi^{-1}(G) \subset (\alpha_1, \alpha_2) - (K, L) Int(\phi^{-1}(\overline{G}^l))$.

(5) $\Rightarrow$ (1): Suppose that $G \in \beta_k$ and $\phi(x) \in G$. Then $x \in \phi^{-1}(G) \subset (\alpha_1, \alpha_2)$ - (K, L) $int(\phi^{-1}(\overline{G}^l))$. Put $U = (\alpha_1, \alpha_2)$ - (K, L) $int(\phi^{-1}(\overline{G}^l))$, U is b-(K, L)-open, $\phi(U) \subset \overline{G}^l$, and $x \in U$. In consequence ϕ is (k, l)-weakly (K, L)-continuous.

Theorem 3.5: *Given* $(E, \alpha_1, \alpha_2, K, L) \xrightarrow{\phi} (F, \beta_1, \beta_2)$, *then:*

(1) ϕ (k, l)-weakly (K, L)-continuous;

(2) $\forall A \subseteq E$, $\phi((\alpha_1, \alpha_2)$-$(K, L)$ $cl(A)) \subset (k, l)$- $cl_\theta(\phi(A))$;

(3) $\forall B \subseteq F$, (α_1, α_2)-(K, L) $cl(\phi^{-1}(B)) \subset (\phi^{-1}(k, l)$- $cl_\theta(B))$;

(4) $\forall B \subseteq F$, (α_1, α_2)-(K, L) $cl(\phi^{-1}(((k, l)$-$Cl_\theta(B)))^{\circ l}) \subset \phi^{-1}((k, l)$- $cl_\theta(B))$.

are equivalent.

Proof: (1) $\Rightarrow$ (2): If $A \subseteq E$, given $x \in (\alpha_1, \alpha_2)$ - (K, L) $cl(A)$ and $V \in \beta_k$ containing $\phi(x)$. Then $\phi(U) \subset \overline{V}^l$, for some b - (K, L) -open subset U containing x. Consequently $\emptyset \neq \phi(U) \cap \phi(A) \subset \overline{V}^l \cap \phi(A)$, because $x \in (\alpha_1, \alpha_2)$-$(K, L)$ $cl(A)$ and then $U \cap A \neq \emptyset$. Hence $\phi(x) \in (k, l)$- $cl_\theta(\phi(A))$.

(2) $\Rightarrow$ (3): Suppose that $B \subseteq F$. Then $\phi((\alpha_1, \alpha_2)$ - (K, L) $cl\phi^{-1}(B)))$ $\subseteq (k, l)$ - $cl_\theta(\phi(f^{-1}(B))) \subseteq (k, l)$ - $cl_\theta(B)$, and (α_1, α_2) - (K, L) $cl(\phi^{-1}(B))$ $\subseteq \phi^{-1}$ ((*k, l*)$-cl_\theta(B)$).

(3) $\Rightarrow$ (4): $Since$(k,l)− cl_(B)is_k−closed in *F*, but as consequence of Lemma,

$$
\begin{aligned}
&(\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}(((k, l) -\ cl_\theta(B)))^{\circ l}) \\
&\qquad \subseteq \quad \phi^{-1}((k, l) -\ cl_\theta(((k, l) - Cl_\theta(B)))^{\circ l}) \\
&\qquad = \quad \phi^{-1}(\overline{(((k, l) - Cl_\theta(B)))^{\circ l}}^k \\
&\qquad \subseteq \quad f^{-1}(\overline{((k, l) -\ cl_\theta(B))}^k \\
&\qquad = \quad \phi^{-1}((k, l) -\ cl_\theta(B)).
\end{aligned}
$$

(4) $\Rightarrow$ (1): Given $V \in \beta_l$, by Lemma 2.9, $V \subset (\overline{V}^k)^{\circ l} = ((k, l)$-$Cl_\theta(V))^{\circ l}$ and obtain

$$
\begin{aligned}
&(\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}(V)) \\
&\qquad \subseteq \quad (\alpha_1, \alpha_2) - (K, L)\ cl(\phi^{-1}(((k, l) - Cl_\theta(V)))^{\circ l}) \\
&\qquad \subseteq \quad \phi^{-1}((k, l) -\ cl_\theta(V)) = \phi^{-1}(\overline{V}^k)
\end{aligned}
$$

In consequence (α_1, α_2) - (K, L) $cl(\phi^{-1}(V)) \subset \phi^{-1}(\overline{V}^k)$. Now using Theorem 3.4, ϕ is (k, l)-weakly (K, L)-continuous.

Theorem 3.6: *The following conditions:*

(1) ϕ (k, l)-*weakly* (K, L)-*continuous;*

(2) (α_1,α_2) - $(K,L)\ cl(\phi^{-1}(V)) \subset \phi^{-1}(\overline{V}^{l})$ *for every* (k,l) *-preopen set* $V \subseteq F$;

(3) $\phi^{-1}(V) \subset (\alpha_1,\alpha_2)$ - $(K,L)\ int(\phi^{-1}(\overline{V}^{l}))$ *for every* (k,l) *-preopen set* $V \subseteq F$;

are equivalent for $(E,\alpha_1,\alpha_2,K,L) \xrightarrow{\phi} (F,\beta_1,\beta_2)$.

Proof: (1) $\Rightarrow$ (2): Assuming that $x \notin \phi^{-1}(\overline{V}^{k})$, with V an arbitrary (k,l)-preopen subset of F, we have that $\phi(x) \in W$ and $W \cap V = \emptyset$ for some $W \in \beta_k$, and then $\overline{W \cap V}^{k} = \emptyset$. Now, since V is (k,l)-preopen, follows

$$\begin{aligned} V \cap \overline{W}^{l} &\subseteq (\overline{V}^{k})^{\circ_l} \cap \overline{W}^{l}, \\ &\subseteq \overline{(\overline{V}^{k})^{\circ_l}}^{l} \cap W, \\ &\subseteq \overline{\overline{V}^{k}}^{l} \cap W, \\ &\subseteq \overline{\overline{V \cap W}^{k}}^{l} = \emptyset. \end{aligned}$$

Being ϕ (k,l)-weakly (K,L)-continuous and $\phi(x) \in W \in \beta_k$, there exists a b-(K,L)-open set U such that $x \in U$ and $\phi(U) \subset \overline{W}^{l}$. Thus $\phi(U) \cap V = \emptyset$ and hence $U \cap \phi^{-1}(V) = \emptyset$. This shows that $x \notin (\alpha_1,\alpha_2)$-$(K,L)Cl(\phi^{-1}(V))$. In consequence, (α_1,α_2)-$(K,L)\ cl(\phi^{-1}(V)) \subset \phi^{-1}(\overline{V}^{k})$.

(2) $\Rightarrow$ (3): If $V \subseteq F$ is a (k,l)-preopen set. By (2), $\phi^{-1}(V) \subset \phi^{-1}((V^{l})^{\circ_k}) = E \backslash \phi^{-1}(\overline{(F \backslash V^{l})}^{k} \subset E \backslash (\alpha_1,\alpha_2)$ - $(K,L)\ cl(\phi^{-1}(F \backslash \overline{V}^{l})) = (\alpha_1,\alpha_2)$ - $(K,L)\ int(\phi^{-1}(\overline{V}^{l}))$.

(3) $\Rightarrow$ (1): Being V a β_k-open subset of F, then V is (k,l)-preopen set in F and $\phi^{-1}(V) \subset (\alpha_1,\alpha_2)$ - $(K,L)\ int(\phi^{-1}(\overline{V}^{l}))$. By Theorem 3.4, ϕ is (k,l)-weakly (K,L)-continuous.

Lemma 3.7: *If* $(E,\alpha_1,\alpha_2,K,L) \xrightarrow{\phi} (F,\beta_1,\beta_2)$ *is* (k,l)*-weakly* (K,L)*-continuous and* $(F,\beta_1,\beta_2) \xrightarrow{\psi} (Z,\eta_1,\eta_2)$ *is pairwise continuous, then* $(E,\alpha_1,\alpha_2) \xrightarrow{\psi\circ\phi} (Z,\eta_1,\eta_2)$ *is a* (k,l)*-weakly* (K,L)*-continuous.*

Proof: Suppose that $\psi(\phi(x)) \in W$ and $W \in \eta_k$. Then $\psi^{-1}(W) \in \beta_k$ and $\phi(x) \in \psi^{-1}(W)$, so $\phi(U) \subset \overline{\psi^{-1}(W)}^{l}$ for some b-(K,L)-open subset U containing x. Being ψ is pairwise continuous, then

$$(\psi \circ \phi)(U) \subset \psi(jCl(\psi^{-1}(W))) \subset \psi(\psi^{-1}(\overline{\psi^{-1}(W)}^{l})) \subset \overline{\psi^{-1}(W)}^{l}.$$

Definition 3.8: If the condition $x \in U \in \alpha_k$, implies that $\overline{V}^l \subset U$ for some $V \in \alpha_k$ containing x, (E, α_1, α_2) is called (k,l)-regular [1].

Lemma 3.9: [15] *If* (E, α_1, α_2) *is a* (k,l)*-regular bitopological space, then* (k,l)- $cl_\theta(A) = A$ *for every* α_k*-closed set* $A \subseteq E$.

Theorem 3.10: *Given* $(E, \alpha_1, \alpha_2, K, L) \xrightarrow{\phi} (F, \beta_1, \beta_2)$ *with* (F, β_1, β_2) a (k,l)-*regular bitopological space. Then the following statements:*

(1) ϕ b-(K,L)*-continuous;*

(2) $\phi^{-1}((k,l)\text{- } cl_\theta(B))$ *is* b-(K,L)*-closed, for every all* $B \subseteq F$;

(3) ϕ (k,l)*-weakly* (K,L)*-continuous;*

(4) $\phi^{-1}(B)$ *is* b-(K,L)*-closed, for any* (k,l)-θ*-closed set* $B \subseteq F$;

(5) $\phi^{-1}(V)$ *is* b-(K,L)*-open, for every* (k,l)-θ*-closed set* $V \subseteq F$.

are equivalent.

Proof: (1) $\Rightarrow$ (2): Note that (k,l)- $cl_\theta(B)$ is β_k-closed for every $B \subseteq F$, then $\phi^{-1}((k,l)\text{- } cl_\theta(B))$ is b-(K,L)-closed in E.

(2) $\Rightarrow$ (3): Observe that for any $B \subseteq F$, we have

$$\begin{aligned}(\alpha_1, \alpha_2) - (K,L)\, cl(\phi^{-1}(B)) \\ &\subseteq (\alpha_1, \alpha_2) - (K,L)\, cl(\phi^{-1}((k,l) - cl_\theta(B))) \\ &= \phi^{-1}((k,l) - cl_\theta(B)).\end{aligned}$$

Thus, by Theorem 3.5, ϕ is (k,l)-weakly (K,L)-continuous.

(3) $\Rightarrow$ (4): If $B \subseteq F$ is a (k,l) - θ -closed set, by Theorem 3.5, (α_1, α_2)-$(K,L)\, cl(\phi^{-1}(B)) \subset \phi^{-1}((k,l)\text{- } cl_\theta(B)) = \phi^{-1}(B)$. Hence, according to Lemma 2.4, $\phi^{-1}(B)$ is b-(K,L)-closed in E.

(4) $\Rightarrow$ (5): Suppose that $V \subseteq F$ is a (k,l)-θ-open set. According to (4), $\phi^{-1}(F - V) = E \backslash \phi^{-1}(V)$ is b - (K,L) -closed and then $\phi^{-1}(V)$ is b-(K,L)-open.

(5) $\Rightarrow$ (1): Being F (k,l)-regular and using Lemma 3.9, (k,l)- $cl_\theta(B) = B$ for all β_k-closed subset $B \subseteq F$, and then every β_k-open set is (k,l)-θ-open. Thus $\phi^{-1}(V)$ is b-(K,L)-open for all $V \in \beta_k$, and by Definition 2.10, ϕ is b-(K,L)-continuous function.

Definition 3.11: $(E, \alpha_1, \alpha_2, K, L) \xrightarrow{\phi} (F, \beta_1, \beta_2)$ satisfies the b-(K,L)-interiority condition if (α_1, α_2)-$(K,L)\, int(\phi^{-1}(\overline{V}^l)) \subseteq \phi^{-1}(V)$ for every $V \in \beta_k$.

Example 3.12: In Example 3.3, the function ϕ, does not satisfies the b-(K,L)-interiority condition.

Example 3.13: Take $E = \mathbb{R}$ with topologies $\alpha_1 = \alpha_2 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and ideals $K = L = \{\emptyset\}$ and $F = \{1,2,3\}$ with topologies $\beta_1 = \beta_2 = \{\emptyset, F, \{1\}, \{1,2\}\}$.

Define $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\rightarrow} (F, \beta_1, \beta_2)$ as

$$\phi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Note that ϕ satisfies the b-(K, L)-interiority condition, also ϕ is (k, l)-weakly (K, L)-continuous function.

Remark 3.14: If $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\rightarrow} (F, \beta_1, \beta_2)$ have the b-(K, L)-interiority condition, ϕ is b-(K, L)-continuous iff ϕ is (k, l)-weakly (K, L)-continuous as shows the following result.

Theorem 3.15: $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\rightarrow} (F, \beta_1, \beta_2)$ *is* b-(K, L)*-continuous, if* ϕ *is* (k, l)*-weakly* (K, L)*-continuous and have the* b-(K, L)*-interiority property*.

Proof: If $V \in \beta_k$, since ϕ is a (k, l)-weakly (K, L)-continuous function, by Theorem 3.4, $\phi^{-1}(V) \subset (\alpha_1, \alpha_2)$ - $(K, L)\ int(\phi^{-1}(\overline{V}^l))$. Now, using the b-(K, L)-interiority condition of ϕ,

$$(\alpha_1, \alpha_2) - (K, L)\ int(\phi^{-1}(\overline{V}^l)) \subset \phi^{-1}(V),$$

and therefore $\phi^{-1}(V) = (\alpha_1, \alpha_2) - (K, L)\ int(\phi^{-1}(\overline{V}^l))$. So, from Lemma 2.4, $\phi^{-1}(V)$ is b-(K, L)-open. Then ϕ is b-(K, L)-continuous.

Definition 3.16: Given $(Z, \gamma_1, \gamma_2, K, L)$ and $S \subseteq Z$. The (γ_1, γ_2)-(K, L)-frontier of S is defined as:

$$(\gamma_1, \gamma_2) - (K, L)\ Fr(S) = (\gamma_1, \gamma_2) - (K, L)\ cl(S) \cup (\gamma_1, \gamma_2) - (K, L)\ cl(Z \backslash S).$$

Observe that,

$$(\gamma_1, \gamma_2) - (K, L)\ Fr(S) = (\gamma_1, \gamma_2) - (K, L)\ cl(S) \backslash (\gamma_1, \gamma_2) - (K, L)\ int(S).$$

Theorem 3.17; *Given* $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\rightarrow} (F, \beta_1, \beta_2)$. *The sets:*

$$\cup \{(\alpha_1, \alpha_2) - (K, L)\ Fr(\phi^{-1}(\overline{V}^l) : x \in E and \phi(x) \in V \in \tau_k\}$$

and

$$\{x \in E : \phi\ is\ not\ (k, l) - weakly\ (K, L) - continuous\},$$

are equals.

Proof: If ϕ is not (k, l)-weakly (K, L)-continuous at $x \in E$, then for every b-(K, L)-open subset U containing x, we can find $V \in \beta_k$ containing $\phi(x)$ such that $U \cap (E \backslash \phi^{-1}(\overline{V}^l)) \neq \emptyset$. By Lemma 2.5, $x \in (\alpha_1, \alpha_2)$-$(K, L)\ cl(E \backslash \phi^{-1}(\overline{V}^l))$. Since $x \in \phi^{-1}(\overline{V}^l))$, $x \in (\alpha_1, \alpha_2)$-$(K, L)\ cl(\phi^{-1}(\overline{V}^l))$ and hence $x \in (\alpha_1, \alpha_2)$ - $(K, L) Fr(\phi^{-1}(\overline{V}^l))$. Reciprocally, by (k, l)-weakly (K, L)-continuity of ϕ at x, for all $V \in \beta_k$ containing $\phi(x)$, we can find a

b-(K,L)-open subset U in such a manner that $x \in U$ and $\phi(U) \subseteq \overline{V}^l$ and hence $x \in U \subseteq \phi^{-1}(\overline{V}^l)$. Thus $x \in (\alpha_1,\alpha_2)$ - $(K,L)\ int(\phi^{-1}(\overline{V}^l))$, but this is a contradiction because $x \in (\alpha_1,\alpha_2)$-$(K,L)Fr(\phi^{-1}(\overline{V}^l))$.

Definition 3.18: [14] $(E,\alpha_1,\alpha_2,K,L) \xrightarrow{\phi} (F,\beta_1,\beta_2)$ is said to be (k,l)-almost (K,L)-continuous if given any $V \in \beta_k$ containing $\phi(x)$, with $x \in E$, there exists a b-(K,L)-open subset U such that $x \in U$ and $\phi(U) \subseteq (\overline{V}^l)^{\circ_k}$.

Remark 3.19: Clearly,

$$(k,l) - \text{almost}(K,L) - \text{continuity} \Rightarrow (k,l) - \text{weakly}(K,L) - \text{continuity}$$

Lemma 3.20: [14] $(E,\alpha_1,\alpha_2,K,L) \xrightarrow{\phi} (F,\beta_1,\beta_2)$ *is* (k,l)*-almost* (K,L)*-continuous iff for all* (k,l)*-regular open subset* $G \subseteq F$, $\phi^{-1}(G)$ *is* b-(K,L)*-open in* E.

Definition 3.21: If the condition $x \in U$, with U a set of (i,j) regular open, implies that $x \in V \subseteq \overline{V}^l \subseteq U$ for some (k,l) regular open V, (E,α_1,α_2) is called (k,l) almost regular [16].

Remark 3.22: The following theorem shows that given a function $(E,\alpha_1,\alpha_2,K,L) \xrightarrow{\phi} (F,\beta_1,\beta_2)$ such that (F,β_1,β_2) is an (k,l)-almost regular bitopological space, both notions (k,l)-almost (K,L)-continuity and (k,l)-weakly (K,L)-continuity are equivalent.

Theorem 3.23: *Given* $(E,\alpha_1,\alpha_2,K,L) \xrightarrow{\phi} (F,\beta_1,\beta_2)$, *with* (F,β_1,β_2) (k,l)*-almost regular. We have*

$$\phi\ (k,l) - \text{almost}(K,L) - \text{continuous}$$
$$\Updownarrow$$
$$\phi\ (k,l) - \text{weakly}(K,L) - \text{continuous}$$

Proof: $(\Rightarrow)$ Given any (k,l)-regular open subset $G \subseteq F$ such that $x \in \phi^{-1}(G)$, then $\phi(x) \in G$. Using almost (k,l)-regularity of codomain, we find that $\phi(x) \in V_0 \subset \overline{V_0}^l \subseteq G$ for some (k,l)-regular open $V_0 \subseteq F$. By the (k,l)-weakly (K,L)-continuity of ϕ, there exists a b-(K,L)-open $U \subseteq E$ such that $x \in U$ and $\phi(U) \subseteq \overline{V_0}^l \subseteq G$. Consequently $x \in U \subseteq \phi^{-1}(G) \subseteq (\alpha_1,\alpha_2)$-$int(\phi^{-1}(G))$ and $\phi^{-1}(G) = (\alpha_1,\alpha_2)$-$(K,L)\ int(\phi^{-1}(G))$. Now, from Lemma 2.4, $\phi^{-1}(G)$ is b-(K,L)-open and then, by Lemma 3.20, ϕ is (k,l)-almost (K,L)-continuous.

$(\Leftarrow)$ This is obvious.

Definition 3.24: [17] Given (E, α_1, α_2) and $C \subseteq E$. C is called a set (k, l)-quasi H-closed relative to E, if every collection $\{U_\alpha\}_{\alpha \in \Omega} \subseteq \alpha_k$ such that $C \subseteq \bigcup_{\alpha \in \Omega} U_\alpha$ contain a finite subcollection $\{U_\alpha\}_{\alpha \in \Omega_0}$, where $\Omega_0 \subset \Omega$ and $C \subseteq \bigcup_{\alpha \in \Omega_0} U_\alpha$

Definition 3.25: Given $(E, \alpha_1, \alpha_2, K, L)$ and $C \subseteq E$. C is called b-(K, L)-compact relative if every b-(K, L)-open cover of C has a finite subcover.

Theorem 3.26: *If* $(E, \alpha_1, \alpha_2, K, L) \overset{\phi}{\to} (F, \beta_1, \beta_2)$ *is* (k, l) *-weakly* (K, L) *-continuous and* C *is* b - (K, L) *-compact relative, then* $\phi(C)$ *is* (k, l)*-quasi* H*-closed relative.*

Proof: Being C be b-(K, L)-compact relative, for every $\{V_\alpha\}_{\alpha \in \Omega}$ cover of $\phi(C)$ by β_k-open subsets of (F, β_1, β_2) $\phi(C) \subseteq \bigcup_{\alpha \in \Omega} V_\alpha$ and so $C \subseteq \bigcup_{\alpha \in \Omega} \phi^{-1}(V_\alpha)$. Now, by (k, l)-weakly (K, L)-continuity of ϕ and Theorem 3.4, for each $\alpha \in \Omega$

$$\phi^{-1}(V_\alpha) \subset (\alpha_1, \alpha_2) - (K, L)\ int(\phi^{-1}(\overline{V_\alpha}^{l})).$$

Therefore, $C \subset \bigcup \{(\alpha_1, \alpha_2)$ - $(K, L)\ int(\phi^{-1}(\overline{V_\alpha}^{l}))): \alpha \in \Omega\}$. Since C is b-(K, L)-compact relative and (α_1, α_2)-$(K, L)\ int(\phi^{-1}(\overline{V_\alpha}^{l}))$ is a b-(K, L)-open cover, we can find a finite set $\Omega_0 \subset \Omega$ and $C \subseteq \bigcup \{(\alpha_1, \alpha_2)$ - $(K, L)\ int(\phi^{-1}(\overline{V_\alpha}^{l})): \alpha \in \Omega_0\}$. Then

$$\begin{aligned} \phi(C) &\subseteq \bigcup \{\phi((\alpha_1, \alpha_2) - (K, L)\ int(\phi^{-1}(\overline{V_\alpha}^{l}))): \alpha \in \Omega_0\}, \\ &\subseteq \bigcup \{\phi(\phi^{-1}(\overline{V_\alpha}^{l})): \alpha \in \Omega_0\}, \\ &\subseteq \bigcup \{\overline{V_\alpha}^{l} : \alpha \in \Omega_0\}. \end{aligned}$$

In consequence, the image $\phi(K)$ is (k, l)-quasi H-closed relative.

Definition 3.27: Given $(Z, \gamma_1, \gamma_2, K, L) \overset{\phi}{\to} (F, \beta_1, \beta_2)$,

$$D_{w(K,L)}(\phi) = \{x \in Z: \phi \text{ is not } (k, l) - \text{weakly } (K, L) - \text{continuous at } x\}.$$

Example 3.28: According with Example 3.3, $D_{w(K,L)}(\phi) = \{\emptyset\}$.

Example 3.29: Consider $(E, \alpha_1, \alpha_2, K, L)$ with $E = \{1,2,3\}$, $\alpha_1 = \{\emptyset, X, \{1\}, \{1,2\}\}$, $\alpha_2 = \{\emptyset, E, \{2,3\}\}$, $K = \{\emptyset\}$, $L = \{\emptyset, \{2\}\}$, and (F, β_1, β_2) with $F = \{1,2,3\}$, $\beta_1 = \{\emptyset, F, \{1\}\}$ and $\beta_2 = \{\emptyset, Y, \{2\}\}$.

Observe that:

$KO(E, \alpha_1) = \{\emptyset, E, \{1\}, \{1,2\}, \{1,3\}\}$.

$LO(E, \alpha_2) = \{\emptyset, E, \{3\}, \{1,3\}, \{2,3\}\}$.

$KO(E, \alpha_1) \cup LO(E, \alpha_2) = \{\emptyset, E, \{1\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.

$KC(E, \alpha_1) \cap LC(E, \alpha_2) = \{\emptyset, E, \{2\}\}$.

Define $(E,\alpha_1,\alpha_2,K,L)\overset{\phi}{\rightarrow}(F,\beta_1,\beta_2)$, where $\phi(1)=2$, $\phi(2)=1$, $\phi(3)=3$. Easily $D_{w(K,L)}(\phi)=\{\emptyset\}$ and then ϕ is (k,l)-weakly (K,L)-continuous but ϕ is not (k,l)-almost (K,L)-continuous.

Theorem 3.30: *Given a function* $(E,\alpha_1,\alpha_2,K,L)\overset{\phi}{\rightarrow}(F,\beta_1,\beta_2)$, *the following properties hold:*

$$\begin{aligned} &D_{w(K,L)}(\phi)\\ &=\cup_{G\in P_{(F,\beta_1,\beta_2)}}\{(\alpha_1,\alpha_2)\text{-}(K,L)\ cl(\phi^{-1}((\overline{G}^{k})^{\circ l}))\setminus\phi^{-1}(\overline{G}^{k})\}\\ &=\cup_{G\in(k,l)-RC_{(F,\beta_1,\beta_2)}}\{(\alpha_1,\alpha_2)\text{-}(K,L)\ cl(\phi^{-1}((G)^{\circ l}))\setminus\phi^{-1}(G)\}\\ &=\cup_{G\in\beta_l}\{(\alpha_1,\alpha_2)\text{-}(K,L)\ cl(\phi^{-1}(G))\setminus\phi^{-1}(\overline{G}^{k})\}\\ &=\cup_{G\in\beta_k}\{\phi^{-1}(G)\setminus(\alpha_1,\alpha_2)\text{-}(K,L)Int(\phi^{-1}(\overline{G}^{l}))\}\\ &=\cup_{G\in P_{(E,\tau_1,\tau_2)}}\{\phi((\alpha_1,\alpha_2)\text{-}(K,L)\ cl(G))\setminus(k,l)\text{-}cl_\theta(\phi(G))\}\\ &=\cup_{G\in P_{(F,\beta_1,\beta_2)}}\{(\alpha_1,\alpha_2)\text{-}(K,L)\ cl(\phi^{-1}(G))\setminus(\phi^{-1}(k,l)\text{-}cl_\theta(G))\}\\ &=\cup_{G\in P_{(F,\beta_1,\beta_2)}}\{(\alpha_1,\alpha_2)\text{-}(K,L)\ cl(\phi^{-1}(((k,l)\text{-}Cl_\theta(G))^{\circ l}))\setminus\phi^{-1}((k,l)\text{-}cl_\theta(G))\}\\ &=\cup_{G\in(k,l)-PO_{(F,\beta_1,\beta_2)}}\{(\alpha_1,\alpha_2)\text{-}(K,L)\ cl(\phi^{-1}(G))\setminus\phi^{-1}(\overline{G}^{k})\}\\ &=\cup_{G\in(k,l)-PO_{(F,\beta_1,\beta_2)}}\{\phi^{-1}(G)\setminus(\alpha_1,\alpha_2)\text{-}(K,L)\ int(\phi^{-1}(\overline{G}^{l}))\} \end{aligned}$$

Proof: It is a consequence of Theorems 3.4, 3.5 and 3.6.

Conclusion

(1) The (k,l)-weakly (K,L)-continuous functions are characterized.

(2) Every b-(K,L)-continuous function is (k,l)-weakly (K,L)-continuous. But not the converse.

(3) If $(E,\alpha_1,\alpha_2,K,L)\overset{\phi}{\rightarrow}(F,\beta_1,\beta_2)$ is (k,l)-weakly (K,L)-continuous and (F,β_1,β_2) is an (k,l)-regular bitopological space then the notions of b-(K,L)-continuous and (k,l)-weakly (K,L)-continuous are equivalent.

(4) If $(E,\alpha_1,\alpha_2,K,L)\overset{\phi}{\rightarrow}(F,\beta_1,\beta_2)$ satisfies the b - (K,L) -interiority condition, b-(K,L)-continuity and (k,l)-weakly (K,L)-continuity are equivalent.

(5) Every (k,l)-almost (K,L)-continuous is (k,l)-weakly (K,L)-continuous. But not the converse.

(6) If (F,β_1,β_2) is an (k,l)-almost regular bitopological space, then ϕ is (k,l)-almost (K,L)-continuous iff ϕ is (k,l)-weakly (K,L)-continuous.

(7) If $(E,\alpha_1,\alpha_2,K,L)\overset{\phi}{\rightarrow}(F,\beta_1,\beta_2)$ a function and (F,β_1,β_2) is an (k,l)-almost regular bitopological space, then ϕ is (k,l)-almost (K,L)-continuous iff ϕ is (k,l)-weakly (K,L)-continuous

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